An Orlicz extension of difference modular sequence spaces

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ABSTRACT: In this paper we construct some new difference modular sequence spaces defined by a sequence of Orlicz functions over \(n\)-normed spaces. We also study several properties relevant to topological structures and interrelationship between these spaces.

Key Words: sequence space, difference sequence space, modular sequence space, paranormed space, Orlicz function, \(n\)-normed space, BK-space.

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1. Introduction and Preliminaries

Let \(X\) be a linear metric space. A function \(p : X \to \mathbb{R}\) is called paranorm, if

1. \(p(x) \geq 0\) for all \(x \in X\),
2. \(p(-x) = p(x)\) for all \(x \in X\),
3. \(p(x + y) \leq p(x) + p(y)\) for all \(x, y \in X\),
4. if \((\lambda_n)\) is a sequence of scalars with \(\lambda_n \to \lambda\) as \(n \to \infty\) and \((x_n)\) is a sequence of vectors with \(p(x_n - x) \to 0\) as \(n \to \infty\), then \(p(\lambda_n x_n - \lambda x) \to 0\) as \(n \to \infty\).

A paranorm \(p\) for which \(p(x) = 0\) implies \(x = 0\) is called total paranorm and the pair \((X, p)\) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, pp. 183).

The notion of difference sequence spaces was introduced by Kizmaz [16], who studied the difference sequence spaces \(l_\infty(\Delta), c(\Delta)\) and \(c_0(\Delta)\). The notion was further generalized by Et and Çolak [7] by introducing the spaces \(l_\infty(\Delta^n), c(\Delta^n)\) and \(c_0(\Delta^n)\). Later the concept have been studied by Bektaş et al. [3] and Et et al. [8]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [29] who studied the spaces \(l_\infty(\Delta_v), c(\Delta_v)\) and \(c_0(\Delta_v)\). Recently, Esi et
al. [9] and Tripathy et al. [30] have introduced a new type of generalized difference operators and unified those as follows. Let \( v \), \( n \) be non-negative integers, then for \( Z \) a given sequence space, we have

\[
Z(\Delta^n_v) = \{ x = (x_k) \in w : (\Delta^n_v x_k) \in Z \}
\]

for \( Z = c, c_0 \) and \( l_\infty \) where \( \Delta^n_v x = (\Delta^n_v x_k) = (\Delta^{n-1}_v x_k - \Delta^{n-1}_v x_{k+n}) \) and \( \Delta^n_v x_k = x_k \) for all \( k \in \mathbb{N} \), which is equivalent to the following binomial representation

\[
\Delta^n_v x_k = \sum_{m=0}^{n} (-1)^m \binom{n}{m} x_{k+vm}.
\]

Taking \( v = 1 \), we get the spaces \( l_\infty(\Delta^n) \), \( c(\Delta^n) \) and \( c_0(\Delta^n) \) studied by Et and Çolak [7]. Taking \( v = n = 1 \), we get the spaces \( l_\infty(\Delta) \), \( c(\Delta) \) and \( c_0(\Delta) \) introduced and studied by Kizmaz [16]. For more details about difference sequence spaces (see [1], [4], [5], [19], [20], [27]) and references therein.

Let \( \omega \) be the family of all real or complex sequences, which is a vector space with the usual pointwise addition and scalar multiplication. We write \( e^n(n \geq 1) \) for the \( n^{th} \) unit vector in \( \omega \), i.e \( e^n = \{ \delta_{nj} \}_{j=1}^\infty \) where \( \delta_{nj} \) is the Kronecker delta, and \( \varphi \) for the subspace of \( \omega \) generated by \( e^n \)’s, \( n \geq 1 \), i.e \( \varphi = \text{span}\{e^n : n \geq 1 \} \). A sequence space \( \eta \) is a subspace of \( \omega \) containing \( \varphi \). The sequence space \( \eta \) is said to be solid if \( (\alpha_k x_k) \in \eta \) whenever \( (x_k) \in \eta \) for all sequences \( (\alpha_k) \) of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \). A sequence space \( \eta \) is said to be monotone if \( \eta \) contains the canonical pre images of all its step spaces. A Banach sequence space \( (\eta, S) \) is called a BK-space if the topology \( S \) of \( \eta \) is finer than the coordinatewise convergence topology, or equivalently, the projection maps \( P_i : \eta \to K \), \( P_i(x) = x_i \), \( i \geq 1 \) are continuous, where \( K \) is the scalar field \( \mathbb{R} \) or \( \mathbb{C} \). For \( x = (x_1, ..., x_n, ...) \) and \( n \in \mathbb{N} \), we write the \( n^{th} \) section of \( x \) as \( x^{(n)} = (x_1, ..., x_n, 0, 0, ...) \). If \( x^{(n)} \to x \) in \( (\eta, S) \) for each \( x \in \eta \), we say that \( (\eta, S) \) is an AK-space. The norm \( \|\|_\eta \) generating the topology \( S \) of \( \eta \) is said to be monotone if \( \|x\|_\eta \leq \|y\|_\eta \) for \( x = \{x_i\}, \ y = \{y_i\} \in \eta \) with \( |x_i| \leq |y_i| \), for all \( i \geq 1 \) (see [14]).

An Orlicz function \( M \) is a function, which is continuous, non-decreasing and convex with \( M(0) = 0 \), \( M(x) > 0 \) for \( x > 0 \) and \( M(x) \to \infty \) as \( x \to \infty \). Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

\[
\ell_M = \{ x \in \omega : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \}
\]

which is called as an Orlicz sequence space. The space \( \ell_M \) is a Banach space with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^\infty M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.
\]

It is shown in [15] that every Orlicz sequence space \( \ell_M \) contains a subspace isomorphic to \( \ell_p(p \geq 1) \). In the later stage different Orlicz sequence spaces were
introduced and studied by Parashar and Choudhary [25], Esi and Et [6], Tripathy and Mahanta [31], Mursaleen [21] and many others. The $\Delta_2$--condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$.

A sequence $M = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [22], [23]). A sequence $N = (N_k)$ defined by

$$N_k(v) = \sup \{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function $M$. For a given Musielak-Orlicz function $M$, the Musielak-Orlicz sequence space $t_M$ and its subspace $h_M$ are defined as follows;

$$t_M = \{x \in \omega : I_M(cx) < \infty \text{ for some } c > 0\},$$
$$h_M = \{x \in \omega : I_M(cx) < \infty \text{ for all } c > 0\},$$

where $I_M$ is a convex modular defined by

$$I_M(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_M.$$ We consider $t_M$ equipped with the Luxemburg norm

$$||x|| = \inf \left\{ k > 0 : I_M\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf \left\{ \frac{1}{k} \left(1 + I_M(kx)\right) : k > 0 \right\}.$$ Any Orlicz function $M_k$ can always be represented in the following integral form

$$M_k(x) = \int_0^x \eta_k(t) dt,$$

where $\eta_k$ is known as the kernel of $M_k$, is a right differentiable for $t \geq 0$, $\eta_k(0) = 0$, $\eta_k(t) > 0$, $\eta_k$ is non-decreasing and $\eta_k(t) \to \infty$ as $t \to \infty$.

