Bol. Soc. Paran. Mat. (3s.) **v. 33** 2 (2015): 97–109.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v33i2.23637

On Integral Representation for Solution of Generalized Sturm-Liouville Equation with Discontinuity Conditions

Y. Güldü and S. Gülyaz

ABSTRACT: In this paper, some properties of kernel and integral representation of Jost solution are studied for Sturm-Liouville operator with diffusion potential and discontinuity on the half line.

Key Words: Transformation operator, Jost solution, Integral equation

Contents

1. Introduction

This paper is concerned with the generalized Sturm-Liouville equation,

$$-y''(x)+\left[q(x)+2\lambda p(x)-\lambda^2\right]y(x)=0,\quad x\in(0,a)\cup(a,\infty) \tag{1}$$

with discontinuity conditions

$$\begin{aligned} y(a-0)&=\alpha y(a+0)\\ y'(a-0)&=\alpha^{-1}y'(a+0)\end{aligned} \tag{2}$$

where $\alpha\in\mathbb{R}^+\backslash\{1\}$, λ is a complex parameter, $q(x)$ is a complex valued function, $q(x)\in L_2(0,\infty)$ and satisfies condition

$$\int\limits_0^\infty (1+x)\,|q(x)|\,dx<\infty \tag{3}$$

$p(x)\in W_2^1(0,\infty)$ is a complex valued function.

The direct and inverse scattering problems for Schrödinger equations of the type (1) without discontinuity conditions have been studied extensively (see e.g., [1,5,7,10,17,18,19,20,21,22,23] and extended reference lists).

Transformation operators have been used in the spectral analysis of Sturm-Liouville operators and other classical operators of mathematical physics [5,11,26, and the references therein]. Particularly, for Sturm-Liouville and Dirac operators,

2000 *Mathematics Subject Classification:* Primary 34A55, Secondary 34B24, 34L05

transformation operator is so much important for solution of inverse spectral problems (see the original papers [11,26] and the monographs [6,11,28] for extended reference lists).

The inverse scattering problem for equation (1) was studied by Jaulent and Jean [17,18,19,20]. They derived a Gelfand-Levitan-Marchenko integral equation coupled with a nonlinear ordinary differential equation to recover the potentials from the scattering data. Therefore, they proved that the potentials are uniquely determined by the scattering data. After that Sattinger and Szmigielski made easy the Jaulent-Jean method to illustrate the existence of isospectral flows in a nonlinear evolution equation [7]. Furthermore, the inverse spectral problem of recovering pencils of second-order differential operators on the half-line with turning points was studied by Yurko [29]. Moreover, integral representation for the Jost solution of the Sturm-Liouville equation with discontinuity conditions and some properties of the kernel of the obtained integral representation were investigated in 2007 [10].

In the present paper, we establish the integral representation for the Jost solution of equation (1) with discontinuity conditions (2) and study some properties for the kernel of the integral representation.

2. Representation for the Solution

Problem (1)-(2) is called an impulsive problem.

The function $e(x,\lambda)$ satisfying (1), conditions (2) and condition at infinity $\lim\limits_{x\to\infty} e(x,\lambda)\, e^{-i\lambda x} = 1$ is said to be Jost solution of equation (1). It is easy to show that Jost solution is as follows;

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x} R_1(x), & x > a \\ \alpha^+ e^{i\lambda x} R_1(x) + \alpha^- e^{i\lambda(2a-x)} R_2(x), & 0 < x < a \end{cases} \tag{4}$$

where $\alpha^{\pm} = \dfrac{1}{2}\left(\alpha \pm \dfrac{1}{\alpha}\right)$, $R_1(x) = e^{i\int\limits_x^\infty p(t)dt}$, $R_2(x) = e^{-i\int\limits_x^a p(t)dt}$.

