Bol. Soc. Paran. Mat. (3s.) **v. 33** 2 (2015): 59–67.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v33i2.21670

Statistical convergence of double sequences on probabilistic normed spaces defined by $[V, \lambda, \mu]$-summability

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abstract: In this paper, we aim to generalize the notion of statistical convergence for double sequences on probabilistic normed spaces with the help of two nondecreasing sequences of positive real numbers $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ such that each tending to ∞, also $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$, and $\mu_{n+1} \leq \mu_n + 1, \mu_1 = 1$. We also define generalized statistically Cauchy double sequences on PN space and establish the Cauchy convergence criteria in these spaces.

Key Words: Statistical convergence; λ-statistical convergence; Probabilistic normed spaces.

Contents

1. Introduction

Before we go into the motivation for this paper and present main results, we move through the background of the topic. Menger [12] provoked a crucial generalization of a metric space and called it a probabilistic metric space. This concept was further developed by various authors [2,3,4], [6], [11] and [23,24]. Probabilistic normed space, which is an important family of probabilistic metric spaces, were firstly defined by *Šternev* [25]. Alsina et al. [1] gave a new definition of probabilistic normed space making *Šternev* definition a special case. As a result, a productive theory agreeable with ordinary normed spaces and probabilistic metric spaces originated.

The notion of statistical convergence of sequence of numbers was introduced by Fast [5] and Schoenberg [22] independently in 1951 and discussed by [7], [13,14], [16,17,18,19,20,21], [26,27], [29] and [31]. During last few years, statistical convergence has been applied in various fields like fourier analysis, ergodic theory and number theory. Mursaleen [15] generalized the notion of statistical convergence with the help of a non-decreasing sequence $\lambda = (\lambda_n)$ of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$ and called respectively λ-statistical convergence. Karakus extended the concept of the statistical convergence for single and double

2000 *Mathematics Subject Classification:* 40A05, 40C05, 46A45

sequences on probabilistic normed spaces in [8] and [9]. Tripathy et al. [28] discussed the double sequence spaces with the help of Orlicz function and in [30], they extended the concept to double sequence spaces of fuzzy numbers. Recently, Kumar and Mursaleen [10] defined (λ, μ)-statistical convergence of double sequences on intuitionistic fuzzy normed spaces. Following Kumar and Mursaleen [10], in this paper, we aim to define strongly (λ, μ)-statistical convergence of double sequences on probabilistic normed spaces.

2. Background and preliminaries

First, We recall some notations and basic definitions those will be used in this paper. By a distribution function we mean a function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ that is left-continuous and non-decreasing on $\mathbb{R}$ with $F(-\infty) = 0$ and $F(\infty) = 1$. We normalize all distribution functions to be left continuous on unextended real line $\mathbb{R} = (-\infty, +\infty)$. Moreover, for any $a \geq 0$, ε_a is the distribution function defined by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a \\ 1, & x > a. \end{cases}$$

Let Δ denotes the set of all the distribution functions, $\Delta^+ = \{F : F \in \Delta \text{ with } F(0) = 0\}$ and $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$ where $l^- f(x) = \lim_{t \to x^-} f(t)$. For $F, G \in \Delta^+$, $F \leq G$ iff $F(x) \leq G(x)$ for all $x \in \mathbb{R}$ and $(\Delta^+, \leq)$ is a partially ordered set. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

Definition 2.1. *A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+,*

(i) $\tau(F, \varepsilon_0) = F$;
(ii) $\tau(F, G) = \tau(G, F)$;
(iii) $\tau(F, G) \leq \tau(H, K)$ *whenever* $F \leq H, G \leq K$;
(iv) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Particular and relevant triangle functions are the functions τ_T , τ_{T^*} and those of the form Π_T which, for any continuous t-norm T, and any $x > 0$, are given by

$$\tau_T(F, G)(x) = \sup\{T(F(s), G(t)) : s + t = x\}$$
$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(s), G(t)) : s + t = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

In 1993, using triangle functions, Alsina et al. [1] defined probabilistic normed spaces as follows:

Definition 2.2. *[1] A probabilistic normed space, briefly PN-space, is a quadruple $(V, v, \tau, \tau*)$ where V is a real linear space, τ and $\tau*$ are continuous triangle functions with $\tau \leq \tau*$ and v, the probabilistic norm, is a mapping from V into the space of distribution function Δ^+ such that writing v_p for $v(p)$ for all p, q in V, the following conditions hold:*

(i) $v_p = \varepsilon_0$ if and only if $p = \theta$, the null vector in V,
(ii) $v_{-p} = v_p$,
(iii) $v_{p+q} \geq \tau(v_p, v_q)$,
(iv) $v_p \leq \tau^*(v_{\alpha p}, v_{(1-\alpha)p})$ for every $\alpha \in [0, 1]$.

