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An Orlicz extension of difference modular sequence spaces

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abstract: In this paper we construct some new difference modular sequence spaces defined by a sequence of Orlicz functions over n-normed spaces. We also study several properties relevant to topological structures and interrelationship between these spaces.

Key Words: sequence space, difference sequence space, modular sequence space, paranormed space, Orlicz function, n-normed space, BK-space.

Contents

1. Introduction and Preliminaries

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, pp. 183).
The notion of difference sequence spaces was introduced by Kızmaz [16], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [7] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Later the concept have been studied by Bektaş et al. [3] and Et et al. [8]. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [29] who studied the spaces $l_\infty(\Delta_v)$, $c(\Delta_v)$ and $c_0(\Delta_v)$. Recently, Esi et

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al. [9] and Tripathy et al. [30] have introduced a new type of generalized difference operators and unified those as follows.
Let v, n be non-negative integers, then for Z a given sequence space, we have

$$Z(\Delta_v^n) = \{x = (x_k) \in w : (\Delta_v^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_v^n x = (\Delta_v^n x_k) = (\Delta_v^{n-1} x_k - \Delta_v^{n-1} x_{k+v})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^n x_k = \sum_{m=0}^{n} (-1)^m \begin{pmatrix} n \\ m \end{pmatrix} x_{k+vm}.$$

Taking $v = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Çolak [7]. Taking $v = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [16]. For more details about difference sequence spaces (see [1], [4], [5], [19], [20], [27]) and references therein.

Let ω be the family of all real or complex sequences, which is a vector space with the usual pointwise addition and scalar multiplication. We write $e^n (n \geq 1)$ for the n^{th} unit vector in ω, i.e $e^n = \{\delta_{nj}\}_{j=1}^\infty$ where δ_{nj} is the Kronecker delta, and φ for the subspace of ω generated by e^n's, $n \geq 1$, i.e, $\varphi =$ span$\{e^n : n \geq 1\}$. A sequence space η is subspace of ω containing φ. The sequence space η is said to be solid if $(\alpha_k x_k) \in \eta$ whenever $(x_k) \in \eta$ for all sequences (α_k) of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. A sequence space η is said to be monotone if η contains the canonical pre images of all its step spaces. A Banach sequence space (η, S) is called a BK-space if the topology S of η is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps $P_i : \eta \to K,\ \ P_i(x) = x_i,\ i \geq 1$ are continuous, where K is the scalar field $\mathbb{R}$ or $\mathbb{C}$. For $x = (x_1, ..., x_n, ...)$ and $n \in \mathbb{N}$, we write the n^{th} section of x as $x^{(n)} = (x_1, ..., x_n, 0, 0, ...)$. If $x^{(n)} \to x$ in (η, S) for each $x \in \eta$, we say that (η, S) is an AK-space. The norm $\|.\|_\eta$ generating the topology S of η is said to be monotone if $\|x\|_\eta \leq \|y\|_\eta$ for $x = \{x_i\},\ \ y = \{y_i\} \in \eta$ with $|x_i| \leq |y_i|$, for all $i \geq 1$ (see [14]).
An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [17] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \Big\{x \in \omega : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [15] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. In the later stage different Orlicz sequence spaces were

introduced and studied by Parashar and Choudhary [25], Esi and Et [6], Tripathy and Mahanta [31], Mursaleen [21] and many others. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [22], [23]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows;

$$t_{\mathcal{M}} = \Big\{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in \omega : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ \ x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Any Orlicz function M_k can always be represented in the following integral form

$$M_k(x) = \int_0^x \eta_k(t)dt,$$

where η_k is known as the kernel of M_k, is a right differentiable for $t \geq 0$, $\eta_k(0) = 0$, $\eta_k(t) > 0$, η_k is non-decreasing and $\eta_k(t) \to \infty$ as $t \to \infty$.
Given an Orlicz function M_k with kernel $\eta_k(t)$, define

$$\nu_k(s) = \sup\{t : \eta_k(t) \leq s, s \geq 0\}.$$

Then $\nu_k(s)$ possesses the same properties as $\eta_k(t)$ and the function N_k defined as

$$N_k(x) = \int_0^x \nu_k(s)ds$$

is an Orlicz function. The functions M_k and N_k are called mutually complementary Orlicz functions.