Given an Orlicz function $M_k$ with kernel $\eta_k(t)$, define

$$\nu_k(s) = \sup \{t : \eta_k(t) \leq s, s \geq 0\}.$$ Then $\nu_k(s)$ possesses the same properties as $\eta_k(t)$ and the function $N_k$ defined as

$$N_k(x) = \int_0^x \nu_k(s) ds$$

is an Orlicz function. The functions $M_k$ and $N_k$ are called mutually complementary Orlicz functions.
For a sequence $M = (M_k)$ of Orlicz functions, the modular sequence class $\tilde{l}(M)$ is defined by

$$\tilde{l}(M) = \{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} M_k(|x_k|) < \infty \}.$$  

Using the sequence $N = (N_k)$ of Orlicz functions, similarly we define $\tilde{l}(N)$. The class $l(M)$ is defined by

$$l(M) = \{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} x_k y_k \text{ converges, for all } y \in \tilde{l}(N) \}.$$ 

For a sequence $M = (M_k)$ of Orlicz functions, the modular sequence space $l(M)$ is also defined as

$$l(M) = \{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq 1 \}.$$ 

These spaces were introduced by Woo [33] around the year 1973 and generalizes the Orlicz sequence space $l^M$ and the modulared sequence spaces considered earlier by Nakano [24]. For more details about modular sequence spaces (see [15], [28]) and references therein.

An important subspace of $l(M)$, which is an AK-space, is the space $h(M)$ defined as

$$h(M) = \{ x \in l(M) : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \}.$$ 

A sequence $(M_k)$ of Orlicz functions is said to satisfy uniform $\Delta_2$-condition at $0'$ if there exist $p > 0$ and $k_0 \in \mathbb{N}$ such that for all $x \in (0, 1)$ and $k > k_0$, we have $M_k(x) \leq p$, or equivalently, there exists a constant $K > 1$ and $k_0 \in \mathbb{N}$ such that $\frac{M_k(x)}{M_k(y)} \leq K$ for all $x, y \in (0, \frac{1}{2}]$. If the sequence $(M_k)$ satisfy uniform $\Delta_2$-condition, then $h(M) = l(M)$ and vice-versa (see [33]).

Let $M_k$ and $N_k$ be mutually complementary Orlicz functions for each $k$ and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Bektaş and Atici [2] define the following sequence spaces:

$$l^M_N(\Delta^m) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M_k\left(\frac{|\Delta^m x_k|}{\lambda_k \rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$ 

and

$$l^N_N(\Delta^m) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} N_k\left(\frac{\lambda_k |\Delta^m x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$
Let $M = (M_k)$ and $N = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$l^M_n[\Delta_n^m, p] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} M_k\left(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}$$

and

$$l^N_n[\Delta_n^m, p] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} N_k\left(\frac{|\Delta_n^m x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $(p_k) = 1$, for all $k$ then

$$l^M_n[\Delta_n^m, 1] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} M_k\left(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

and

$$l^N_n[\Delta_n^m, 1] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} N_k\left(\frac{|\Delta_n^m x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$

If $(\lambda_k) = 1$ for all $k \in \mathbb{N}$, then

$$l^M_n[\Delta_n^m, p] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} M_k\left(\frac{|\Delta_n^m x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}$$

and

$$l^N_n[\Delta_n^m, p] = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} N_k\left(\frac{|\Delta_n^m x_k|}{\rho}\right)^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.$$

If we take $(p_k) = 1$, for all $k$ and $n=1$ we get the spaces defined by Bektaş and Atıcı [2].

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$, and let $D = \max\{1, 2^{H-1}\}$. Then, for the factorable sequences $(a_k)$ and $(b_k)$ in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \quad (1.1)$$

Throughout the paper we write $M_k(1) = 1$ and $N_k(1) = 1$ for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some difference new modular sequence spaces defined by a sequence of Orlicz functions over $n-$normed spaces. We shall study some topological, algebraic properties of the sequence spaces $l^M_n[\Delta_n^m, p]$ and $l^N_n[\Delta_n^m, p]$ in the second section of the paper. In the third section we shall determine the dual spaces of $h(M)$, $l(M, \lambda, p)$ and $l(N, \lambda, p)$. Finally, we shall study some sequence spaces over $n-$normed spaces in the fourth section of the paper. We have also made an attempt to study some topological, algebraic properties and inclusion relations between the sequence spaces $l^M_n[\Delta_n^m, p, ||\cdot\cdot\cdot||]$ and $l^N_n[\Delta_n^m, p, ||\cdot\cdot\cdot||]$. 
2. Some topological properties of the spaces $l^{M}_{\lambda} [\Delta_{n}, p]$ and $l^{N}_{\lambda} [\Delta_{m}, p]$

The purpose of this section is to study the properties like linearity, paranorm, solidity and relevant inclusion relations in the spaces $l^{M}_{\lambda} [\Delta_{n}, p]$ and $l^{N}_{\lambda} [\Delta_{m}, p]$.

**Theorem 2.1.** Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $l^{M}_{\lambda} [\Delta_{n}, p]$ and $l^{N}_{\lambda} [\Delta_{m}, p]$ are linear spaces over the complex field $\mathbb{C}$.

**Proof:** Let $x = (x_k)$ and $y = (y_k) \in l^{M}_{\lambda} [\Delta_{n}, p]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m} x_k}{\lambda_k \rho_1} \right) \right]^{p_k} < \infty$$

and

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m} y_k}{\lambda_k \rho_2} \right) \right]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M_k$'s are non-decreasing and convex function so by using inequality (1.1), we have

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m}(\alpha x_k + \beta y_k)}{\lambda_k \rho_3} \right) \right]^{p_k} \leq \sum_{k \geq 1} \left[ M_k \left( \frac{\alpha \Delta_{m} x_k}{\lambda_k \rho_3} + \frac{\beta \Delta_{m} y_k}{\lambda_k \rho_3} \right) \right]^{p_k}$$

$$\leq D \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m} x_k}{\lambda_k \rho_1} \right) \right]^{p_k} + D \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m} y_k}{\lambda_k \rho_2} \right) \right]^{p_k}$$

$$< \infty.$$

Therefore, $\alpha x + \beta y \in l^{M}_{\lambda} [\Delta_{n}, p]$ and hence, $l^{M}_{\lambda} [\Delta_{n}, p]$ is a linear space. Similarly, we can prove that $l^{N}_{\lambda} [\Delta_{m}, p]$ is a linear space. This completes the proof. $\square$

**Theorem 2.2.** Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence space $l^{M}_{\lambda} [\Delta_{n}, p]$ is a paranormed space with paranorm defined by

$$g(x) = \inf \left\{ \varphi \in \mathbb{P} : \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta_{m} x_k}{\lambda_k \rho} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1 \right\}$$

where $H = \max(1, G)$, $0 < p_k \leq \sup_{k} p_k = G$. 


Proof: Clearly $g(x) \geq 0$, for $x = (x_k) \in l^\infty_\lambda[\Delta_n^m, p]$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf \left\{ (\rho)^{\frac{1}{p}} : \left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho} \right) \right]^p \right)^{\frac{1}{p}} \leq 1 \right\} = 0,$$

this implies that for a given $\epsilon > 0$, there exist some $\rho_\epsilon$ ($0 < \rho_\epsilon < \epsilon$) such that

$$\left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon} \right) \right]^p \right)^{\frac{1}{p}} \leq 1.$$

Thus,

$$\left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon} \right) \right]^p \right)^{\frac{1}{p}} \leq \left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon} \right) \right]^p \right)^{\frac{1}{p}} \leq 1.$$

Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_n^m x_k \neq 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$, then $\frac{|\Delta_n^m x_k|}{\lambda_k \epsilon} \to \infty$. It follows that

$$\left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon} \right) \right]^p \right)^{\frac{1}{p}} \to \infty,$$

which is a contradiction. Therefore, $\Delta_n^m x_k = 0$ for each $k$ and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_1} \right) \right]^p \right)^{\frac{1}{p}} \leq 1$$

and

$$\left( \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho_2} \right) \right]^p \right)^{\frac{1}{p}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski’s inequality, we have
\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho} \right)^p \right] \right)^{\frac{1}{p}} \leq \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k (\rho_1 + \rho_2)} \right)^p \right] \right)^{\frac{1}{p}} \\
\leq \left( \sum_{k \geq 1} \left[ \frac{\rho_1}{\rho_1 + \rho_2} M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho_1} \right)^p \right] \right)^{\frac{1}{p}} + \left( \frac{\rho_2}{\rho_1 + \rho_2} M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho_2} \right)^p \right)^{\frac{1}{p}} \\
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho_1} \right)^p \right] \right)^{\frac{1}{p}} + \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho_2} \right)^p \right] \right)^{\frac{1}{p}} \\
\leq 1.
\]