Let us prove that representation

$$e(x,\lambda) = e_0(x,\lambda) + \int\limits_x^\infty K(x,t) e^{i\lambda t} dt \tag{5}$$

is true. For this, let

$$e(x,\lambda) = \begin{cases} e_1(x,\lambda), & x > a \\ e_2(x,\lambda), & 0 < x < a. \end{cases} \tag{6}$$

Then it is clearly shown that integral equation for the solution $e(x,\lambda)$ is as follows;

$$\begin{cases} e_1\left(x,\lambda\right) = e^{i\lambda x} - \dfrac{1}{\lambda}\displaystyle\int\limits_x^\infty \sin\lambda(x-t)\left[q(t)+2\lambda p(t)\right]e_1\left(t,\lambda\right)dt \\ e_2\left(x,\lambda\right) = \alpha^+ e^{i\lambda x} + \alpha^- e^{i\lambda(2a-x)} \\ -\dfrac{1}{\lambda}\displaystyle\int\limits_a^\infty \left[\alpha^+\sin\lambda(x-t) - \alpha^-\sin\lambda(x+t-2a)\right]\left[q(t)+2\lambda p(t)\right]e_1\left(t,\lambda\right)dt \\ -\dfrac{1}{\lambda}\displaystyle\int\limits_x^a \sin\lambda(x-t)\left[q(t)+2\lambda p(t)\right]e_2\left(t,\lambda\right)dt. \end{cases} \tag{7}$$

In order to be solution of equation (7) of the function which has representation (5), the following equality is satisfied;

$$\int\limits_x^\infty K(x,t)e^{i\lambda t}dt = -\alpha^+\int\limits_a^\infty \frac{e^{i\lambda x}}{2i\lambda}q(t)R_1(t)dt + \alpha^+\int\limits_a^\infty \frac{e^{-i\lambda(x-2t)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]R_1(t)dt$$

$$-\alpha^+\int\limits_a^\infty \frac{e^{i\lambda(x-t)}-e^{-i\lambda(x-t)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]\int\limits_t^\infty K(t,\xi)e^{i\lambda\xi}d\xi dt$$

$$+\alpha^-\int\limits_a^\infty \frac{e^{i\lambda(x+2t-2a)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]R_1(t)dt - \alpha^-\int\limits_a^\infty \frac{e^{i\lambda(2a-x)}}{2i\lambda}q(t)R_1(t)dt$$

$$+\alpha^-\int\limits_a^\infty \frac{e^{i\lambda(x+t-2a)}-e^{-i\lambda(x+t-2a)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]\int\limits_t^\infty K(t,\xi)e^{i\lambda\xi}d\xi dt$$

$$-\alpha^+\int\limits_x^a \frac{e^{i\lambda x}}{2i\lambda}q(t)R_1(t)dt + \alpha^+\int\limits_x^a \frac{e^{-i\lambda(x-2t)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]R_1(t)dt$$

$$+\alpha^-\int\limits_x^a \frac{e^{i\lambda(2a-x)}}{2i\lambda}q(t)R_2(t)dt - \alpha^-\int\limits_x^a \frac{e^{i\lambda(x-2t+2a)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]R_2(t)dt$$

$$-\int\limits_x^a \frac{e^{i\lambda(x-t)}-e^{-i\lambda(x-t)}}{2i\lambda}\left[q(t)+2\lambda p(t)\right]\int\limits_t^\infty K(t,\xi)e^{i\lambda\xi}d\xi dt.$$

$$\int\limits_x^\infty K(x,t)e^{i\lambda t}dt = \frac{\alpha^+}{2}\int\limits_x^{2a-x}\left\{\int\limits_a^\infty q(s)R_1(s)ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{2}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{\frac{t+x}{2}}^{\infty}q(s)R_1(s)ds\right\}e^{i\lambda t}dt+\frac{\alpha^+}{2i}\int\limits_{2a-x}^{\infty}p\left(\frac{x+t}{2}\right)R_1\left(\frac{x+t}{2}\right)e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{2}\int\limits_{x}^{2a-x}\left\{\int\limits_{a}^{\infty}q(s)\left[\int\limits_{s}^{t-x+s}K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{2}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{\frac{t+x}{2}}^{\infty}q(s)\left[\int\limits_{s}^{t-x+s}K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{2}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{a}^{\frac{t+x}{2}}q(s)\left[\int\limits_{t-s+x}^{t-x+s}K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$-\frac{\alpha^+}{i}\int\limits_{x}^{\infty}\left\{\int\limits_{a}^{\infty}p(s)K(s,t+s-x)ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{i}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{a}^{\frac{t+x}{2}}p(s)K(s,t-s+x)ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^-}{2}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{\frac{t-x+2a}{2}}^{\infty}q(s)R_1(s)ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^-}{2i}\int\limits_{x}^{\infty}p\left(\frac{t-x+2a}{2}\right)R_1\left(\frac{t-x+2a}{2}\right)e^{i\lambda t}dt$$