For a sequence $\mathcal{M} = (M_k)$ of Orlicz functions, the modular sequence class $\tilde{l}(\mathcal{M})$ is defined by

$$\tilde{l}(\mathcal{M}) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} M_k(|x_k|) < \infty\}.$$

Using the sequence $\mathcal{N} = (N_k)$ of Orlicz functions, similarly we define $\tilde{l}(\mathcal{N})$. The class $l(\mathcal{M})$ is defined by

$$l(\mathcal{M}) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} x_k y_k \text{ converges, for all } y \in \tilde{l}(\mathcal{N})\}.$$

For a sequence $\mathcal{M} = (M_k)$ of Orlicz functions, the modular sequence space $l(\mathcal{M})$ is also defined as

$$l(\mathcal{M}) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} M_k\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\}.$$

The space $l(\mathcal{M})$ is a Banach space with respect to the norm $\|x\|_{\mathcal{M}}$ defined as

$$\|x\|_{\mathcal{M}} = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M_k\Big(\frac{|x_k|}{\rho}\Big) \leq 1\}.$$

These spaces were introduced by Woo [33] around the year 1973 and generalizes the Orlicz sequence space l_M and the modulared sequence spaces considered earlier by Nakano [24]. For more details about modular sequence spaces (see [15], [28]) and references therein.

An important subspace of $l(\mathcal{M})$, which is an AK-space, is the space $h(\mathcal{M})$ defined as

$$h(\mathcal{M}) = \{x \in l(\mathcal{M}) : \sum_{k=1}^{\infty} M_k\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\}.$$

A sequence (M_k) of Orlicz functions is said to satisfy uniform Δ_2- condition at $'0'$ if there exist $p > 0$ and $k_0 \in \mathbb{N}$ such that for all $x \in (0,1)$ and $k > k_0$, we have $\frac{xM_k'(x)}{M_k(x)} \leq p$, or equivalently, there exists a constant $K > 1$ and $k_0 \in \mathbb{N}$ such that $\frac{M_k(2x)}{M_k(x)} \leq K$ for all $x \in (0, \frac{1}{2}]$. If the sequence (M_k) satisfy uniform Δ_2- condition, then $h(\mathcal{M}) = l(\mathcal{M})$ and vice-versa (see [33]).

Let M_k and N_k be mutually complementary Orlicz functions for each k and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Bektaş and Atici [2] define the following sequence spaces:

$$l_\lambda^{\mathcal{M}}(\Delta^m) = \Big\{x = (x_k) : \sum_{k \geq 1} M_k\Big(\frac{|\Delta^m x_k|}{\lambda_k \rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}^{\lambda}(\Delta^m) = \Big\{x = (x_k) : \sum_{k \geq 1} N_k\Big(\frac{\lambda_k|\Delta^m x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}.$$

Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}^\lambda\big[\Delta_n^m, p\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[N_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $(p_k) = 1$, for all k then

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}^\lambda\big[\Delta_n^m\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[N_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho}\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}.$$

If $(\lambda_k) = 1$ for all $k \in \mathbb{N}$, then

$$l^{\mathcal{M}}\big[\Delta_n^m, p\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}\big[\Delta_n^m, p\big] = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[N_k\Big(\frac{|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $(p_k) = 1$, for all k and n=1 we get the spaces defined by Bektaş and Atici [2].
The following inequality will be used throughout the paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = H$, and let $D = \max\big\{1, 2^{H-1}\big\}$. Then, for the factorable sequences (a_k) and (b_k) in the complex plane, we have

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k}). \tag{1.1}$$

Throughout the paper we write $M_k(1) = 1$ and $N_k(1) = 1$ for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some difference new modular sequence spaces defined by a sequence of Orlicz functions over $n-$normed spaces. We shall study some topological, algebraic properties of the sequence spaces $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $l_{\mathcal{N}}^\lambda\big[\Delta_n^m, p\big]$ in the second section of the paper. In the third section we shall determine the dual spaces of $h(\mathcal{M})$, $l(\mathcal{M}, \lambda, p)$ and $l(\mathcal{N}, \lambda, p)$. Finally, we shall study some sequence spaces over $n-$ normed spaces in the fourth section of the paper. We have also made an attempt to study some topological, algebraic properties and inclusion relations between the sequence spaces $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ and $l_{\mathcal{N}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$.

2. Some topological properties of the spaces $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$

The purpose of this section is to study the properties like linearity, paranorm, solidity and relevant inclusion relations in the spaces $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ are linear spaces over the complex field $\mathbb{C}$.*

Proof: Let $x = (x_k)$ and $y = (y_k) \in l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} < \infty$$

and

$$\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m y_k|}{\lambda_k \rho_2}\Big)\Big]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $M_k's$ are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned}
\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m(\alpha x_k + \beta y_k)|}{\lambda_k \rho_3}\Big)\Big]^{p_k} &\leq \sum_{k\geq 1}\Big[M_k\Big(\frac{|\alpha\Delta_n^m x_k|}{\lambda_k \rho_3} + \frac{|\beta\Delta_n^m y_k|}{\lambda_k \rho_3}\Big)\Big]^{p_k} \\
&\leq D\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} \\
&\quad + D\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m y_k|}{\lambda_k \rho_2}\Big)\Big]^{p_k} \\
&< \infty.
\end{aligned}$$

Therefore, $\alpha x + \beta y \in l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and hence, $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ is a linear space. Similarly, we can prove that $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ is a linear space. This completes the proof. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence space $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ is a paranormed space with paranorm defined by*

$$g(x) = \inf\left\{(\rho)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

where $H = \max(1, G)$, $0 < p_k \leq \sup_k p_k = G$.