Since \( \rho \)'s are non-negative, so we have
\[
g(x + y) = \inf \left\{ (\rho_1 + \rho_2) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n (x_k + y_k) \|}{\lambda_k (\rho_1 + \rho_2)} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}
\leq \inf \left\{ \rho_1 \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k \rho_1} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}
+ \inf \left\{ \rho_2 \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n y_k \|}{\lambda_k \rho_2} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}.
\]

Therefore, \( g(x + y) \leq g(x) + g(y) \). Finally, we prove that the scalar multiplication is continuous. Let \( \mu \) be any complex number, therefore, by definition
\[
g(\mu x) = \inf \left\{ (\rho) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n \mu x_k \|}{\lambda_k \rho} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}
\]
thus,
\[
g(\mu x) = \inf \left\{ (|\mu| t) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k t} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}
\]
where \( t = \frac{\rho}{|\mu|} \). Since \( |\mu|^p \leq \max(1, |\mu| \sup p_k) \). Hence,
\[
g(\mu x) = \max(1, |\mu| \sup p_k) \inf \left\{ (t) \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n x_k \|}{\lambda_k t} \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \right\}.
\]
So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof. □

Theorem 2.3. Suppose \( M = (M_k) \) be a sequence of Orlicz functions, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( \lambda = (\lambda_k) \) be a sequence of strictly positive real numbers. If \( 0 < p_k \leq q_k < \infty \), for each \( k \in \mathbb{N} \), then \( l^M_\lambda[\Delta^m_n, p] \subseteq l^M_\lambda[\Delta^m_n, q] \).

Proof: Suppose that \( x = (x_k) \in l^M_\lambda[\Delta^m_n, p] \). This implies that

\[
\sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} \leq 1
\]

for sufficiently large value of \( k \) say \( k \geq k_0 \), for some fixed \( k_0 \in \mathbb{N} \). Since \( M = (M_k) \) is non decreasing, we have

\[
\sum_{k=k_0}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} \leq \sum_{k=k_0}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} < \infty.
\]

Hence, \( x = (x_k) \in l^M_\lambda[\Delta^m_n, q] \). This completes the proof. □

Theorem 2.4. (i) If \( 0 < \inf p_k \leq p_k < 1 \) for each \( k \), then \( l^M_\lambda[\Delta^m_n, p] \subseteq l^M_\lambda[\Delta^m_n] \).
(ii) If \( 1 \leq p_k \leq \sup p_k < \infty \) for each \( k \), then \( l^M_\lambda[\Delta^m_n] \subseteq l^M_\lambda[\Delta^m_n, p] \).

Proof: (i) Let \( x = (x_k) \in l^M_\lambda[\Delta^m_n, p] \). Since \( 0 < \inf p_k < 1 \), we have

\[
\sum_{k=1}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k}
\]

and hence, \( x = (x_k) \in l^M_\lambda[\Delta^m_n] \).

(ii) Suppose \( p_k \) for each \( k \) \( \sup p_k < \infty \) and let \( x = (x_k) \in l^M_\lambda[\Delta^m_n] \). Then for each \( 0 < \epsilon < 1 \), there exists a positive integer \( N \) such that

\[
\sum_{k=1}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} \leq \epsilon < 1, \text{ for all } k \in \mathbb{N},
\]

this implies that

\[
\sum_{k=1}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} \left[ M_k \left( \frac{|\Delta^m_n x_k|}{\lambda_k \rho} \right) \right].
\]

Thus, \( x = (x_k) \in l^M_\lambda[\Delta^m_n, p] \). This completes the proof. □
Theorem 2.5. The sequence space $l^N_X\left[\Delta_n^m, p\right]$ is solid.

Proof: Let $x = (x_k) \in l^N_X\left[\Delta_n^m, p\right]$. Then

$$\sum_{k=1}^{\infty} \left[ M_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho} \right) \right]^{p_k} < \infty.$$ 

Let $(\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{k=1}^{\infty} \left[ M_k \left( \frac{\left| \Delta_n^m \alpha_k x_k \right|}{\lambda_k \rho} \right) \right]^{p_k} \leq \sum_{k=1}^{\infty} \left[ M_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho} \right) \right]^{p_k}.$$ 

This completes the proof. \qed

Corollary 2.6. The sequence space $l^N_X\left[\Delta_n^m, p\right]$ is monotone.

Proof: It is obvious so we omit the proof. \qed

Theorem 2.7. Let $\mathcal{M} = (M_k)$ and $\mathcal{M}' = (M'_k)$ be two sequences of Orlicz functions. Then, we have

$$l^N_X\left[\Delta_n^m, p\right] \cap l^N_X^M\left[\Delta_n^m, p\right] \subseteq l^N_X^{\mathcal{M} + \mathcal{M}'}\left[\Delta_n^m, p\right].$$

Proof: Let $x = (x_k) \in l^N_X\left[\Delta_n^m, p\right] \cap l^{N'}_X\left[\Delta_n^m, p\right]$. Then

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_1} \right) \right]^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sum_{k \geq 1} \left[ M'_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_2} \right) \right]^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$ 

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the inequality

$$\sum_{k \geq 1} \left[ (M_k + M'_k) \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho} \right) \right]^{p_k} = \sum_{k \geq 1} \left[ M_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_1} \right) \right]^{p_k} + M'_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_2} \right) \leq D \sum_{k \geq 1} \left[ M_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_1} \right) \right]^{p_k} + D \sum_{k \geq 1} \left[ M'_k \left( \frac{\left| \Delta_n^m x_k \right|}{\lambda_k \rho_2} \right) \right]^{p_k}.$$ 

This completes the proof. \qed

The proof of the following theorems are easy so omitted.
Theorem 2.8. The sequence space \( l^M_\lambda [\Delta^m_n, p] \) is a normed space with norm
\[
\|x\|_M^\lambda = \sum_{i=1}^m |x_i| + \inf \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho} \right)^p \right] \leq 1 \right\}.
\]

Theorem 2.9. The sequence space \( l^\lambda_N [\Delta^m_n, p] \) is a normed space with norm
\[
\|x\|_N^\lambda = \sum_{i=1}^m |x_i| + \inf \left\{ \rho > 0 : \sum_{k \geq 1} \left[ N_k \left( \frac{\lambda_k |\Delta^m_n x_k|}{\rho} \right)^p \right] \leq 1 \right\}.
\]

Theorem 2.10. The spaces \( (l^M_\lambda [\Delta^m_n, p], \|\cdot\|_M^\lambda) \) and \( (l^\lambda_N [\Delta^m_n, p], \|\cdot\|_N^\lambda) \) are Banach spaces.

Theorem 2.11. The space \( l^M_\lambda [\Delta^m_n, p] \) equipped with the norm \( \|\cdot\|_M^\lambda \) and the space \( l^\lambda_N [\Delta^m_n, p] \) equipped with the norm \( \|\cdot\|_N^\lambda \) are BK-spaces.