$$+\frac{\alpha^-}{2}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{a}^{\infty}q(s)\left[\int\limits_{t-x-s+2a}^{t+x+s-2a}K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^-}{i}\int\limits_{x}^{\infty}\left\{\int\limits_{a}^{\frac{t-x+2a}{2}}p(s)K(s,t-x-s+2a)ds\right\}e^{i\lambda t}dt$$

$$-\frac{\alpha^-}{i}\int\limits_{2a-x}^{\infty}\left\{\int\limits_{a}^{\infty}p(s)K(s,t+x+s-2a)ds\right\}e^{i\lambda t}dt$$

$$+\frac{\alpha^+}{2}\int_x^{2a-x}\left\{\int_{\frac{t+x}{2}}^{a} q(s)R_1(s)ds\right\}e^{i\lambda t}dt+\frac{\alpha^+}{2i}\int_x^{2a-x} p\left(\frac{x+t}{2}\right)R_1\left(\frac{x+t}{2}\right)e^{i\lambda t}dt$$

$$+\frac{\alpha^-}{2}\int_x^{2a-x}\left\{\int_{\frac{x+2a-t}{2}}^{a} q(s)R_2(s)ds\right\}e^{i\lambda t}dt$$

$$-\frac{\alpha^-}{2i}\int_x^{2a-x} p\left(\frac{x+2a-t}{2}\right)R_2\left(\frac{x+2a-t}{2}\right)e^{i\lambda t}dt$$

$$-\frac{1}{i}\int_x^{\infty}\left\{\int_x^{a} p(s)K(s,t-x+s)ds\right\}e^{i\lambda t}dt$$

$$+\frac{1}{i}\int_x^{2a-x}\left\{\int_x^{\frac{t+x}{2}} p(s)K(s,t+x-s)ds\right\}e^{i\lambda t}dt$$

$$+\frac{1}{i}\int_{2a-x}^{\infty}\left\{\int_x^{a} p(s)K(s,t+x-s)ds\right\}e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_x^{2a-x}\left\{\int_{\frac{t+x}{2}}^{a} q(s)\left[\int_s^{t-x+s} K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_x^{2a-x}\left\{\int_x^{\frac{t+x}{2}} q(s)\left[\int_{t-s+x}^{t-x+s} K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt$$

$$+\frac{1}{2}\int_{2a-x}^{\infty}\left\{\int_x^{a} q(s)\left[\int_{t-s+x}^{t-x+s} K(s,\xi)d\xi\right]ds\right\}e^{i\lambda t}dt.$$

The following integral equations from the last equality are obtained;
(1) for $x<t<2a-x$,

$$K(x,t)=\frac{\alpha^+}{2}\int_{\frac{t+x}{2}}^{\infty} q(s)R_1(s)ds+\frac{\alpha^+}{2}\int_a^{\infty} q(s)\left[\int_s^{t-x+s} K(s,\xi)d\xi\right]ds$$

$$-\frac{\alpha^+}{i}\int_a^\infty p(s)K(s,t+s-x)ds+\frac{\alpha^-}{2i}p\left(\frac{t-x+2a}{2}\right)R_1\left(\frac{t-x+2a}{2}\right)$$

$$+\frac{\alpha^-}{i}\int_a^{\frac{t-x+2a}{2}} p(s)K(s,t-x-s+2a)ds+\frac{\alpha^+}{2i}p\left(\frac{x+t}{2}\right)R_1\left(\frac{x+t}{2}\right)$$

$$+\frac{\alpha^-}{2}\int_{\frac{x+2a-t}{2}}^a q(s)R_2(s)ds-\frac{\alpha^-}{2i}p\left(\frac{x+2a-t}{2}\right)R_2\left(\frac{x+2a-t}{2}\right)$$

$$-\frac{1}{i}\int_x^a p(s)K(s,t-x+s)ds+\frac{1}{i}\int_x^{\frac{t+x}{2}} p(s)K(s,t+x-s)ds$$

$$+\frac{1}{2}\int_{\frac{t+x}{2}}^a q(s)\left[\int_s^{t-x+s} K(s,\xi)d\xi\right]ds+\frac{1}{2}\int_x^{\frac{t+x}{2}} q(s)\left[\int_{t-s+x}^{t-x+s} K(s,\xi)d\xi\right]ds$$