Proof: Clearly $g(x) \geq 0$, for $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf\left\{(\rho)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\} = 0,$$

this implies that for a given $\epsilon > 0$, there exist some ρ_ϵ $(0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Thus,

$$\begin{aligned}\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \epsilon}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} &\leq \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \\ &\leq 1.\end{aligned}$$

Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_n^m x_k \neq 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$, then $\frac{|\Delta_n^m x_k|}{\lambda_k \epsilon} \to \infty$. It follows that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_\epsilon}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \to \infty,$$

which is a contradiction. Therefore, $\Delta_n^m x_k = 0$ for each k and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_2}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by Minkowski's inequality, we have

$$
\begin{aligned}
\left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} &\leq \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k(\rho_1+\rho_2)}\Big)\Big]^{p_k}\right)^{\frac{1}{H}}\\
&\leq \left(\sum_{k\geq 1}\Bigg[\frac{\rho_1}{\rho_1+\rho_2}M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho_1}\Big)\right.\\
&\quad \left.+\frac{\rho_2}{\rho_1+\rho_2}M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho_2}\Big)\Bigg]^{p_k}\right)^{\frac{1}{H}}\\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\sum_{k\geq 1}\Bigg[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho_1}\Big)\Bigg]^{p_k}\right)^{\frac{1}{H}}\\
&\quad +\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\sum_{k\geq 1}\Bigg[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho_2}\Big)\Bigg]^{p_k}\right)^{\frac{1}{H}}\\
&\leq 1.
\end{aligned}
$$

Since ρ's are non-negative, so we have

$$
\begin{aligned}
g(x+y) &= \inf\left\{(\rho_1+\rho_2)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m (x_k+y_k)|}{\lambda_k(\rho_1+\rho_2)}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}\\
&\leq \inf\left\{(\rho_1)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho_1}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}\\
&+ \inf\left\{(\rho_2)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m y_k|}{\lambda_k\rho_2}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}.
\end{aligned}
$$

Therefore, $g(x+y) \leq g(x)+g(y)$. Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number, therefore, by definition

$$
g(\mu x) = \inf\left\{(\rho)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m \mu x_k|}{\lambda_k\rho}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\} \quad \text{and}
$$

thus,

$$
g(\mu x) = \inf\left\{(|\mu| t)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k t}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}
$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_k} \leq \max(1, |\mu| \sup p_k)$. Hence,

$$
g(\mu x) = \max(1, |\mu| \sup p_k) \inf\left\{(t)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k t}\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}.
$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof. □

Theorem 2.3. *Suppose* $\mathcal{M} = (M_k)$ *be a sequence of Orlicz functions,* $p = (p_k)$ *be a bounded sequence of positive real numbers and* $\lambda = (\lambda_k)$ *be a sequence of strictly positive real numbers. If* $0 < p_k \leq q_k < \infty$*, for each* $k \in \mathbb{N}$*, then* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m, q\big]$.

Proof: Suppose that $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$. This implies that

$$\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} \leq 1$$

for sufficiently large value of k say $k \geq k_0$, for some fixed $k_0 \in \mathbb{N}$. Since $M = (M_k)$ is non decreasing, we have

$$\begin{aligned}\sum_{k=k_0}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{q_k} &\leq \sum_{k=k_0}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k}\\ &< \infty.\end{aligned}$$

Hence, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, q\big]$. This completes the proof. □

Theorem 2.4. (i) *If* $0 < \inf p_k \leq p_k < 1$ *for each* k*, then* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m\big]$.
(ii) *If* $1 \leq p_k \leq \sup p_k < \infty$ *for each* k*, then* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$.

Proof: (i) Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$. Since $0 < \inf p_k < 1$, we have

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big] \leq \sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k}$$

and hence, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m\big]$.

(ii) Suppose p_k for each k $\sup p_k < \infty$ and let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m\big]$. Then for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big] \leq \epsilon < 1, \quad \text{for all } k \in \mathbb{N},$$

this implies that

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} \leq \sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big].$$

Thus, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$. This completes the proof. □

Theorem 2.5. *The sequence space* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$ *is solid.*

Proof: Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$. Then

$$\sum_{k=1}^{\infty} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} < \infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{k=1}^{\infty} \Big[M_k\Big(\frac{|\Delta_n^m \alpha_k x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} \leq \sum_{k=1}^{\infty} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k}.$$

This completes the proof. □

Corollary 2.6. *The sequence space* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big]$ *is monotone.*

Proof: It is obvious so we omit the proof. □

Theorem 2.7. *Let* $\mathcal{M} = (M_k)$ *and* $\mathcal{M}' = (M_k')$ *be two sequences of Orlicz functions. Then, we have*