Proof: The space \( (l^M_\lambda [\Delta^m_n, p], \|\cdot\|_M^\lambda) \) is a Banach space by the Theorem 2.10. Now let \( \|x^l - x\|_M^\lambda \to 0 \) as \( l \to \infty \). Then
\[
|x^l_k - x_k| \to 0 \text{ as } l \to \infty, \text{ for each } k \leq m
\]
and
\[
\inf \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x^l_k - \Delta^m_n x_k}{\lambda_k \rho} \right)^p \right] \leq 1 \right\} \to 0
\]
as \( l \to \infty \) for all \( k \in \mathbb{N} \). If \( M_k \left( \frac{\Delta^m_n x^l_k - \Delta^m_n x_k}{\lambda_k \|x\|_M^\lambda} \right)^p \leq 1 \) then \( \frac{|\Delta^m_n x^l_k - \Delta^m_n x_k|}{\lambda_k \|x\|_M^\lambda} \leq 1 \) for all \( k \). Therefore, we also obtain
\[
|\Delta^m_n x^l_k - \Delta^m_n x_k| \leq \lambda_k \|x^l - x\|_M^\lambda.
\]
Since \( \|x^l - x\|_M^\lambda \to 0 \), then \( |\Delta^m_n x^l_k - \Delta^m_n x_k| \to 0 \) and
\[
\left| \sum_{v=0}^m (-1)^v \binom{m}{v} (x^l_{k+nv} - x_{k+nv}) \right| \to 0
\]
as \( l \to \infty \) for all \( k \in \mathbb{N} \). On the other hand, since we may write
\[
|x^l_{k+nv} - x_{k+nv}| \leq \left| \sum_{v=0}^m (-1)^v \binom{m}{v} (x^l_{k+nv} - x_{k+nv}) + \left( \binom{m}{0} (x^l_k - x_k) \right. \right.
\]

\[
+ \ldots + \left. \left( \binom{m}{m-1} (x^l_{k+n(m+1)} - x_{k+n(m+1)}) \right) \right|
\]
Then \( |x^l_k - x_k| \to 0 \) as \( l \to \infty \) for each \( k \in \mathbb{N} \). Hence, \( (l^M_\lambda [\Delta^m_n, p], \|\cdot\|_M^\lambda) \) is a BK-space. Similarly we can prove \( (l^\lambda_N [\Delta^m_n, p], \|\cdot\|_N^\lambda) \) is a BK-space. This completes the proof. \( \square \)
Theorem 2.12. If \( Z \) is a normal sequence space containing \( \lambda \), then \( l_\lambda^M [\Delta_n^m, p] \) is a proper subspace of \( Z \). In addition, if \( Z \) is equipped with the monotone norm (quasi-norm) \( \| \cdot \|_z \). The inclusion \( R : l_\lambda^M [\Delta_n^m, p] \to Z[\Delta_n^m, p] \) is continuous with \( \| R \| \leq \| \{ \lambda_k \} \|_z \).

Proof: Let \( x \in l_\lambda^M [\Delta_n^m, p] \), then
\[
\sum_{k=1}^{\infty} \left[ M_k \left( \frac{\| \Delta_n^m x_k \|}{\lambda_k^p} \right) \right]^{p_k} < \infty, \quad \text{for some } \rho > 0.
\]
So there exists a constant \( K > 0 \) such that
\[
\frac{\| \Delta_n^m x_k \|}{\lambda_k^p} \leq K \quad \text{for all } k \in \mathbb{N}.
\]
Since \( Z \) is a normal sequence space containing \( \lambda \), we have \( [\Delta_n^m x_k]^{p_k} \in Z \) and so that \( x \in Z[\Delta_n^m, p] \). Hence, \( l_\lambda^M [\Delta_n^m, p] \subseteq Z[\Delta_n^m, p] \). Further, since \( M_k(1) = 1 \) for all \( k \in \mathbb{N} \) then
\[
\sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n^m x_k \|}{\lambda_k^p} \right) \right]^{p_k} \leq 1
\]
so that \( \| \Delta_n^m x_k \| \leq \lambda_k^p \| x \|^M_\lambda \). As \( \| \cdot \|_z \) is monotone, \( \| Rx \|_z = \| \Delta_n^m x_k \|_z \leq \| \{ \lambda_k \} \|_z \| x \|^M_\lambda \) and hence, \( \| R \| \leq \| \{ \lambda_k \} \|_z \). This completes the proof.

Theorem 2.13. If \( Y \) is a normal sequence space containing \( \lambda^{-1} = \{ \frac{1}{\lambda} \} \), then \( l_\lambda^M [\Delta_n^m, p] \) is a proper subspace of \( Y \). In addition, if \( Y \) is equipped with the monotone norm (quasi-norm) \( \| \cdot \|_y \). The inclusion \( S : l_\lambda^M [\Delta_n^m, p] \to Y[\Delta_n^m, p] \) is continuous with \( \| S \| \leq \| \{ \lambda_k^{-1} \} \|_y \).

Proof: The proof of the theorem is similar to that of Theorem 2.12 and so is omitted.

Theorem 2.14. If \( \lambda = (\lambda_k) \) is a bounded sequence such that \( \inf \lambda_k > 0 \) (i.e both \( \lambda \) and \( \lambda^{-1} \) are in \( l_\infty \)). Then \( l_\lambda^M [\Delta_n^m, p] = l_\lambda^M [\Delta_n^m, p] = l_\lambda^M [\Delta_n^m, p] \).

Proof: Let \( x = (x_k) \in l_\lambda^M [\Delta_n^m, p] \), then
\[
\sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n^m x_k \|}{\rho} \right) \right]^{p_k} < \infty, \quad \text{for some } \rho > 0.
\]
Since \( \lambda = (\lambda_k) \) is bounded, we can write \( a \leq \lambda_k \leq b \) for some \( b > a \geq 0 \). Define \( \rho_1 = \rho b \). Also since \( M_k \)'s are increasing, it follows that
\[
\sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n^m x_k \|}{\rho_1} \right) \right]^{p_k} \leq \sum_{k \geq 1} \left[ M_k \left( \frac{\| \Delta_n^m x_k \|}{\rho} \right) \right]^{p_k} < \infty.
\]
Hence, \( l_M[\Delta_n^m,p] \subseteq l_M^\lambda[\Delta_n^m,p] \). The other inclusion \( l_M^\lambda[\Delta_n^m,p] \subseteq l_M[\Delta_n^m,p] \) follows from the inequality
\[
\sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\rho} \right)^p \right] \leq \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho} \right)^p \right] < \infty.
\]
Therefore, \( l_M^\lambda[\Delta_n^m,p] = l_M[\Delta_n^m,p] \). Similarly one can prove that \( l_M^\lambda[\Delta_n^m,p] = l_M[\Delta_n^m,p] \). This completes the proof. \( \square \)

**Theorem 2.15.** If \( \{\lambda_k\} \in l_\infty \) with \( c = \sup_{k \geq 1} \lambda_k \geq 1 \) and \( \{\lambda_k^{-1}\} \) is unbounded, then \( l_M^\lambda[\Delta_n^m,p] \) is properly contained in \( l_M^\lambda[\Delta_n^m,p] \) and the inclusion map \( U : l_M^\lambda[\Delta_n^m,p] \to l_M^\lambda[\Delta_n^m,p] \) is continuous with \( \|U\| \leq c^2 \).