(2) for $2a-x<t$,

$$K(x,t)=\frac{\alpha^+}{2}\int_{\frac{t+x}{2}}^\infty q(s)R_1(s)ds+\frac{\alpha^+}{2i}p\left(\frac{x+t}{2}\right)R_1\left(\frac{x+t}{2}\right)$$

$$+\frac{\alpha^+}{2}\int_{\frac{t+x}{2}}^\infty q(s)\left[\int_s^{t-x+s} K(s,\xi)d\xi\right]ds+\frac{\alpha^+}{2}\int_a^{\frac{t+x}{2}} q(s)\left[\int_{t-s+x}^{t-x+s} K(s,\xi)d\xi\right]ds$$

$$-\frac{\alpha^+}{i}\int_a^\infty p(s)K(s,t+s-x)ds+\frac{\alpha^+}{i}\int_a^{\frac{t+x}{2}} p(s)K(s,t-s+x)ds$$

$$+\frac{\alpha^-}{2}\int_{\frac{t-x+2a}{2}}^\infty q(s)R_1(s)ds+\frac{\alpha^-}{2i}p\left(\frac{t-x+2a}{2}\right)R_1\left(\frac{t-x+2a}{2}\right)$$

$$+\frac{\alpha^-}{2}\int_a^\infty q(s)\left[\int_{t-x-s+2a}^{t+x+s-2a} K(s,\xi)d\xi\right]ds+\frac{\alpha^-}{i}\int_a^{\frac{t-x+2a}{2}} p(s)K(s,t-x-s+2a)ds$$

$$-\frac{\alpha^-}{i}\int_a^\infty p(s)K(s,t+x+s-2a)ds-\frac{1}{i}\int_x^a p(s)K(s,t-x+s)ds$$

$$+\frac{1}{i}\int_{x}^{a} p(s)K(s,t+x-s)ds+\frac{1}{2}\int_{x}^{a} q(s)\left[\int_{t-s+x}^{t-x+s} K(s,\xi)d\xi\right] ds$$

3. Existence of Solution of Integral Equations System

The successive approximations method can be applied for existence of solution of integral equations system;

for $0<x<a$,

$$K_0(x,t)=\frac{\alpha^+}{2}\int_{\frac{t+x}{2}}^{\infty} q(s)R_1(s)ds-\frac{1}{2}\alpha^- ip\left(\frac{t-x+2a}{2}\right)R_1\left(\frac{t-x+2a}{2}\right)$$

$$-\frac{\alpha^+}{2}ip\left(\frac{t+x}{2}\right)R_1\left(\frac{t+x}{2}\right)+\frac{\alpha^-}{2}\int_{\frac{x+2a-t}{2}}^{a} q(s)R_2(s)ds$$

$$+\frac{\alpha^-}{2}ip\left(\frac{x-t+2a}{2}\right)R_2\left(\frac{x-t+2a}{2}\right)+\frac{\alpha^+}{2}\int_{\frac{x+t}{2}}^{\infty} q(s)R_1(s)ds$$

$$-\frac{\alpha^+}{2}ip\left(\frac{t+x}{2}\right)R_1\left(\frac{t+x}{2}\right)+\frac{\alpha^-}{2}\int_{\frac{t-x+2a}{2}}^{\infty} q(s)R_1(s)ds$$

$$-\frac{\alpha^-}{2}ip\left(\frac{t-x+2a}{2}\right)R_1\left(\frac{t-x+2a}{2}\right)$$

$$\int_{x}^{\infty}\left|K_0(x,t)\right|dt\le\left(\alpha^+ +\left|\alpha^-\right|\right)\int_{x}^{\infty}\left[\left(1+|s|\right)|q(s)|+2\,|p(s)|\right]ds=\sigma(x)$$

$$K_1(x,t)=\frac{\alpha^+}{2}\int_{a}^{\infty} q(s)\left(\int_{s}^{t-x+s} K_0(s,\xi)d\xi\right)ds+\alpha^+ i\int_{a}^{\infty} p(s)K_0(s,t+s-x)ds$$

$$-\alpha^- i\int_{a}^{\frac{t-x+2a}{2}} p(s)K_0(s,t-s-x+2a)ds+i\int_{x}^{a} p(s)K_0(s,t+s-x)ds$$

$$-i\int_{x}^{\frac{x+t}{2}} p(s)K_0(s,t+x-s)ds+\frac{1}{2}\int_{\frac{x+t}{2}}^{a} q(s)\left(\int_{s}^{t-x+s} K_0(s,\xi)d\xi\right)ds$$

