



Solution of Integral Equations by Generalized Wavelet Transform

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ABSTRACT: Wavelet transform and distributional wavelet transform are used to obtain the solution of few integral equations in this paper.

Key Words: Wavelet transform, distribution spaces, Volterra integral equation, Fredholm integral equation of convolution type, convolution.

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1. Introduction

During the preceding years, wavelet analysis has enveloped many areas in pure and applied mathematics, which can be justified through [2,3,4,5,7]. In pure mathematics, the concept and formulae of wavelet transform and wavelets (of specific type) are invoked to study distribution spaces and quotient spaces (Boehmians) [1,15], which are found fruitful in obtaining solutions of integral and differential equations.

Different methods, associated with numerical approximations and certain algorithms, wavelets (viz. Haar, Mexican, Morlet to name few) are used to solve certain integral equations, values of which have established graphical verifications [8,12,13]. Integral transforms, viz. Fourier, Laplace, Hankel among other, in distributional sense, have been used [6,9,10,11,14] to obtain solutions of certain type of integral equations. In this paper, similarly, wavelet transform and distributional wavelet transform are employed in evaluating solutions of certain integral equations [6,15].

To study the applications of wavelet transform on distribution spaces and Boehmians, one can refer to [1,9,10,11,15]. In what follow, this section describes notations, definitions, and properties related to wavelet transform and generalized wavelet transform. In the following section, we solve certain integral equations using wavelet transform and distributional wavelet transform.

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The continuous wavelet transform (defined in the space $L^2(\mathbb{R})$) of a function f with respect to the wavelet ψ is defined by [15, p. 12]

$$(W_\psi f)(b, a) = \int_{-\infty}^{\infty} f(t) \overline{\psi_{b,a}(t)} dt \quad , \quad b \in \mathbb{R}, a \in \mathbb{R} \setminus \{0\} := \mathbb{R}_0 \quad (1.1)$$

where

$$\psi_{b,a}(t) = \frac{1}{\sqrt{|a|}} \psi \left(\frac{t-b}{a} \right) \quad ,$$

provided the integral exists. When $f \in L^p(\mathbb{R})$, $\psi \in L^q(\mathbb{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, and, in particular, for $p = q = 2$, the existence of the integral in (1) is justified for

$$|W_\psi f(b, a)| \leq \|f\|_2 \|\psi\|_2. \quad (1.2)$$

The convolution of continuous wavelet transform is defined by [15, p. 119]

$$W_\varphi[f \# g](b, a) = (W_\psi f)(b, a)(W_\theta g)(b, a), \quad (1.3)$$

whereas the associated convolution is defined by

$$(f \# g)(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} D(x, y, z) f(z) g(y) dz dy, \quad (1.4)$$

where $D(x, y, z)$ is a measure or basic generalized function [15, pp. 110-111].

If $\theta = \psi = \varphi$, then the commutative and associative properties hold good for the representation (1.4) as

$$f \# g = g \# f \quad (1.5)$$

and

$$(f \# g) \# h = f \# (g \# h) \quad . \quad (1.6)$$

The differentiability for the Schwartz space of rapidly decreasing functions [15, p. 120] is given by

$$\mathcal{D}_x^k(f \# g)(x) = (\mathcal{D}^k f \# g)(x) + (f \# \mathcal{D}^k g)(x), k \in \mathbb{N}_0, \quad (1.7)$$

$$\Delta_x^k(f \# g)(x) = (\Delta^k f \# g)(x) + (f \# \Delta^k g)(x), k \in \mathbb{N}_0, \quad (1.8)$$

where $\Delta_x f(x) = x \left(\frac{d}{dx} \right) f(x)$.

Synonym to the Fourier transform convolution

$$(f * g)^\wedge = \widehat{f\widehat{g}} \quad , \quad (1.9)$$

the convolution product associated with wavelet transform is given by

$$W_\psi(f \# g)(b, a) = (W_\psi f)(b, a)(W_\psi g)(b, a). \tag{1.10}$$

Existence for wavelet transform of convolution in a generalized Sobolev space are investigated in [15, pp. 130 - 136]. Applications of wavelet transform to different types of distribution spaces such as tempered, ultradistribution can be seen in [15]. For more applications of the wavelet transform, one may refer to [1,2,3,4,5,9,10,11,15].

2. Solution of Integral Equations, Wavelet Transform and Distributional Wavelet Transform

In this section we use wavelet transform to obtain solution of certain integral equation. The solution, so obtained, is then defined on distribution spaces.

1. Consider the **Volterra integral equation of first kind with a convolution type kernel**

$$f(x) = \int_0^x k(s-t)g(t)dt \quad , \tag{2.1}$$

where $k(s-t)$ depends only on the difference $(x-t)$. Invoking the wavelet transform (1.1) and convolution expression (1.3), the expression (2.1) yields

$$W_\psi[f](b, a) = (W_\psi k)(b, a)(W_\psi g)(b, a)$$

i.e.

$$W_\psi[g] = \frac{W_\psi[f]}{W_\psi[k]}. \tag{2.2}$$

Taking inverse wavelet transform, we obtain

$$g(x) = W_\psi^{-1} \left[\frac{W_\psi[f](b, a)}{W_\psi[k](b, a)} \right]. \tag{2.3}$$

2. Consider the **Volterra integral equation of second kind with a convolution type kernel**

$$g(x) = f(x) + \int_0^x k(s-t)g(t)dt \quad . \tag{2.4}$$

On applying the wavelet transform (1.1) to both the sides, and using convolution formula (1.3), equation (2.4) gives

$$W_\psi[g] = W_\psi[f] + W_\psi[k]W_\psi[g]$$

i.e.

$$W_\psi[g][1 - W_\psi[k]] = W_\psi[f]$$

i.e.

$$W_\psi[g] = \frac{W_\psi[f]}{[1 - W_\psi[k]]} \quad (2.5)$$

and the inverse wavelet transform, when invoked, gives

$$g(x) = W^{-1} \left[\frac{W_\psi[f]}{[1 - W_\psi[k]]} \right], \quad (2.6)$$

which is the required solution of (2.4).