**Proof:** For any \( \rho > 0 \) and \( \rho' = \rho c^2 \), we have
\[
\sum_{k \geq 1} \left[ M_k \left( \frac{C_{n,M} |\Delta_n^m x_k|}{\rho'} \right)^p \right] \leq \sum_{k \geq 1} \left[ M_k \left( \frac{C_{n,M} |\Delta_n^m x_k|}{\lambda_k \rho} \right)^p \right] < \infty.
\]
for \( x = \{x_k\} \). Hence, \( l_M^\lambda[\Delta_n^m,p] \subset l_M^\lambda[\Delta_n^m,p] \). We now show that the containment \( l_M^\lambda[\Delta_n^m,p] \subset l_M^\lambda[\Delta_n^m,p] \) is proper. From the unboundedness of the sequence \( \{\lambda_k^{-1}\} \), choose a sequence \( \{k_i\} \) of positive integers such that \( \lambda_k^{-1} \geq l \). Now define
\[
\Delta_n^m x_k = \begin{cases} \frac{1}{l}, & k = k_i, \quad l = 1, 2, \ldots; \\ 0, & \text{otherwise}. \end{cases}
\]
Then \( x \in l_M^\lambda[\Delta_n^m,p] \), but \( x \notin l_M^\lambda[\Delta_n^m,p] \). To prove the continuity of the inclusion map \( U \), let us first consider the case obtained for \( c = 1 \). For \( x \in l_M^\lambda[\Delta_n^m,p] \), we write
\[
A_M^\lambda[\Delta_n^m,p] = \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \frac{|\Delta_n^m x_k|}{\lambda_k \rho} \right)^p \right] \leq 1 \right\}
\]
and
\[
B_M^\lambda[\Delta_n^m,p] = \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \frac{\lambda_k |\Delta_n^m x_k|}{\rho} \right)^p \right] \leq 1 \right\}.
\]
Since \( M_k \)'s are increasing and \( c = 1 \), we get \( A_M^\lambda[\Delta_n^m,p] \subset B_M^\lambda[\Delta_n^m,p] \). Hence,
\[
\|x\|_{l_M}^\lambda = \inf B_M^\lambda[\Delta_n^m,p] \leq \inf A_M^\lambda[\Delta_n^m,p] = \|x\|_{l_M}^\lambda
\]
\( (2.1) \)
i.e. \( \|U(x)\|_{l_M}^\lambda \leq \|x\|_{l_M}^\lambda \). Thus, \( U \) is continuous with \( \|U\| \leq 1 = c^2 \). If \( c \neq 1 \), define \( \delta_k = \frac{1}{\lambda_k}, \quad k \in \mathbb{N} \). Then \( \delta_k \leq 1 \) and from (2.1), it follows that
\[
\|x\|_{l_M}^\delta \leq \|x\|_{l_M}^\delta \quad \text{for} \quad x \in l_M^\lambda[\Delta_n^m,p].
\]
\( (2.2) \)
Proposition 3.1. \( \eta \)

**Proposition 3.2.** Let \( \eta \) if \( \alpha \) and \( \eta \) is monotone (see [13]).

Hence, from (2.2)

\[
\|U(x)\|_M^\lambda = \|x\|_M^\lambda \leq c^2\|x\|_\lambda^M
\]

i.e. \( U \) is continuous with \( \|U\| \leq c^2 \). This completes the proof.

**Theorem 2.16.** If \( \{\lambda_k\} \) is unbounded with \( \sup_{k \geq 1} \lambda_k^{-1} = d \geq 1, \lambda_k > 0 \) for all \( k \), then \( l_\lambda^M[\Delta_n^m, p] \) is properly contained in \( l_\lambda^M[\Delta_n^m, p] \) and the inclusion map \( V : l_\lambda^M[\Delta_n^m, p] \rightarrow l_\lambda^M[\Delta_n^m, p] \) is continuous with \( \|V\| \leq d^2 \).

**Proof:** The proof of the theorem is similar to that of Theorem 2.15 and so is omitted.

**3. Dual spaces of** \( h(M), \ l(M, \lambda, p) \text{ and } l(N, \lambda, p) \)

Let \( \eta \) be a sequence space and defined

\[
\eta^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_kx_k| < \infty, \text{ for all } x \in \eta \},
\]

\[
\eta^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_kx_k \text{ converges for all } x \in \eta \},
\]

\[
\eta^\gamma = \{ a = (a_k) : \sup_{n \geq 1} \sum_{k=1}^{\infty} a_kx_k | < \infty, \text{ for all } x \in \eta \} \text{ (see [13]).}
\]

Then \( \eta^\alpha, \eta^\beta, \eta^\gamma \) are called \( \alpha-, \beta-, \gamma- \) dual spaces of \( \eta \) respectively. It is easy to show that \( \phi \subset \eta^\alpha \subset \eta^\beta \subset \eta^\gamma \). If \( \eta \subset \nu \), then \( \nu^\sigma \subset \eta^\sigma \) for \( \sigma = \alpha, \beta, \gamma \). We shall write \( \eta^\alpha = (\eta^\alpha)^\alpha \).

Let \( \eta \) be a sequence space. Then \( \eta \) is called perfect if \( \eta = \eta^\alpha \) (see [15]).

For \( m = n = 0 \) we write \( l(M, \lambda, p) \) and \( l(N, \lambda, p) \) instead of \( l_\lambda^M[\Delta_n^m, p] \) and \( l_\lambda^N[\Delta_n^m, p] \) respectively which we define as:

\[
l(M, \lambda, p) = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} \left[M_k \left( \frac{|x_k|}{\lambda_k \rho} \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}
\]

\[
l(N, \lambda, p) = \left\{ x = (x_k) \in \omega : \sum_{k \geq 1} N_k \left( \frac{\lambda_k |x_k|}{\rho} \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.
\]

In this section we shall obtain \( \alpha-, \beta-, \gamma- \) duals of the sequence space \( h(M) \) and \( \alpha-, \beta-, \gamma- \) duals of \( l(M, \lambda, p) \) and \( l(N, \lambda, p) \).

**Proposition 3.1.** \( \eta \) is perfect \( \Rightarrow \) \( \eta \) is normal \( \Rightarrow \) \( \eta \) is monotone (see [15]).

**Proposition 3.2.** Let \( \eta \) be a sequence space. If \( \eta \) is monotone, then \( \eta^\alpha = \eta^\beta \) and if \( \eta \) is normal, then \( \eta^\alpha = \eta^\gamma \).
Proposition 3.3. The sequence space \( h(M) \) is normal for any sequence \( (M_k) \) of Orlicz functions.

**Proof:** Let \( x \in h(M) \) and \( |y_k| \leq |x_k| \), for each \( k \in \mathbb{N} \). Since \( M'_k \)'s are non-decreasing we have
\[
\sum_{k=1}^{\infty} M_k \left( \frac{|y_k|}{\rho} \right) \leq \sum_{k=1}^{\infty} M_k \left( \frac{|x_k|}{\rho} \right) < \infty.
\]
Hence, \( y \in h(M) \). Thus, \( h(M) \) is normal. \( \square \)

Theorem 3.1. Let \( (M_k) \) and \( (N_k) \) for each \( k \) be mutually complementary Orlicz functions. Then
\[
[h(M)]^\beta = [h(M)]^\alpha = [h(M)]^\gamma = l(N).
\]
The proof is seen from Proposition 3.1, Proposition 3.2 and Proposition 3.3.

Theorem 3.2. If the sequence \( (M_k) \) satisfies uniform \( \Delta_2 \)-condition, then
\[
[l(M, \lambda, p)]^\alpha = l(N, \lambda, p)
\]
**Proof:** Let the sequence \( (M_k) \) satisfies uniform \( \Delta_2 \)-condition. Then for any \( x \in l(M, \lambda, p) \) and \( a \in l(N, \lambda, p) \), we have
\[
\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} \left( M_k \left( \frac{|x_k|}{\lambda_k \rho} \right) \right)^p_k + \sum_{k=1}^{\infty} \left( N_k \left( \frac{\lambda_k |a_k|}{\rho'} \right) \right)^p_k < \infty
\]
where \( \rho' = \frac{1}{\rho} \) and \( \rho > 0 \). Thus, \( a \in [l(M, \lambda, p)]^\alpha \). Hence, \( l(N, \lambda, p) \subset [l(M, \lambda, p)]^\alpha \). To prove the inclusion \( [l(M, \lambda, p)]^\alpha \subset l(N, \lambda, p) \), let \( a \in [l(M, \lambda, p)]^\alpha \). Then for all \( \{x_k\} \) with \( \left( \frac{x_k}{\lambda_k} \right) \in l(M) \) we have
\[
\sum_{k=1}^{\infty} |a_k x_k| < \infty. \tag{3.1}
\]
Since the sequence satisfies uniform \( \Delta_2 \)-condition, then \( l(M) = h(M) \) and so for \( (y_k) \in h(M) \) we have
\[
\sum_{k=1}^{\infty} |\lambda_k y_k a_k| < \infty \text{ by } (3.1). \]
Thus, \( (\lambda_k a_k) \in [h(M)]^\alpha = l(N) \) and hence, \( (a_k) \in l(N, \lambda, p) \). Therefore, \( [l(M, \lambda, p)]^\alpha = l(N, \lambda, p) \). \( \square \)