$$+\frac{1}{2}\int_{x}^{\frac{x+t}{2}} q(s)\left(\int_{t-s+x}^{t-x+s} K_0(s,\xi)d\xi\right)ds+\frac{\alpha^+}{2}\int_{\frac{x+t}{2}}^{\infty} q(s)\left(\int_{s}^{t-x+s} K_0(s,\xi)d\xi\right)ds$$

$$+\frac{\alpha^{+}}{2}\int\limits_{a}^{\frac{x+t}{2}} q(s)\left(\int\limits_{t-s+x}^{t-x+s} K_0(s,\xi)d\xi\right)ds+\alpha^{+}i\int\limits_{a}^{\infty} p(s)K_0(s,t+s-x)ds$$

$$-\alpha^{+}i\int\limits_{a}^{\frac{x+t}{2}} p(s)K_0(s,t-s+x)ds+\frac{\alpha^{-}}{2}\int\limits_{a}^{\infty} q(s)\left(\int\limits_{t-s-x+2a}^{t+x+s-2a} K_0(s,\xi)d\xi\right)ds$$

$$-\alpha^{-}i\int\limits_{a}^{\frac{t-x+2a}{2}} p(s)K_0(s,t-s-x+2a)ds+\alpha^{-}i\int\limits_{a}^{\infty} p(s)K_0(s,t+s+x-2a)ds$$

$$+i\int\limits_{x}^{a} p(s)K_0(s,t+s-x)ds-ip(s)K_0(s,t-s+x)ds+\frac{1}{2}\int\limits_{x}^{a} q(s)\left(\int\limits_{t-s+x}^{t-x+s} K_0(s,\xi)d\xi\right)ds$$

$$\int\limits_{x}^{\infty}|K_1(x,t)|\,dt\le 3\alpha^{+}\int\limits_{a}^{\infty}|(1+s)\,q(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds$$

$$+3\int\limits_{x}^{a}|(1+s)\,q(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds+\left|\alpha^{-}\right|\int\limits_{a}^{\infty}|(1+s)\,q(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds$$

$$+2\alpha^{+}\int\limits_{a}^{\infty}|p(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds+2\int\limits_{x}^{a}|p(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds$$

$$+2\left|\alpha^{-}\right|\int\limits_{a}^{\infty}|p(s)|\int\limits_{s}^{\infty}|K_0(s,\xi)|\,d\xi ds$$

$$\le 3(\alpha^{+}+|\alpha^{-}|)\int\limits_{a}^{\infty}(1+|s|)\,|q(s)|\,\sigma(s)ds+3\int\limits_{x}^{a}(1+|s|)\,|q(s)|\,\sigma(s)ds$$

$$+\,2(\alpha^{+}+\left|\alpha^{-}\right|)\int\limits_{a}^{\infty}|p(s)|\,\sigma(s)ds+2\int\limits_{x}^{a}|p(s)|\,\sigma(s)ds$$

$$\int\limits_{x}^{\infty}|K_1(x,t)|\,dt\le -3(\alpha^{+}+\left|\alpha^{-}\right|)\int\limits_{x}^{\infty}\sigma(s)d\left(\int\limits_{s}^{\infty}(1+|t|)\,|q(t)|\,dt\right)$$

$$-3\int\limits_{x}^{\infty}\sigma(s)d\left(\int\limits_{s}^{\infty}(1+|t|)\,|q(t)|\,dt\right)-2(\alpha^{+}+\left|\alpha^{-}\right|)\int\limits_{x}^{\infty}\sigma(s)d\left(\int\limits_{s}^{\infty}2\,|p(t)|\,dt\right)$$

$$-\,2\int\limits_{x}^{\infty}\sigma(s)d\left(\int\limits_{s}^{\infty}2\,|p(t)|\,dt\right)$$

$$\leq 3(\alpha^+ + |\alpha^-|)\int\limits_x^\infty \sigma(s)d\sigma(s)$$

$$+3\int\limits_x^\infty \sigma(s)d\sigma(s) + 2(\alpha^+ + |\alpha^-|)\int\limits_x^\infty \sigma(s)d\sigma(s) + 2\int\limits_x^\infty \sigma(s)d\sigma(s).$$

Since $\lim\limits_{x\to\infty}\int\limits_x^\infty \xi\,|q(\xi)|\,d\xi = 0$,

$$\int\limits_x^\infty |K_1(x,t)|\,dt \leq 5(\alpha^+ + |\alpha^-| + 1)\frac{\sigma^2(x)}{2}.$$