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big] \cap l_\lambda^{\mathcal{M}'}\big[\Delta_n^m, p\big] \subseteq l_\lambda^{\mathcal{M}+\mathcal{M}'}\big[\Delta_n^m, p\big].$$

Proof: Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p\big] \cap l_\lambda^{\mathcal{M}'}\big[\Delta_n^m, p\big]$. Then

$$\sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} < \infty, \ \text{ for some } \ \rho_1 > 0$$

and

$$\sum_{k\geq 1} \Big[M_k'\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_2}\Big)\Big]^{p_k} < \infty, \ \text{ for some } \ \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the inequality

$$\begin{aligned}
\sum_{k\geq 1} \Big[(M_k + M_k')\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} &= \sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big) + M_k'\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} \\
&\leq D\sum_{k\geq 1} \Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k} \\
&\quad + D\sum_{k\geq 1} \Big[M_k'\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho_1}\Big)\Big]^{p_k}.
\end{aligned}$$

This completes the proof. □

The proof of the following theorems are easy so omitted.

Theorem 2.8. *The sequence space* $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ *is a normed space with norm*

$$\|x\|_{\lambda}^{\mathcal{M}} = \sum_{i=1}^{m} |x_i| + \inf\Big\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} \leq 1\Big\}.$$

Theorem 2.9. *The sequence space* $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ *is a normed space with norm*

$$\|x\|_{\mathcal{N}}^{\lambda} = \sum_{i=1}^{m} |x_i| + \inf\Big\{\rho > 0 : \sum_{k\geq 1}\Big[N_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} \leq 1\Big\}.$$

Theorem 2.10. *The spaces* $\Big(l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big], \|.\|_{\lambda}^{\mathcal{M}}\Big)$ *and* $\Big(l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big], \|.\|_{\mathcal{N}}^{\lambda}\Big)$ *are Banach spaces.*

Theorem 2.11. *The space* $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ *equipped with the norm* $\|.\|_{\lambda}^{\mathcal{M}}$ *and the space* $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ *equipped with the norm* $\|.\|_{\mathcal{N}}^{\lambda}$ *are BK-spaces.*

Proof: The space $\Big(l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big], \|.\|_{\lambda}^{\mathcal{M}}\Big)$ is a Banach space by the Theorem 2.10. Now let $\|x^l - x\|_{\lambda}^{\mathcal{M}} \to 0$ as $l \to \infty$. Then

$$|x_k^l - x_k| \to 0 \text{ as } l \to \infty, \text{ for each } k \leq m$$

and

$$\inf\left\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k^l - \Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} \leq 1\right\} \to 0$$

as $l \to \infty$ for all $k \in \mathbb{N}$.
If $M_k\Big(\frac{|\Delta_n^m x_k^l - \Delta_n^m x_k|}{\lambda_k \|x\|_{\lambda}^{\mathcal{M}}}\Big)^{p_k} \leq 1$ then $\frac{|\Delta_n^m x_k^l - \Delta_n^m x_k|}{\lambda_k \|x\|_{\lambda}^{\mathcal{M}}} \leq 1$ for all k. Therefore, we also obtain

$$|\Delta_n^m x_k^l - \Delta_n^m x_k| \leq \lambda_k \|x^l - x\|_{\lambda}^{\mathcal{M}}.$$

Since $\|x^l - x\|_{\lambda}^{\mathcal{M}} \to 0$, then $|\Delta_n^m x_k^l - \Delta_n^m x_k| \to 0$ and

$$\Big|\sum_{v=0}^{m}(-1)^v \binom{m}{v}(x_{k+nv}^l - x_{k+nv})\Big| \to 0$$

as $l \to \infty$ for all $k \in \mathbb{N}$. On the other hand, since we may write

$$|x_{k+nv}^l - x_{k+nv}| \leq \Big|\sum_{v=0}^{m}(-1)^v \binom{m}{v}(x_{k+nv}^l - x_{k+nv})\Big| + \Big|\binom{m}{0}(x_k^l - x_k)\Big|$$
$$+ \ldots\ldots + \Big|\binom{m}{m-1}(x_{k+n(m+1)}^l - x_{k+n(m+1)})\Big|.$$

Then $|x_k^l - x_k| \to 0$ as $l \to \infty$ for each $k \in \mathbb{N}$. Hence, $\Big(l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big], \|.\|_{\lambda}^{\mathcal{M}}\Big)$ is a BK-space. Similarly we can prove $\Big(l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big], \|.\|_{\mathcal{N}}^{\lambda}\Big)$ is a BK-space. This completes the proof. □

Theorem 2.12. *If $\mathcal{Z}$ is a normal sequence space containing λ, then $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ is a proper subspace of $\mathcal{Z}$. In addition, if $\mathcal{Z}$ is equipped with the monotone norm (quasi-norm) $\|.\|_{\mathcal{Z}}$. The inclusion $R : l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \to \mathcal{Z}\big[\Delta_n^m, p\big]$ is continuous with $\|R\| \leq \|\{\lambda_k\}\|_{\mathcal{Z}}$.*