If, instead of (i), we only have $v_p = \varepsilon_0$, then we shall speak of a probabilistic pseudo normed space, briefly a PPN-space. If the inequality (iv) is replaced by the equality $v_p = \tau_M(v_{\alpha p}, v_{(1-\alpha)p})$, then the PN-space is called a *Šerstnev* space, in this case, a condition stronger than (ii) holds, namely

$v_{\lambda_p} = v_p(\frac{j}{|\lambda|}), \forall \lambda \neq 0 \quad \forall p \in V,$

here j is the identity map on $\mathbb{R}$. A *Šerstnev* space is denoted by (V, v, τ).
There is a natural topology in PN-space (V, v, τ, τ^*), called the strong topology. It is defined, for $t > 0$, by the neighbourhoods

$$N_p(t) = \{q \in V : d_S(v_{q-p}, \varepsilon_0) < t\} = \{q \in V : v_{q-p}(t) > 1 - t\}$$

The strong neighbourhood system for V is the union $\bigcup_{p \in V} \mathcal{N}_p\lambda$ where $\mathcal{N}_p = \{N_p\lambda : \lambda > 0\}$. The strong neighbourhood system for V determines a Hausdroff topology for V.

Definition 2.3. *Let (V, v, τ, τ^*) be a PN-space. A sequence $(p_n)_n$ in V is said to be strongly convergent to p in V if for each $\lambda > 0$, there exists a positive integer N such that $p_n \in N_p(\lambda)$, for $n \geq N$.*

Definition 2.4. *Let (V, v, τ, τ^*) be a PN-space. A sequence $(p_n)_n$ in V is called strongly Cauchy sequence if , for every $\lambda > 0$, there is a positive integer N such that $v_{p_n - p_m}(\lambda) > 1 - \lambda$, whenever $m, n > N$.*

Definition 2.5. *A PN-space (V, v, τ, τ^*) is said to be strongly complete in the strong topology if and only if every strong Cauchy sequence in V is strongly convergent to a point in V.*

Lemma 2.6. *If $|\alpha| \leq |\beta|$, then $v_{\beta_p} \leq v_{\alpha_p}$ for every $p \in V$.*

Definition 2.7. *The natural density of a set K of positive integers is defined by*

$$\delta(K) = \lim_{n \to \infty} \frac{1}{n} |\{k \in K : k \leq n\}|.$$

Where $|\{k \in K : k \leq n\}|$ denotes the number of elements of K not exceeding n.

Definition 2.8. *Let (V, v, τ, τ^*) be a PN-space. A sequence $(p_n)_n$ in V is said to be strongly statistical convergent to p in V if for each $\lambda > 0$,*

$$\delta(\{n \in N : p_n \notin \mathcal{N}_p(\lambda)\}) = 0.$$

The element p is called the statistical limit of the sequence $(p_n)_n$ with respect to the probabilistic norm v and we write $st_v \to \lim p_n = p$

Definition 2.9. *Let (V, v, τ, τ^*) be a PN-space. A sequence $(p_n)_n$ in V is called strongly statistical Cauchy sequence if , for every $\lambda > 0$, there is a positive integer N such that*

$$\delta(\{n \in N : p_n \notin \mathcal{N}_{p_N}(\lambda)\}) = 0.$$

Namely, (p_n) is strong statistically Cauchy if and only if, for every $\lambda > 0$ there exists a number N such that $d_L(v_{p_n - p_N}, \varepsilon_0) < \lambda$ for a.a.n.

3. Strong (λ, μ)-statistical convergence of double sequences on a PN-space

In this section we define and study Strong (λ, μ)-statistical convergence of double sequences on probabilistic normed spaces.