Similarly, since

$$K_n(x,t) = \frac{\alpha^+}{2}\int\limits_a^\infty q(s)\left(\int\limits_s^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds + \alpha^+ i\int\limits_a^\infty p(s)K_{n-1}(s,t+s-x)ds$$

$$-\alpha^- i\int\limits_a^{\frac{t-x+2a}{2}} p(s)K_{n-1}(s,t-s-x+2a)ds + i\int\limits_x^a p(s)K_{n-1}(s,t+s-x)ds$$

$$-i\int\limits_x^{\frac{x+t}{2}} p(s)K_{n-1}(s,t+x-s)ds + \frac{1}{2}\int\limits_{\frac{x+t}{2}}^a q(s)\left(\int\limits_s^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds$$

$$+\frac{1}{2}\int\limits_x^{\frac{x+t}{2}} q(s)\left(\int\limits_{t-s+x}^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds + \frac{\alpha^+}{2}\int\limits_{\frac{x+t}{2}}^\infty q(s)\left(\int\limits_s^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds$$

$$+\frac{\alpha^+}{2}\int\limits_a^{\frac{x+t}{2}} q(s)\left(\int\limits_{t-s+x}^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds + \alpha^+ i\int\limits_a^\infty p(s)K_{n-1}(s,t+s-x)ds$$

$$-\alpha^+ i\int\limits_a^{\frac{x+t}{2}} p(s)K_{n-1}(s,t-s+x)ds + \frac{\alpha^-}{2}\int\limits_a^\infty q(s)\left(\int\limits_{t-s-x+2a}^{t+x+s-2a} K_{n-1}(s,\xi)d\xi\right)ds$$

$$-\alpha^- i\int\limits_a^{\frac{t-x+2a}{2}} p(s)K_{n-1}(s,t-s-x+2a)ds + \alpha^- i\int\limits_a^\infty p(s)K_{n-1}(s,t+s+x-2a)ds$$

$$+i\int\limits_x^a p(s)K_{n-1}(s,t+s-x)ds - i\int\limits_x^a p(s)K_{n-1}(s,t-s+x)ds$$

$$+\frac{1}{2}\int_{x}^{a} q(s)\left(\int_{t-s+x}^{t-x+s} K_{n-1}(s,\xi)d\xi\right)ds,$$

$$\int_{x}^{\infty} |K_n(x,t)|\,dt \le C^{n+1}\frac{\sigma^{n+1}(x)}{(n+1)!} \tag{8}$$

where $C = 5\left(\alpha^{+} + |\alpha^{-}| + 1\right),\ 0 < x < a$. The inequality (8) is proved by mathematical induction method. Truth of inequality is shown for $n = 0$ and $n = 1$.

Truth of this inequality is assumed for $n = m - 1$, i.e.;

$\int_{x}^{\infty} |K_{m-1}(x,t)|\,dt \le C^{m}\frac{\sigma^{m}(x)}{m!}$ is valid.

Now for $n = m$,

$$\int_{x}^{\infty} |K_m(x,t)|\,dt \le 3\alpha^{+}\int_{x}^{\infty}(1+s)\,|q(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds$$

$$+3\int_{x}^{\infty}(1+s)\,|q(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds + \left|\alpha^{-}\right|\int_{x}^{\infty}(1+s)\,|q(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds$$

$$+2\alpha^{+}\int_{x}^{\infty}|p(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds + 2\int_{x}^{\infty}|p(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds$$

$$+2\left|\alpha^{-}\right|\int_{x}^{\infty}|p(s)|\int_{s}^{\infty}|K_{m-1}(s,\xi)|\,d\xi ds$$

$$\le 3\alpha^{+}\int_{x}^{\infty}(1+|s|)\,|q(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds + 3\int_{x}^{\infty}(1+|s|)\,|q(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds$$

$$+\left|\alpha^{-}\right|\int_{x}^{\infty}(1+|s|)\,|q(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds + 2\alpha^{+}\int_{x}^{\infty}|p(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds$$

$$+2\int_{x}^{\infty}|p(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds + 2\left|\alpha^{-}\right|\int_{x}^{\infty}|p(s)|\,C^{m}\frac{\sigma^{m}(s)}{m!}ds$$

$$\le 5(\alpha^{+} + \left|\alpha^{-}\right| + 1)\frac{C^{m}}{m!}\int_{x}^{\infty}\sigma^{m}(s)d\sigma(s) = C^{m+1}\frac{\sigma^{m+1}(x)}{(m+1)!}$$