Similarly, considering Fredholm integral equation of first and second kind of convolution type and using the wavelet transform and its convolution, under similar analysis, solutions for these can be obtained.

When distributional wavelet transform is considered in solving the integral equations, the solution obtained will also be in the sense of distributions.

3. Now let we consider the **Abel integral equation** [6]

$$f(t) = \int_0^x \frac{g(x)}{(t-x)^\alpha} dx, 0 < \alpha < 1 \quad (2.7)$$

i.e.

$$f = g * t_+^{-\alpha}, \quad (2.8)$$

where $t_+^{-\alpha} = t^{-\alpha}H(t)$ and $t_+^{-\alpha}$ is Heaviside unit step function . When $W_\psi[t_+^{-\alpha}]$ is known and the convolution of the wavelet transform is employed, the solution of (2.7) will be as follows

$$W_\psi[f(t)] = W_\psi[g(t)] W_\psi[t_+^{-\alpha}] \quad (2.9)$$

Therefore, the complete solution of (2.7) should be obtained as

$$g(t) = \frac{\sin(\pi\alpha)}{\pi} \frac{d}{dt} \left[\int_0^t (t-x)^{\alpha-1} f(x) dx \right] \quad (2.10)$$

We need to specify the space (or spaces) of generalized function in order to define integral equations on distribution spaces,. Then, we need to give an *interpretation of the equation in terms of an operator defined in that space of distributions*. This

interpretation should be such that when applied to ordinary functions, integral equation can be recovered. One is the space $\mathcal{D}'_{41}[a, \infty)$, which is known as mixed distribution space [cf. [6]] that can be identified with the space of distribution $\mathcal{D}'(R)$ whose support is $[a, \infty)$. Another distribution space is $\mathcal{D}'_{43}[a, \infty)$, which can be identified with the space $S'(R)$ (tempered distribution space) whose support is contained in $[a, \infty)$.

The interpretation of integral equation can be achieved by using the concept of convolution of distributions. If both u and v have supports bounded on the left, then $u * v$ is always defined. Actually, if $\text{supp } u \subseteq [a, \infty)$ and $\text{supp } v \subseteq [b, \infty)$, then $\text{supp } u * v \subseteq [a + b, \infty)$. Thus, the convolution can be considered as a bilinear operation $*$: $\mathcal{D}'_{41}[a, \infty) \times \mathcal{D}'_{41}[b, \infty) \rightarrow \mathcal{D}'_{41}[a + b, \infty)$. If $u \in \mathcal{D}'_{41}[a, \infty)$ and $v \in \mathcal{D}'_{41}[b, \infty)$ are locally integrable functions, then we have

$$(u * v)(t) = \int_a^{t-b} u(\tau)v(t - \tau)d\tau \quad , \quad t > a + b \quad . \quad (2.11)$$

When $b = 0$ and $v \in \mathcal{D}'_{41}[0, \infty)$, we have $u * v \in \mathcal{D}'_{41}[a, \infty)$. Thus the convolution, with v , defines an operator of the space $\mathcal{D}'_{41}[a, \infty)$, which is given by

$$(u * v)(t) = \int_a^t u(\tau)v(t - \tau)d\tau \quad , \quad t > a \quad , \quad (2.12)$$

where u and v are locally integrable functions. *The integral equation and its solution can be interpreted in the distributional sense.*

To define the integral equations on distribution spaces, we need to specify the spaces we intend to use. Such are given in terms of distribution spaces (i.e. tempered, ultradistribution, Schwartz space, Sobolev spaces and many others) [9,10,11,15] and mixed distribution spaces [6]. Wavelet transform and its convolution as defined on different types of distribution spaces as given above in Section 1 and moreover can be refer [15].

Moreover, the solutions of integral equations obtained above by wavelet transform, can be interpreted in distributional sense, either by considering integral equations on distributions spaces or the solutions of integral equations, which can be obtained by using distributional wavelet transform where $f(t), g(t)$ and $t^{-\alpha}$ are locally integrable.

Curiosity : It is observed to be a common practice to solve integral equations by wavelet analysis, invoking the numerical techniques, which is not the same as is employed in the present paper. Meaning thereby, that we have employed the wavelet transform (as in Section 2) to solve integral equations, which is similar

to that when other integral transforms (viz. Laplace, Fourier, Hilbert, for example) are invoked to obtain the solutions of either differential equations or integral equations.

On the other hand, if we take function $f(t) = 1$, then its Fourier transform is $F[1] = (2\pi)^n \delta$. Similarly, if we consider the function $f(t) = 1$, then what will be the value of wavelet transform of it for $\psi_{a,b}(t)$ be a constant (or any value).

Do we have knowledge of the existence of a collection of tables for wavelet transform, as we have [Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. Tables of Integral Transforms, Mc Graw Hill, Vol. 1 and Vol. 2, 1954] for other integral transforms. If so, then using such formulae of wavelet transform, we can easily solve the Abel integral equations or any other integral equations.

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