Proof: Let $x \in l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$, then

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0.$$

So there exists a constant $K > 0$ such that

$$\frac{|\Delta_n^m x_k|}{\lambda_k \rho} \leq K \text{ for all } k \in \mathbb{N}.$$

Since $\mathcal{Z}$ is a normal sequence space containing λ, we have $\big[\Delta_n^m x_k\big]^{p_k} \in \mathcal{Z}$ and so that $x \in \mathcal{Z}\big[\Delta_n^m, p\big]$. Hence, $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \subseteq \mathcal{Z}\big[\Delta_n^m, p\big]$. Further, since $M_k(1) = 1$ for all $k \in \mathbb{N}$ then

$$\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k \|x\|_{\lambda}^{\mathcal{M}}}\Big)\Big]^{p_k} \leq 1$$

so that $|\Delta_n^m x_k| \leq \lambda_k \|x\|_{\lambda}^{\mathcal{M}}$. As $\|.\|_{\mathcal{Z}}$ is monotone, $\|Rx\|_{\mathcal{Z}} = \|\Delta_n^m x_k\|_{\mathcal{Z}} \leq \|\{\lambda_k\}\|_{\mathcal{Z}}\|x\|_{\lambda}^{\mathcal{M}}$ and hence, $\|R\| \leq \|\{\lambda_k\}\|_{\mathcal{Z}}$. This completes the proof. □

Theorem 2.13. *If $\mathcal{Y}$ is a normal sequence space containing $\lambda^{-1} \equiv \{\frac{1}{\lambda_k}\}$, then $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ is a proper subspace of $\mathcal{Y}$. In addition, if $\mathcal{Y}$ is equipped with the monotone norm (quasi-norm) $\|.\|_{\mathcal{Y}}$. The inclusion $S : l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big] \to \mathcal{Y}\big[\Delta_n^m, p\big]$ is continuous with $\|S\| \leq \|\{\lambda_k^{-1}\}\|_{\mathcal{Y}}$.*

Proof: The proof of the theorem is similar to that of Theorem 2.12 and so is omitted. □

Theorem 2.14. *If $\lambda = (\lambda_k)$ is a bounded sequence such that* $\inf \lambda_k > 0$ *(i.e both λ and λ^{-1} are in l_∞). Then $l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] = l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] = l_{\mathcal{M}}\big[\Delta_n^m, p\big]$.*

Proof: Let $x = (x_k) \in l_{\mathcal{M}}\big[\Delta_n^m, p\big]$, then

$$\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0.$$

Since $\lambda = (\lambda_k)$ is bounded, we can write $a \leq \lambda_k \leq b$ for some $b > a \geq 0$. Define $\rho_1 = \rho b$. Also since $M_k's$ are increasing, it follows that

$$\begin{aligned}\sum_{k\geq 1}\Big[M_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho_1}\Big)\Big]^{p_k} &\leq \sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} \\ &< \infty.\end{aligned}$$

Hence, $l_{\mathcal{M}}\big[\Delta_n^m, p\big] \subseteq l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$. The other inclusion $l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] \subseteq l_{\mathcal{M}}\big[\Delta_n^m, p\big]$ follows from the inequality

$$\begin{aligned}\sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\frac{\rho}{a}}\Big)\Big]^{p_k} &\leq \sum_{k\geq 1}\Big[M_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k}\\ &< \infty.\end{aligned}$$

Therefore, $l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] = l_{\mathcal{M}}\big[\Delta_n^m, p\big]$. Similarly one can prove that $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] = l_{\mathcal{M}}\big[\Delta_n^m, p\big]$. This completes the proof. □

Theorem 2.15. *If* $\{\lambda_k\} \in l_\infty$ *with* $c = \sup\limits_{k\geq 1}\lambda_k \geq 1$ *and* $\{\lambda_k^{-1}\}$ *is unbounded, then* $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ *is properly contained in* $l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$ *and the inclusion map* $U : l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \to l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$ *is continuous with* $\|U\| \leq c^2$.

Proof: For any $\rho > 0$ and $\rho' = \rho c^2$, we have

$$\begin{aligned}\sum_{k\geq 1}\Big[M_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho'}\Big)\Big]^{p_k} &\leq \sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho}\Big)\Big]^{p_k}\\ &< \infty.\end{aligned}$$

for $x = \{x_k\}$. Hence, $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \subset l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$. We now show that the containment $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \subset l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$ is proper. From the unboundedness of the sequence $\{\lambda_k^{-1}\}$, choose a sequence $\{k_l\}$ of positive integers such that $\lambda_{k_l}^{-1} \geq l$. Now define

$$\Delta_n^m x_k = \begin{cases} \frac{1}{l}, & k = k_l,\ \ l = 1, 2, ...;\\ 0, & \textit{otherwise}.\end{cases}$$