$$\int_{x}^{\infty}|K_m(x,t)|\,dt \le C^{m+1}\frac{\sigma^{m+1}(x)}{(m+1)!}.$$

Theorem 3.1. *Let $q(x)$ be a complex valued function satisfying the condition* $\int\limits_0^\infty (1+x)\,|q(x)|\,dx < \infty$ *and* $p(x) \in W_2^1(0,\infty)$. *Then equation (1) has a unique Jost solution $e(x,\lambda)$ which is in the form of*

$$e(x,\lambda) = e_0(x,\lambda) + \int\limits_x^\infty K(x,t)e^{i\lambda t}dt$$

for all λ in the upper half plane. For each fix $x \in (0,a) \cup (a,\infty)$, kernel function $K(x,\cdot)$ belongs to the space $L_1(x,\infty)$ and the following inequality is satisfied;

$$\int\limits_x^\infty |K(x,t)|\,dt \le e^{c\sigma(x)} - 1, \text{ where } \quad C = 5(\alpha^+ + |\alpha^-| + 1),$$

$$\sigma(x) = (\alpha^+ + |\alpha^-|)\int\limits_x^\infty [(1+|s|)\,|q(s)| + 2\,|p(s)|]\,ds.$$

When $p(x) \in W_2^1(0,\infty)$, $q(x) \in L_2(0,\infty)$,the following properties are valid;

i) for $0 < x < a$,

$$\frac{dK(x,x)}{dx} = -\frac{1}{2}\alpha^+ R_1(x)\left[q(x) + ip'(x) + p^2(x)\right] - ip(x)K(x,x)$$

$$\frac{d}{dx}\left\{ K(x,t)|_{t=2a-x-0}^{2a-x+0} \right\} = \frac{1}{2}\alpha^- R_2(x)\left[q(x) - ip'(x) + p^2(x)\right]$$

$$+ip(x)\left[K(x,2a-x+0) - K(x,2a-x-0)\right]$$

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = q(x)K(x,t) + 2ip(x)\frac{\partial K(x,t)}{\partial t}$$

$$K(a-0,t) = \alpha K(a+0,t)$$

$$\left.\frac{\partial K(x,t)}{\partial x}\right|_{x=a-0} = \alpha^{-1}\left.\frac{\partial K(x,t)}{\partial x}\right|_{x=a+0}$$

$$R_1(a) = 1, \qquad p(a) = 0$$

$$\lim_{t\to\infty}\frac{\partial K(x,t)}{\partial x} = 0, \quad \lim_{t\to\infty}\frac{\partial K(x,t)}{\partial t} = 0, \quad \lim_{t\to\infty} K(x,t) = 0$$

ii) for $x > a$,

$$\frac{dK(x,x)}{dx} = -\frac{1}{2}R_1(x)\left[q(x) + ip'(x) + p^2(x)\right] - ip(x)K(x,x).$$

References

1. A A Nabiev and I M Guseinov, On the Jost solutions of the Schrödinger-type equations with a polynomial energy-dependent potential, *Inverse Problems,* 22 (2006) 55–67.

2. A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with singular potentials, *Mat. Zametki* (Math. Notes) 66(6) (1999), 897–912 (in Russian).

   TRANSFORMATION OPERATORS 149.

3. A. M. Savchuk, On eigenvalues and eigenfunctions of Sturm–Liouville operators with singular potentials, *Mat. Zametki* (Math. Notes) 69(2) (2001), 277–285 (in Russian).

4. A. M. Savchuk and A. A. Shkalikov, Sturm–Liouville operators with distributional potentials, *Trudy Moskov. Mat. Obshch.* (Trans. Moscow Math. Soc.), 64 (2003), to appear (in Russian).

5. A. Ya. Povzner, On differential Sturm–Liouville operators on semiaxis, *Math. USSR-Sb.* 23(65) (1948), 3–52.

6. B.M. Levitan, Inverse Sturm–Liouville Problems, *Nauka Publ., Moscow,* 1984 (in Russian); Engl. transl.: VNU Science Press, Utrecht, 1987.

7. D.H. Sattinger and J. Szmigielski, Energy dependent scattering theory, Differential Integral Equations 8 (1995), 945 959.

8. G. Sh.Guseinov, Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval. (Russian) *Spectral theory of operators and its applications*, "E'lm", Baku, 7 (1986), 51–101.