Definition 3.1. *Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two nondecreasing sequences of positive real numbers such that each tending to ∞ and*
$\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1,$
$\mu_{n+1} \le \mu_n + 1, \mu_1 = 1.$
Let $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$.
For any set $K \subseteq N \times N$, the number

$$\delta_{\lambda,\mu}(K) = lim_{m,n\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) : i \in I_n, j \in I_m, (i, j) \in K\}|,$$

is called the (λ, μ)-density of the set K provided the limit exists.

A double sequence $x = (x_{ij})$ of numbers is said to be (λ, μ)-statistical convergent to a number ξ provided that for each $\epsilon > 0$,

$$lim_{m,n\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) : i \in I_n, j \in I_m, |x_{ij} - \xi| \ge \epsilon\}| = 0,$$

i.e., the set $K(\epsilon) = \frac{1}{\lambda_n \mu_m} |\{(i, j) : i \in I_n, j \in I_m, |x_{ij} - \xi| \ge \epsilon\}|$ has (λ, μ)-density zero. In this case the number ξ is called the (λ, μ)-statistical limit of the sequence $x = (x_{ij})$ and we write $St_{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = \xi$.

Now we define the strong (λ, μ)-statistical convergence of double sequences with respect to PN-space.

Definition 3.2. *Let (V, v, τ, τ^*) be a PN-space. A double sequence $x = (x_{ij})$ of elements in V is said to be strongly (λ, μ)-statistical convergent to ξ in V if for each $\lambda > 0$,*

$$\delta_{(\lambda,\mu)}(\{(i, j) : i \in I_n, j \in I_m, x_{ij} \notin \mathcal{N}_\xi(\lambda)\}) = 0.$$

equivalently

$$\delta_{(\lambda,\mu)}(\{(i, j) : i \in I_n, j \in I_m, x_{ij} \in \mathcal{N}_\xi(\lambda)\}) = 1.$$

In this case the element ξ is called the strong (λ, μ)-statistical limit of the sequence $x = x_{ij}$ with respect to the probabilistic norm v and we write $st_v^{(\lambda,\mu)} \rightarrow \lim_{i,j\rightarrow\infty} x_{ij} = x$.

Let $St_v^{(\lambda,\mu)}$ denotes the set of all strongly (λ, μ)-statistical convergent double sequences with respect to the probabilistic norm v.

Lemma 3.3. *Let (V, v, τ, τ^*) be a PN-space and $x = x_{ij}$ be a double sequence of elements in V. Then for each $\lambda > 0$, the following statements are equivalent*
(i) $st_v^{(\lambda,\mu)} \rightarrow \lim_{i,j\rightarrow\infty} x_{ij} = x$.
(ii) $\delta_{(\lambda,\mu)}(\{(i,j) : i \in I_n, j \in I_m, x_{ij} \notin \mathcal{N}_\xi(\lambda)\}) = 0$.
(iii) $\delta_{(\lambda,\mu)}(\{(i,j) : i \in I_n, j \in I_m, x_{ij} \in \mathcal{N}_\xi(\lambda)\}) = 1$.
(iv) $st_{(\lambda,\mu)} \rightarrow \lim_{i,j\rightarrow\infty} v_{x_{ij}-\xi} = 1$.

Theorem 3.4. *Let (V, v, τ, τ^*) be a PN-space. If a double sequence $x = x_{ij}$ of elements in V is strongly (λ, μ)-statistical convergent with respect to probabilistic norm v, then its $st_v^{(\lambda,\mu)}$-limit is unique.*

Proof: The proof of the Theorem can be established using standard techniques, so we omit. □

Theorem 3.5. *Let (V, v, τ, τ^*) be a PN-space. If $x = x_{ij}$ be a double sequence of elements in V such that $v - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$ then $st_v^{(\lambda,\mu)} - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$.*

Proof: Let $v - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$. For each $\lambda > 0$, there exists a positive integer m such that $v_{x_{ij}-\xi}(\lambda) > 1 - \lambda$ for every $i, j \geq m$. It follows that the set $\{(i,j) : i \in I_n, j \in I_m, x_{ij} \notin \mathcal{N}_\xi(\lambda)\}$ has atmost finitely many terms. It follows that

$$\delta_{(\lambda,\mu)}\{(i,j) : i \in I_n, j \in I_m, x_{ij} \notin \mathcal{N}_\xi(\lambda)\} = 0$$

This shows that $st_v^{(\lambda,\mu)} - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$. □