Then $x \in l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$, but $x \notin l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$. To prove the continuity of the inclusion map U, let us first consider the case obtained for $c = 1$. For $x \in l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$, we write

$$A_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] = \left\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\frac{|\Delta_n^m x_k|}{\lambda_k\rho}\Big)\Big]^{p_k} \leq 1\right\}$$

and

$$B_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] = \left\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\frac{\lambda_k|\Delta_n^m x_k|}{\rho}\Big)\Big]^{p_k} \leq 1\right\}.$$

Since $M_k' s$ are increasing and $c = 1$, we get $A_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big] \subset B_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$. Hence,

$$\|x\|_{\mathcal{M}}^{\lambda} = \inf B_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] \leq \inf A_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] = \|x\|_{\lambda}^{\mathcal{M}} \tag{2.1}$$

i.e $\|U(x)\|_{\mathcal{M}}^{\lambda} \leq \|x\|_{\lambda}^{\mathcal{M}}$. Thus, U is continuous with $\|U\| \leq 1 = c^2$. If $c \neq 1$, define $\delta_k = \frac{\lambda_k}{c},\ \ k \in \mathbb{N}$. Then $\delta_k \leq 1$ and from (2.1), it follows that

$$\|x\|_{\mathcal{M}}^{\delta} \leq \|x\|_{\delta}^{\mathcal{M}} \ \text{ for } \ x \in l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]. \tag{2.2}$$

Hence, from (2.2)

$$\|U(x)\|_{\mathcal{M}}^{\lambda} = \|x\|_{\mathcal{M}}^{\lambda} \leq c^2 \|x\|_{\lambda}^{\mathcal{M}}$$

i.e. U is continuous with $\|U\| \leq c^2$. This completes the proof. □

Theorem 2.16. *If $\{\lambda_k\}$ is unbounded with $\sup_{k\geq 1} \lambda_k^{-1} = d \geq 1$, $\lambda_k > 0$ for all k, then $l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big]$ is properly contained in $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and the inclusion map $V : l_{\mathcal{M}}^{\lambda}\big[\Delta_n^m, p\big] \to l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ is continuous with $\|V\| \leq d^2$.*

Proof: The proof of the theorem is similar to that of Theorem 2.15 and so is omitted. □

3. Dual spaces of $h(\mathcal{M})$, $l(\mathcal{M}, \lambda, p)$ and $l(\mathcal{N}, \lambda, p)$

Let η be a sequence space and defined

$$\eta^{\alpha} = \Big\{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \ \text{ for all } \ x \in \eta\Big\},$$

$$\eta^{\beta} = \Big\{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \quad \text{converges } \text{ for all } \ x \in \eta\Big\},$$

$$\eta^{\gamma} = \Big\{a = (a_k) : \sup \Big|\sum_{k=1}^{\infty} a_k x_k\Big| < \infty, \ \text{ for all } \ x \in \eta\Big\} \ \text{ (see [13]).}$$

Then η^{α}, η^{β}, η^{γ} are called $\alpha-$, $\beta-$, $\gamma-$ dual spaces of η respectively. It is easy to show that $\phi \subset \eta^{\alpha} \subset \eta^{\beta} \subset \eta^{\gamma}$. If $\eta \subset \nu$, then $\nu^{\sigma} \subset \eta^{\sigma}$ for $\sigma = \alpha, \beta, \gamma$. We shall write $\eta^{\alpha\alpha} = (\eta^{\alpha})^{\alpha}$.
Let η be a sequence space. Then η is called perfect if $\eta = \eta^{\alpha\alpha}$ (see [15]).
For $m = n = 0$ we write $l(\mathcal{M}, \lambda, p)$ and $l(\mathcal{N}, \lambda, p)$ instead of $l_{\lambda}^{\mathcal{M}}\big[\Delta_n^m, p\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p\big]$ repectively which we define as:

$$l(\mathcal{M}, \lambda, p) = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[M_k\Big(\frac{|x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} < \infty, \ \text{ for some } \ \rho > 0\Big\}$$

$$l(\mathcal{N}, \lambda, p) = \Big\{x = (x_k) \in \omega : \sum_{k\geq 1} \Big[N_k\Big(\frac{\lambda_k |x_k|}{\rho}\Big)\Big]^{p_k} < \infty, \ \text{ for some } \ \rho > 0\Big\}.$$

In this section we shall obtain $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence space $h(\mathcal{M})$ and $\alpha-$ duals of $l(\mathcal{M}, \lambda, p)$ and $l(\mathcal{N}, \lambda, p)$.