9. H. Cornille, Existence and uniqueness of crossing symmetric N/D-type equations corresponding to the Klein-Gordon equation, J.Math. Phys. 11 (1970), 79 98.

10. H.M. Huseynov, J.A. Osmanova, On Jost solution of Sturm_Liouville equation with discontinuity conditions, *Transactions of NAS Azerbaijan,* **XXVII 1** (2007), 63-70

11. I. M. Gelfand and B. M. Levitan, On determination of a differential equation by its spectral function, *Izv. Akad. Nauk SSSR Ser. Mat.* 15(4) (1951), 309–360 (in Russian).

12. . I. M. Guseinov, A.A. Nabiev, R.T. Pashaev, Transformation operators and asymptotic formulas for the eigenvalues of a polynomial pencil of Sturm-Liouville operators. (Russian. Russian summary) *Siberian Mathematical Journal*, 41 (2000), no. 3, 554–566.

13. J. Herczy´nski, On Schrödinger operators with distributional potentials, *J. Oper. Theory* 21(2) (1989), 273–295.

14. J. Pöschel. and E. Trubowitz, Inverse Spectral Theory, *Pure Appl. Math.* 130, Academic Press, Orlando, Florida, 1987.

15. K. Chadan and P.C. Sabatier, Inverse problems in quantum scattering theory, 2nd. edition, Springer-Verlag, New York, 1989.

16. M. I. Neiman-zade, and A. A. Shkalikov, Schrödinger operators with singular potentials from the space of multipliers, *Mat. Zametki* (Math. Notes) 66(5) (1999), 723–733.

17. .M. Jaulent and C. Jean, The inverse *s*-ware scattering problem for a class of potentials depending on energy, *Comm. Math. Phys.* **28** (1972), 177-220

18. M. Jaulent, On an inverse scattering problem with an energy-dependent potential, Ann. Inst. Henri Poincar´e 17 (1972), 363 378.

19. M. Jaulent, Sur le probléme inverse de la diffusion pour l'*é*quation de Schrödinger radiale avec un potentiel d*é*pendant de l'*é*nergie, *C.R. Acad. Sci.* Paris 280 (1975), 1467 1470.

20. .M. Jaulent and C. Jean, The inverse problem for the one-dimensional Schrödinger equation with an energy-dependent potential I, II, *Ann. Inst. H. Poincare Sec. a* **25** (1976), 105-118, 119-137.

21. R. O. Hryniv and Ya. V. Mykytyuk, 1D Schrödinger operators with singular periodic potentials, *Meth. Funct. Anal. Topol.* 7(4) (2001), 31–42.

22. R. O. Hryniv and Ya. V. Mykytyuk, 1D Schrödinger operators with singular Gordon potentials, *Meth. Funct. Anal. Topol.* 8(1) (2002), 36–48.

23. R. Weiss and G. Scharf, The inverse problem of potential scattering according to the Klein-Gordon equation, Helv. Phys. Acta 44 (1971), 910 929.

24. S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, Solvable Models in Quantum Mechanics, Springer, New York, 1988.

25. S. Albeverio and P. Kurasov, Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators, Cambridge University Press, Cambridge, 2000.

26. V. A. Marchenko, Some questions of the theory of second order differential operators, *Dokl. Akad. Nauk SSSR,* 72(3) (1950), 457–460 (in Russian).

27. V.A. Marchenko, Sturm-Liouville operators and applications, in Operator theory: Advances and applications 22, Birkhäuser,-Verlag, Basel, 1986.

28. V. A. Marchenko, Sturm–Liouville Operators and Their Applications, *Naukova Dumka Publ., Kyiv,* 1977 (in Russian); Engl. transl.: Birkhäuser, Basel, 1986.

29. .V. A. Yurko, Inverse spectral problems for differential pencils on the half-line with turning points, *J. Math. Anal. Appl.* **320**, (2006) 439-463.

30. Yu. M. Berezanskii, and J. Brasche, Generalized selfadjoint operators and their singular perturbations, *Methods Funct. Anal. Topol.* 8(4) (2002), 1–14.

Y. Güldü
Department of Mathematics, Faculty of Sciences,
Cumhuriyet University, 58140 Sivas, Turkey
E-mail address: yguldu@gmail.com

and

S. Gülyaz
Department of Mathematics, Faculty of Sciences,
Cumhuriyet University, 58140 Sivas, Turkey
E-mail address: slmgulyaz@gmail.com