Theorem 3.6. *Let (V, v, τ, τ^*) be a PN space. The $st_v^{(\lambda,\mu)} - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$, if and only if, there exists a subset $K = \{(i,j) : i, j = 1, 2, 3, ...\}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $v - \lim_{(i,j)\in K, i,j\rightarrow\infty} x_{ij} = \xi$.*

Proof: Necessity– Suppose that $st_v^{(\lambda,\mu)} - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$. For $\lambda > 0$, consider the sets

$$M_v(\lambda) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\lambda) > 1 - \frac{1}{\lambda}\}$$
$$K_v(\lambda) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\lambda) \leq 1 - \frac{1}{\lambda}\}$$

Since $st_v^{(\lambda,\mu)} - \lim_{i,j\rightarrow\infty} x_{ij} = \xi$, it follows that $\delta_{(\lambda,\mu)}(K_v(\lambda)) = 0$. Furthermore, for $\lambda = 1, 2, 3, ...,$ we observe $M_v(\lambda) \supset M_v(\lambda + 1)$ and

$$\delta_{(\lambda,\mu)}(M_v(\lambda)) = 1. \tag{3.1}$$

Now we have to show that for $(i,j) \in M_v(\lambda)$, $v - \lim_{i,j\to\infty} x_{ij} = \xi$. Suppose, for $(i,j) \in M_v(\lambda)$, (x_{ij}) is not convergent to ξ with respect to the probabilistic norm v. Then, there exists some $\beta > 0$ such that

$\{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\lambda) \leq 1-\beta\}$

for infinitely many terms (x_{ij}).

Let $M_v(\beta) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\lambda) > 1-\beta\}$ and $\beta > \frac{1}{\lambda}$ for $\lambda = 1, 2, 3, ...$

Then, we have

$$\delta_{(\lambda,\mu)}(M_v(\beta)) = 0. \tag{3.2}$$

Also, $M_v(\lambda) \subset M_v(\beta)$ implies that $\delta_{(\lambda,\mu)}(M_v(\lambda)) = 0$. In this way, we obtained a contradiction to (3.1) as $\delta_{(\lambda,\mu)}(M_v(\lambda)) = 1$. Hence $v - \lim_{i,j\to\infty} x_{ij} = \xi$.

Sufficiency− Suppose that there exists a subset $K = \{(i,j) : i,j = 1,2,3,...\}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $v - \lim_{(i,j)\in K, i,j\to\infty} x_{ij} = \xi$. But then for $\lambda > 0$, we can find out a positive integer m such that

$v_{xij-\xi}(\lambda) > 1-\lambda$

for all $i, j \geq m$. If we take,

$K_v(\lambda) = \{(i,j) : i \in I_n, j \in I_m, x_{ij} \notin N_\xi(\lambda)\}$

Then, it is easy to see that

$K_v(\lambda) \subseteq N \times N - \{(i,j) : i \in I_n, j \in I_m, x_{ij} \in N_\xi(\lambda)\}$

and consequently

$\delta_{\lambda,\mu} K_v(\lambda) \leq 1 - 1 = 0.$

Hence, $st_v^{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = \xi$. □

Now we define strongly $(\lambda,\mu)-$statistically Cauchy double sequences in PN-space and establish the Cauchy convergence criteria in these spaces.

Definition 3.7. *Let (V, v, τ, τ^*) be a PN-space. A double sequence $x = (x_{ij})$ of elements in V is said to be strongly (λ,μ)-statistically Cauchy with respect to the probabilistic norm v if for each $\lambda > 0$ there exists a positive integers n and m such that for all $i, p \geq n$ and $j, q \geq m$,*

$\delta_{(\lambda,\mu)}(\{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-x_{pq}}(\lambda) \leq 1-\lambda\}) = 0.$

or equivalently

$\delta_{(\lambda,\mu)}(\{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-x_{pq}}(\lambda) > 1-\lambda\}) = 1.$

Theorem 3.8. *Let (V, v, τ, τ^*) be a PN-space. If a double sequence $x = x_{ij}$ of elements in V is strongly (λ,μ)-statistical convergent, if and only if, it is strongly (λ,μ)-statistical Cauchy with respect to probabilistic norm v.*