Proposition 3.1. *η is perfect $\Rightarrow$ η is normal $\Rightarrow$ η is monotone (see [15]).*

Proposition 3.2. *Let η be a sequence space. If η is monotone, then $\eta^{\alpha} = \eta^{\beta}$ and if η is normal, then $\eta^{\alpha} = \eta^{\gamma}$.*

Proposition 3.3. *The sequence space $h(\mathcal{M})$ is normal for any sequence (M_k) of Orlicz functions.*

Proof: Let $x \in h(\mathcal{M})$ and $|y_k| \leq |x_k|$, for each $k \in \mathbb{N}$. Since $M_k's$ are non-decreasing we have

$$\sum_{k=1}^{\infty} M_k\Big(\frac{|y_k|}{\rho}\Big) \leq \sum_{k=1}^{\infty} M_k\Big(\frac{|x_k|}{\rho}\Big) < \infty.$$

Hence, $y \in h(\mathcal{M})$. Thus, $h(\mathcal{M})$ is normal. □

Theorem 3.1. *Let (M_k) and (N_k) for each k be mutually complementary Orlicz functions. Then*

$$[h(\mathcal{M})]^{\beta} = [h(\mathcal{M})]^{\alpha} = [h(\mathcal{M})]^{\gamma} = l(\mathcal{N}).$$

The proof is seen from Proposition 3.1, Proposition 3.2 and Proposition 3.3.

Theorem 3.2. *If the sequence (M_k) satisfies uniform Δ_2- condition, then* $[l(\mathcal{M},\lambda,p)]^{\alpha} = l(\mathcal{N},\lambda,p)$

Proof: Let the sequence (M_k) satisfies uniform Δ_2- condition, Then for any $x \in l(\mathcal{M},\lambda,p)$ and $a \in l(\mathcal{N},\lambda,p)$, we have

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} \Big[M_k\Big(\frac{|x_k|}{\lambda_k \rho}\Big)\Big]^{p_k} + \sum_{k=1}^{\infty} \Big[N_k\Big(\frac{\lambda_k |a_k|}{\rho'}\Big)\Big]^{p_k} < \infty$$

where $\rho' = \frac{1}{\rho}$ and $\rho > 0$. Thus, $a \in [l(\mathcal{M},\lambda,p)]^{\alpha}$. Hence, $l(\mathcal{N},\lambda,p) \subset [l(\mathcal{M},\lambda,p)]^{\alpha}$. To prove the inclusion $[l(\mathcal{M},\lambda,p)]^{\alpha} \subset l(\mathcal{N},\lambda,p)$, let $a \in [l(\mathcal{M},\lambda,p)]^{\alpha}$. Then for all $\{x_k\}$ with $\left(\frac{x_k}{\lambda_k}\right) \in l(\mathcal{M})$ we have

$$\sum_{k=1}^{\infty} |a_k x_k| < \infty. \tag{3.1}$$

Since the sequence satisfies uniform Δ_2-condition, then $l(\mathcal{M}) = h(\mathcal{M})$ and so for $(y_k) \in h(\mathcal{M})$ we have $\sum_{k=1}^{\infty} |\lambda_k y_k a_k| < \infty$ by (3.1). Thus, $(\lambda_k a_k) \in [h(\mathcal{M})]^{\alpha} = l(\mathcal{N})$ and hence, $(a_k) \in l(\mathcal{N},\lambda,p)$. Therefore, $[l(\mathcal{M},\lambda,p)]^{\alpha} = l(\mathcal{N},\lambda,p)$. □

Theorem 3.3. *If the sequence (M_k) satisfies uniform Δ_2- condition, then* $[l(\mathcal{N},\lambda,p)]^{\alpha} = l(\mathcal{M},\lambda,p)$.

Proof: Immediate from Theorem 3.5. □

4. Some new sequence spaces over n-normed space

The concept of 2-normed spaces was initially developed by Gähler [10] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [18]. Since then, many others have studied this concept and obtained various results, see Gunawan ([11], [12]) and Gunawan and Mashadi [13]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X,
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation,
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. For more details about $n-$ normed space (see [26]) and references therein.

Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space and $W(n-X)$ denotes the space of X-valued sequences. Let $p = (p_k)$ be a bounded sequence of positive real numbers, $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions and $\mathcal{N} = (N_k)$ is a complementary function of Orlicz function $\mathcal{M} = (M_k)$. In this section of the paper we define the following sequences:

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[N_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $(p_k) = 1$ for all k then

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}^\lambda\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[N_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}.$$

If $(\lambda_k) = 1$ for all $k \in \mathbb{N}$, then

$$l^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}$$

and

$$l_{\mathcal{N}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \Big\{x = (x_k) \in W(n-X) : \sum_{k\geq 1}\Big[N_k\Big(\Big\|\frac{\Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0\Big\}$$

respectively.

The main aim of this section is to study some topological properties and inclusion relations between the spaces $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$.