Theorem 3.3. If the sequence \( (M_k) \) satisfies uniform \( \Delta_2 \)-condition, then
\[
[l(N, \lambda, p)]^\alpha = l(M, \lambda, p)
\]
**Proof:** Immediate from Theorem 3.5. \( \square \)
4. Some new sequence spaces over \( n \)-normed space

The concept of 2-normed spaces was initially developed by Gähler \([10]\) in the mid of 1960’s, while that of \( n \)-normed spaces one can see in Misiak \([18]\). Since then, many others have studied this concept and obtained various results, see Gunawan \([11], [12]\) and Gunawan and Mashadi \([13]\). Let \( n \in \mathbb{N} \) and \( X \) be a linear space over the field \( \mathbb{K} \), where \( \mathbb{K} \) is field of real or complex numbers of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot, \cdots, || \) on \( X^n \) satisfying the following four conditions:

1. \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \),

2. \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation,

3. \( ||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \cdot ||x_1, x_2, \cdots, x_n|| \) for any \( \alpha \in \mathbb{K} \), and

4. \( ||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n|| \)

is called an \( n \)-norm on \( X \), and the pair \( (X, ||\cdot, \cdots, ||) \) is called a \( n \)-normed space over the field \( \mathbb{K} \).

For example, we may take \( X = \mathbb{R}^n \) being equipped with the \( n \)-norm

\[
||x_1, x_2, \cdots, x_n||_E = \text{the volume of the } n \text{-dimensional parallelopiped spanned by the vectors } x_1, x_2, \cdots, x_n \text{ which may be given explicitly by the formula} \\
||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|, \\
\]

where \( x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n \) for each \( i = 1, 2, \cdots, n \). Let \( (X, ||\cdot, \cdots, ||) \) be an \( n \)-normed space of dimension \( d \geq n \geq 2 \) and \( \{a_1, a_2, \cdots, a_n\} \) be linearly independent set in \( X \). Then the following function \( ||\cdot, \cdots, || \) on \( X^{n-1} \) defined by

\[
||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}
\]

defines an \((n-1)\)-norm on \( X \) with respect to \( \{a_1, a_2, \cdots, a_n\} \).

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot, \cdots, ||)\) is said to converge to some \( L \in X \) if

\[
\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.
\]

A sequence \((x_k)\) in a \( n \)-normed space \((X, ||\cdot, \cdots, ||)\) is said to be Cauchy if

\[
\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.
\]

If every Cauchy sequence in \( X \) converges to some \( L \in X \), then \( X \) is said to be complete with respect to the \( n \)-norm. Any complete \( n \)-normed space is said to be \( n \)-Banach space. For more details about \( n \)-normed space (see [26]) and references therein.
Let \((X, ||·||, \cdots, ||·||)\) be a \(n\)-normed space and \(W(n - X)\) denotes the space of \(X\)-valued sequences. Let \(p = (p_k)\) be a bounded sequence of positive real numbers, \(\lambda = (\lambda_k)\) be a sequence of strictly positive real numbers. Let \(M = (M_k)\) be a sequence of Orlicz functions and \(N = (N_k)\) is a complementary function of Orlicz function \(M = (M_k)\). In this section of the paper we define the following sequences:

\[
l^M_\lambda \left[ \Delta^m_n, p, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}
\]

and

\[
l^N_\lambda \left[ \Delta^m_n, p, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ N_k \left( \left\| \frac{\lambda_k \Delta^m_n x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}.
\]

If we take \((p_k) = 1\) for all \(k\) then

\[
l^M_\lambda \left[ \Delta^m_n, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] < \infty, \text{ for some } \rho > 0 \right\}
\]

and

\[
l^N_\lambda \left[ \Delta^m_n, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ N_k \left( \left\| \frac{\lambda_k \Delta^m_n x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.
\]

If \((\lambda_k) = 1\) for all \(k \in \mathbb{N}\), then

\[
l^M_\lambda \left[ \Delta^m_n, p, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta^m_n x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}
\]

and

\[
l^N_\lambda \left[ \Delta^m_n, p, ||·||, \cdots, ||·|| \right] = \left\{ x = (x_k) \in W(n - X) : \sum_{k \geq 1} \left[ N_k \left( \left\| \frac{\Delta^m_n x_k}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}
\]
Theorem 4.1. Let $M = (M_k)$ and $N = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $l_M^m[\Delta^m_n, p, ||\cdot||]$ and $l_N^m[\Delta^m_n, p, ||\cdot||]$ are linear spaces over the field $\mathbb{C}$ of complex numbers.

Proof: Let $x = (x_k)$ and $y = (y_k) \in l_M^m[\Delta^m_n, p, ||\cdot||]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{k \geq 1} [M_k(\|\Delta^m_n x_k + \beta y_k\|) + \|\Delta^m_n y_k\|)]^{p_k} < \infty$$

and

$$\sum_{k \geq 1} [M_k(\|\Delta^m_n x_k + \beta y_k\|) + \|\Delta^m_n y_k\|)]^{p_k} < \infty.$$ 

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot\|$ is a $\alpha$-norm on $X$ and $M_k$’s are non-decreasing and convex function so by using inequality (1.1), we have

$$\sum_{k \geq 1} [M_k(\|\Delta^m_n x_k + \beta y_k\|) + \|\Delta^m_n y_k\|)]^{p_k} < \infty.$$ 

Therefore, $ax + \beta y \in l_M^m[\Delta^m_n, p, ||\cdot||]$ and hence, $l_M^m[\Delta^m_n, p, ||\cdot||]$ is a linear space. Similarly, we can prove $l_N^m[\Delta^m_n, p, ||\cdot||]$ is a linear space. This completes the proof.

Theorem 4.2. Let $M = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence space $l_M^m[\Delta^m_n, p, ||\cdot||]$ is a paranormed space with paranorm defined by

$$g(x) = \inf \left\{ (\rho)^{\frac{1}{p_k}} : \left( \sum_{k \geq 1} [M_k(\|\Delta^m_n x_k\|) + \|\Delta^m_n x_k\|)]^{p_k} \right)^{\frac{1}{p_k}} \leq 1 \right\},$$

where $H = \max(1, G)$, $0 < p_k \leq \sup_k p_k = G$. 

respectively.
**Proof:** Clearly \( g(x) \geq 0 \), for \( x = (x_k) \in l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|] \). Since \( M_k(0) = 0 \), we get \( g(0) = 0 \). Again, if \( g(x) = 0 \), then

\[
g(x) = \inf \left\{ (\rho)^{\frac{p_k}{p}} : \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \right\} = 0,
\]

this implies that for a given \( \epsilon > 0 \), there exist some \( \rho_\epsilon (0 < \rho_\epsilon < \epsilon) \) such that

\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1.
\]

Thus,

\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq \left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_\epsilon}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1.
\]

Suppose that \( x_k \neq 0 \) for each \( k \in \mathbb{N} \). This implies that \( \Delta^m_n x_k \neq 0 \) for each \( k \in \mathbb{N} \). Let \( \epsilon \to 0 \), then \( \left\| \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right\| \to \infty \). It follows that