Proof: First suppose that there exists $\xi \in V$ such that $st_v^{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = \xi$. Let $\lambda > 0$ be given. Choose $\gamma > 0$ such that

$$\tau(1-\gamma, 1-\gamma) > 1-\lambda \tag{3.3}$$

For $\lambda > 0$, if we define

$A(\gamma) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\frac{1}{\lambda}) \leq 1-\gamma\}$

then

$A^C(\gamma) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\frac{1}{\lambda}) > 1-\gamma\}$
Since $st_v^{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = \xi$, it follows that $\delta_{(\lambda,\mu)}(A(\gamma)) = 0$ and consequently $\delta_{(\lambda,\mu)}(A^C(\gamma)) = 1$. Let $(p,q) \in (A^C(\lambda))$. Then

$$v_{x_{pq}-\xi}(\frac{1}{\lambda}) > 1-\gamma. \tag{3.4}$$

If we take
$B(\lambda) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-x_{pq}}(\lambda) \leq 1-\lambda\}$,
then to prove the result it is sufficient to prove that $B(\lambda) \subseteq A(\gamma)$. For $(m,n) \in B(\lambda)$,
$v_{x_{mn}-x_{pq}}(\lambda) \leq 1-\lambda$
If $v_{x_{mn}-x_{pq}}(\lambda) \leq 1-\lambda$, then we have $v_{x_{mn}-\xi}(\frac{1}{\lambda}) \leq 1-\gamma$ and therefore $(m,n) \in A(\gamma)$. As otherwise i.e., if $v_{x_{mn}-\xi}(\lambda) > 1-\lambda$, then by using (3.3) and (3.4) we have

$$1-\lambda \geq v_{x_{ij}-x_{pq}}(\lambda) \geq \tau(v_{x_{mn}-\xi}(\frac{1}{\lambda}), v_{x_{pq}-\xi}(\frac{1}{\lambda})) > \tau(1-\gamma, 1-\gamma) > 1-\lambda,$$

which is not possible. Hence $B(\lambda) \subseteq A(\gamma)$.
Conversely− Suppose that $x = (x_{ij})$ is strongly $(\lambda,\mu)-$statistical Cauchy but not strongly $(\lambda,\mu)-$statistical convergent with respect to the probabilistic norm v. Then there exists positive integers p and q such that if we take
$A(\lambda) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-x_{pq}}(\lambda) \leq 1-\lambda\}$
and
$B(\lambda) = \{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-\xi}(\frac{1}{\lambda}) > 1-\lambda\}$.
then $\delta_{(\lambda,\mu)}(A(\lambda)) = \delta_{(\lambda,\mu)}(B(\lambda)) = 0$ and consequently

$$\delta_{(\lambda,\mu)}(A^C(\lambda)) = \delta_{(\lambda,\mu)}(B^C(\lambda)) = 1. \tag{3.5}$$

Since
$v_{x_{ij}-x_{pq}}(\lambda) \geq 2v_{x_{ij}-\xi}(\frac{1}{\lambda}) > 1-\lambda$
If $v_{x_{ij}-\xi}(\frac{1}{\lambda}) > \frac{1-\lambda}{2}$.
It follows that
$\delta_{(\lambda,\mu)}(\{(i,j) : i \in I_n, j \in I_m, v_{x_{ij}-x_{pq}}(\lambda) > 1-\lambda) = 0\}$
i.e., $\delta_{(\lambda,\mu)}(A^C(\lambda)) = 0$. But then we obtained a contradiction to (3.5) as $\delta_{(\lambda,\mu)}(A^C(\lambda)) = 1$. Hence, (x_{ij}) is strongly $(\lambda,\mu)-$statistical convergent with respect to the probabilistic norm v. □

On combining Theorem 3.6 and Theorem 3.8, we obtain the following result.

Theorem 3.9. *Let (V, v, τ, τ^*) be a PN-space and $x = x_{ij}$ be a double sequence of elements in V. Then, the following conditions are equivalent:*

(i) x is a strongly $(\lambda,\mu)-$statistical convergent with respect to the probabilistic norm v.

(ii) x is a strongly $(\lambda,\mu)-$statistical Cauchy with respect to the probabilistic norm v.

(iii) there exists a subset $K = \{(i,j) : i,j = 1,2,3,...\}$ such that $\delta_{(\lambda,\mu)}(K) = 1$ and $v - lim_{(i,j)\in K, i,j\to\infty} x_{ij} = \xi$.

Acknowledgments

The authors are grateful to the referees of the papers for their valuable suggestions which improved the readability of the paper.

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