\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_\epsilon}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \to \infty
\]

which is a contradiction. Therefore, \( \Delta^m_n x_k = 0 \) for each \( k \) and thus \( x_k = 0 \) for each

\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_1}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1
\]

and

\[
\left( \sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_2}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \right)^{\frac{1}{p_k}} \leq 1.
\]
Let \( \rho = \rho_1 + \rho_2 \). Then by Minkowski’s inequality, we have
\[
\left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \\
\leq \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k (\rho_1 + \rho_2)}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \\
\leq \left( \sum_{k \geq 1} \left[ \frac{\rho_1}{\rho_1 + \rho_2} M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \\
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \\
\leq \left( \frac{\rho_1}{\rho_1 + \rho_2} \right) \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \\
+ \left( \frac{\rho_2}{\rho_1 + \rho_2} \right) \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \leq 1.
\]
Since \( \rho \)'s are non-negative, so we have
\[
g(x + y) = \inf \left\{ (\rho_1 + \rho_2)^p \right\} : \\
\left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n (x_k + y_k)}{\lambda_k (\rho_1 + \rho_2)}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \} \\
\leq \inf \left\{ (\rho_1)^p : \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \} \\
+ \inf \left\{ (\rho_2)^p : \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n x_k}{\lambda_k \rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \}.
\]
Therefore, \( g(x + y) \leq g(x) + g(y) \). Finally, we prove that the scalar multiplication is continuous. Let \( \mu \) be any complex number, therefore, by definition
\[
g(\mu x) = \inf \left\{ (\rho)^p : \left( \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta_n \mu x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1} \right\| \right)^p \right] \right)^{\frac{1}{p}} \leq 1 \} \quad \text{and}
\]
thus,
\[ g(\mu x) = \inf \left\{ (|\mu| t)^{\frac{1}{p_k}} : \left( \sum_{k \geq 1} [M_k \left( \| \Delta^m x_k / \lambda_k \rho, z_1, \ldots, z_{n-1} \| \right)]^{p_k} \right)^{\frac{1}{p_k}} \leq 1 \right\} \]

where \( t = \frac{\mu}{|\mu|} \). Since \( |\mu|^{p_k} \leq \max(1, |\mu| \sup p_k) \). Hence,

\[ g(\mu x) = \max(1, |\mu| \sup p_k) \inf \left\{ (t)^{\frac{1}{p_k}} : \left( \sum_{k \geq 1} [M_k \left( \| \Delta^m x_k / \lambda_k \rho, z_1, \ldots, z_{n-1} \| \right)]^{p_k} \right)^{\frac{1}{p_k}} \leq 1 \right\} \]

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof. \( \blacksquare \)

**Theorem 4.3.** Suppose \( M = (M_k) \) be a sequence of Orlicz functions, \( p = (p_k) \) be a bounded sequence of positive real numbers and \( \lambda = (\lambda_k) \) be a sequence of strictly positive real numbers. If \( 0 < p_k \leq q_k < \infty \), for each \( k \in \mathbb{N} \), then \( l_\lambda^M [\Delta^m_n, p, \|., \cdots, \|] \subseteq l_\lambda^M [\Delta^m_n, q, \|., \cdots, \|] \).

**Proof:** Suppose that \( x = (x_k) \in l_\lambda^M [\Delta^m_n, p, \|., \cdots, \|] \), this implies that

\[ \sum_{k \geq 1} [M_k \left( \| \Delta^m x_k / \lambda_k \rho, z_1, \ldots, z_{n-1} \| \right)]^{p_k} \leq 1 \]

for sufficiently large value of \( k \) say \( k \geq k_0 \), for some fixed \( k_0 \in \mathbb{N} \). Since \( M = (M_k) \) is non decreasing, we have

\[ \sum_{k=k_0}^{\infty} [M_k \left( \| \Delta^m x_k / \lambda_k \rho, z_1, \ldots, z_{n-1} \| \right)]^{q_k} \leq \sum_{k=k_0}^{\infty} [M_k \left( \| \Delta^m x_k / \lambda_k \rho, z_1, \ldots, z_{n-1} \| \right)]^{p_k} \leq \infty. \]

Hence, \( x = (x_k) \in l_\lambda^M [\Delta^m_n, q, \|., \cdots, \|] \). This completes the proof. \( \blacksquare \)

**Theorem 4.4.** (i) If \( 0 < \inf p_k \leq p_k < 1 \) for each \( k \), then \( l_\lambda^M [\Delta^m_n, p, \|., \cdots, \|] \subset l_\lambda^M [\Delta^m_n, \|., \cdots, \|] \).

(ii) If \( 1 \leq p_k \leq \sup p_k < \infty \) for each \( k \), then \( l_\lambda^M [\Delta^m_n, \|., \cdots, \|] \subseteq l_\lambda^M [\Delta^m_n, p, \|., \cdots, \|] \).
Proof: (i) Let \( x = (x_k) \in \ell^M_X[\Delta^m_n, p, ||\cdot||] \). Since \( 0 < \inf p_k < 1 \), we have
\[
\sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} \leq \sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k}
\]
and hence, \( x = (x_k) \in \ell^M_X[\Delta^m_n, ||\cdot||] \).

(ii) Suppose \( p_k \) for each \( k \) sup \( p_k < \infty \) and let \( x = (x_k) \in \ell^M_X[\Delta^m_n, ||\cdot||] \). Then for each \( 0 < \epsilon < 1 \), there exists a positive integer \( N \) such that
\[
\sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} \leq \sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} < \epsilon, \text{ for all } k \in \mathbb{N}.
\]
This implies that
\[
\sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} \leq \sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k}.
\]
Thus, \( x = (x_k) \in \ell^M_X[\Delta^m_n, p, ||\cdot||] \). This completes the proof. \( \square \)

Theorem 4.5. The sequence space \( \ell^M_X[\Delta^m_n, p, ||\cdot||] \) is solid.

Proof: Let \( x = (x_k) \in \ell^M_X[\Delta^m_n, p, ||\cdot||] \). Then
\[
\sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} < \infty.
\]
Let \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \) for all \( k \in \mathbb{N} \). Then the result follows from the following inequality
\[
\sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n \alpha_k x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k} \leq \sum_{k=1}^{\infty} [M_k \left( \left\| \frac{\Delta^m_n x_k}{\lambda_k p} \right\|, z_1, \ldots, z_{n-1} \right) ]^{p_k}.
\]
This completes the proof. \( \square \)

Corollary 4.6. The sequence space \( \ell^M_X[\Delta^m_n, p, ||\cdot||] \) is monotone.

Proof: It is obvious so we omit the proof. \( \square \)

Theorem 4.7. Let \( M = (M_k) \) and \( M' = (M'_k) \) be two sequences of Orlicz functions. Then, we have
\[
\ell^M_X[\Delta^m_n, p, ||\cdot||] \cap \ell^{M'}_X[\Delta^m_n, p, ||\cdot||] \subseteq \ell^{M+M'}_X[\Delta^m_n, p, ||\cdot||].
\]
Let $K > 0$, so there exists a constant $\rho_1 > 0$ for some $\rho_1 > 0$.

Proof: Let $x = (x_k) \in l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|] \cap l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|]$. Then

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_1}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} < \infty,$$

and

$$\sum_{k \geq 1} \left[ M'_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho_2}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} < \infty,$$

for some $\rho_2 > 0$.

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the inequality

$$\sum_{k \geq 1} \left[ (M_k + M'_k) \left( \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k}.$$

This completes the proof.

**Theorem 4.8.** If $\mathcal{Z}$ is a normal sequence space containing $\lambda$, then $l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|]$ is a proper subspace of $\mathcal{Z}$. In addition, if $\mathcal{Z}$ is equipped with the monotone norm (quasi-norm) $\| \cdot \|_\mathcal{Z}$. The inclusion $R : l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|] \to \mathcal{Z}[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|]$ is continuous with $\| R \| \leq \| \lambda_k \|_\mathcal{Z}$.

Proof: Let $x \in l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|]$, then

$$\sum_{k=1}^{\infty} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} < \infty,$$

for some $\rho > 0$.

So there exists a constant $K > 0$ such that

$$\frac{\Delta^m_n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \leq K$$

for all $k \in \mathbb{N}$.

Since $\mathcal{Z}$ is a normal sequence space containing $\lambda$, we have $\| \Delta^m_n x_k, z_1, \ldots, z_{n-1} \|^{p_k} \in \mathcal{Z}$ and so that $x \in \mathcal{Z}[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|]$. Hence,

$$l^M_\lambda[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|] \subseteq \mathcal{Z}[\Delta^m_n, p, \| \cdot \|, \ldots, \| \cdot \|].$$

Further, since $M_k(1) = 1$ for all $k \in \mathbb{N}$ then

$$\sum_{k \geq 1} \left[ M_k \left( \frac{\Delta^m_n x_k}{\lambda_k \| x \|_\lambda}, z_1, \ldots, z_{n-1} \right) \right]^{p_k} \leq 1.$$
Theorem 4.9. If $\mathcal{Y}$ is a normal sequence space containing $\lambda^{-1} \equiv \{ \frac{1}{\lambda_k} \}$, then $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ is a proper subspace of $\mathcal{Y}$. In addition, if $\mathcal{Y}$ is equipped with the monotone norm (quasi-norm) $\| \cdot \|$. The inclusion $S : l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] \to \mathcal{Y}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ is continuous with $\|S\| \leq \|\{\lambda_k\}\|$. 

**Proof:** The proof of the theorem is similar to that of Theorem 4.8 and so is omitted. 

Theorem 4.10. If $\lambda = (\lambda_k)$ is a bounded sequence such that inf $\lambda_k > 0$ (i.e. both $\lambda$ and $\lambda^{-1}$ are in $l_\infty$). Then $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] = l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$. 

**Proof:** Let $x = (x_k) \in l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$. Then 

$$\sum_{k \geq 1} \left[ M_k \left( \| \frac{\lambda_k \Delta_n^m x_k}{\rho} \|, z_1, \cdots, z_{n-1} \| \right) \right]^{p_k} < \infty, \text{ for some } \rho > 0.$$ 

Since $\lambda = (\lambda_k)$ is bounded, we can write $a \leq \lambda_k \leq b$ for some $b > a \geq 0$. Define $\rho_1 = \rho b$. Also since $M^\ell_{\lambda}$s are increasing, it follows that 

$$\sum_{k \geq 1} \left[ M_k \left( \| \frac{\lambda_k \Delta_n^m x_k}{\rho_1} \|, z_1, \cdots, z_{n-1} \| \right) \right]^{p_k} \leq \sum_{k \geq 1} \left[ M_k \left( \| \frac{\lambda_k \Delta_n^m x_k}{\rho} \|, z_1, \cdots, z_{n-1} \| \right) \right]^{p_k} < \infty.$$ 

Hence, $l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] \subseteq l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$. The other inclusion $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] \subseteq l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ follows from the inequality 

$$\sum_{k \geq 1} \left[ M_k \left( \| \frac{\lambda_k \Delta_n^m x_k}{\rho} \|, z_1, \cdots, z_{n-1} \| \right) \right]^{p_k} \leq \sum_{k \geq 1} \left[ M_k \left( \| \frac{\lambda_k \Delta_n^m x_k}{\rho} \|, z_1, \cdots, z_{n-1} \| \right) \right]^{p_k} < \infty.$$ 

Therefore, $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] = l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$. Similarly one can prove that $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] = l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$. This completes the proof. 

Theorem 4.11. If $\{\lambda_k\} \in l_\infty$ with $c = \sup_{k \geq 1} \lambda_k \geq 1$ and $\{\lambda_k^{-1}\}$ is unbounded, then $l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ is properly contained in $l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ and the inclusion map $U : l^M_{\lambda}[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|] \to l^M[\Delta_n^m, p, \| \cdot \|, \cdots, \| \cdot \|]$ is continuous with $\|U\| \leq c^2$. 


Proof: For any $\rho > 0$ and $\rho' = \rho c^2$, we have
\[
\sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\lambda_k \Delta^n x_k}{\rho'}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \leq \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta^n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \leq \infty,
\]
for $x = \{x_k\}$. Hence, $l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|] \subset l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]$. We now show that the containment $l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|] \subset l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]$ is proper. From the unboundedness of the sequence $\{\lambda_k^{-1}\}$, choose a sequence $\{k_l\}$ of positive integers such that $\lambda_k^{-1} \geq l$. Now define
\[
\Delta^n_m x_k = \begin{cases} \frac{1}{k}, & k = k_l, \ l = 1, 2, \ldots; \\ 0, & \text{otherwise}. \end{cases}
\]
Then $x \in l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]$ but $x \notin l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]$. To prove the continuity of the inclusion map $U$, let us first consider the case obtained for $c = 1$. For $x \in l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]$, we write
\[
A_\lambda^M[\Delta^n_m, p, \|, \cdots, \|] = \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\Delta^n x_k}{\lambda_k \rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1 \right\}
\]
and
\[
B_\lambda^M[\Delta^n_m, p, \|, \cdots, \|] = \left\{ \rho > 0 : \sum_{k \geq 1} \left[ M_k \left( \left\| \frac{\lambda_k \Delta^n x_k}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_k} \leq 1 \right\}.
\]
Since $M_k$'s are increasing and $c = 1$, we get $A_\lambda^M[\Delta^n_m, p, \|, \cdots, \|] \subset B_\lambda^M[\Delta^n_m, p, \|, \cdots, \|]$. Hence,
\[
\|x\|_M^\lambda = \inf B_\lambda^M[\Delta^n_m, p, \|, \cdots, \|] \leq \inf A_\lambda^M[\Delta^n_m, p, \|, \cdots, \|] = \|x\|_M^\lambda \quad (4.1)
\]
i.e. $\|U(x)\|_M^\lambda \leq \|x\|_M^\lambda$. Thus, $U$ is continuous with $\|U\| \leq 1 = c^2$. If $c = 1$, define $\delta_k = \frac{1}{\lambda_k}$, $k \in \mathbb{N}$. Then $\delta_k \leq 1$ and from (4.1), it follows that
\[
\|x\|_{s\lambda}^\delta \leq \|x\|_M^\lambda \quad \text{for} \ \ x \in l^M_\lambda[\Delta^n_m, p, \|, \cdots, \|]. \quad (4.2)
\]
Hence, from (4.2)
\[
\|U(x)\|_{s\lambda}^\delta = \|x\|_{s\lambda}^\delta \leq c^2 \|x\|_M^\lambda,
\]
thus, $U$ is continuous with $\|U\| \leq c^2$. This completes the proof.

Theorem 4.12. If $\{\lambda_k\}$ is unbounded with $\sup_{k \geq 1} \lambda_k^{-1} = d \geq 1$, $\lambda_k > 0$ for all $k$, then $l^M_{\lambda_n}[\Delta^n_m, p, \|, \cdots, \|]$ is properly contained in $l^M_{\lambda_n}[\Delta^n_m, p, \|, \cdots, \|]$ and the inclusion map $V : l^M_{\lambda_n}[\Delta^n_m, p, \|, \cdots, \|] \rightarrow l^M_{\lambda_n}[\Delta^n_m, p, \|, \cdots, \|]$ is continuous with $\|V\| \leq d^2$.

Proof: The proof of the theorem is similar to that of Theorem 4.11 and so is omitted.
References


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