Theorem 4.1. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence spaces $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ and $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ are linear spaces over the field $\mathbb{C}$ of complex numbers.*

Proof: Let $x = (x_k)$ and $y = (y_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty$$

and

$$\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m y_k}{\lambda_k\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\|\cdot,\cdots,\cdot\|$ is a n-norm on X and M_k's are non-decreasing and convex function so by using inequality (1.1), we have

$$\begin{aligned}
&\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m(\alpha x_k+\beta y_k)}{\lambda_k\rho_3}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\leq \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\alpha\Delta_n^m x_k}{\lambda_k\rho_3}, z_1,\cdots,z_{n-1}\Big\| + \Big\|\frac{\beta\Delta_n^m y_k}{\lambda_k\rho_3}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\leq D\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\quad +D\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m y_k}{\lambda_k\rho_2}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&< \infty.
\end{aligned}$$

Therefore, $\alpha x + \beta y \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ and hence, $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ is a linear space. Similarly, we can prove $l_{\mathcal{N}}^{\lambda}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ is a linear space. This completes the proof. □

Theorem 4.2. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. Then the sequence space $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot,\cdots,\cdot\|\big]$ is a paranormed space with paranorm defined by*

$$g(x) = \inf\left\{(\rho)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho}, z_1,\cdots,z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

where $H = \max(1, G)$, $0 < p_k \leq \sup_k p_k = G$.

Proof: Clearly $g(x) \geq 0$, for $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Since $M_k(0) = 0$, we get $g(0) = 0$. Again, if $g(x) = 0$, then

$$g(x) = \inf\left\{(\rho)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\} = 0,$$

this implies that for a given $\epsilon > 0$, there exist some ρ_ϵ $(0 < \rho_\epsilon < \epsilon)$ such that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho_\epsilon}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Thus,

$$\begin{aligned}\left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \epsilon}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} & \\ \leq & \left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho_\epsilon}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \\ \leq & \ 1.\end{aligned}$$

Suppose that $x_k \neq 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_n^m x_k \neq 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$, then $\Big\|\frac{\Delta_n^m x_k}{\lambda_k \epsilon}, z_1, \cdots, z_{n-1}\Big\| \to \infty$. It follows that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho_\epsilon}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \to \infty$$

which is a contradiction. Therefore, $\Delta_n^m x_k = 0$ for each k and thus $x_k = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1$$

and

$$\left(\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\right)^{\frac{1}{H}} \leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then by Minkowski's inequality, we have

$$
\begin{aligned}
&\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\\
&\leq \left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k(\rho_1+\rho_2)},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\\
&\leq \left(\sum_{k\geq 1}\left[\frac{\rho_1}{\rho_1+\rho_2}M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1},z_1,\cdots,z_{n-1}\right\|\right)\right.\right.\\
&\qquad\left.\left.+\frac{\rho_2}{\rho_1+\rho_2}M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho_2},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\\
&+ \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho_2},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\\
&\leq 1.
\end{aligned}
$$

Since ρ's are non-negative, so we have

$$
\begin{aligned}
g(x+y) &= \inf\Bigg\{(\rho_1+\rho_2)^{\frac{p_k}{H}}:\\
&\qquad \left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m (x_k+y_k)}{\lambda_k(\rho_1+\rho_2)},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1\Bigg\}\\
&\leq \inf\left\{(\rho_1)^{\frac{p_k}{H}}:\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1\right\}\\
&+ \inf\left\{(\rho_2)^{\frac{p_k}{H}}:\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m y_k}{\lambda_k\rho_2},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1\right\}.
\end{aligned}
$$

Therefore, $g(x+y)\leq g(x)+g(y)$. Finally, we prove that the scalar multiplication is continuous. Let μ be any complex number, therefore, by definition

$$
g(\mu x)=\inf\left\{(\rho)^{\frac{p_k}{H}}:\left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m \mu x_k}{\lambda_k\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}}\leq 1\right\} \text{ and}
$$

thus,

$$g(\mu x) = \inf\left\{(|\mu|t)^{\frac{p_k}{H}} : \left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k t}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\right\}$$

where $t = \frac{\rho}{|\mu|}$. Since $|\mu|^{p_k} \leq \max(1, |\mu| \sup p_k)$. Hence,

$$\begin{aligned} g(\mu x) &= \max(1, |\mu| \sup p_k) \inf\Bigg\{(t)^{\frac{p_k}{H}} : \\ &\qquad \left(\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k t}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k}\right)^{\frac{1}{H}} \leq 1\Bigg\}. \end{aligned}$$

So, the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof. □

Theorem 4.3. *Suppose $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be a bounded sequence of positive real numbers and $\lambda = (\lambda_k)$ be a sequence of strictly positive real numbers. If $0 < p_k \leq q_k < \infty$, for each $k \in \mathbb{N}$, then $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m, q, \|\cdot, \cdots, \cdot\|\big]$.*

Proof: Suppose that $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$, this implies that

$$\sum_{k\geq 1}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \leq 1$$

for sufficiently large value of k say $k \geq k_0$, for some fixed $k_0 \in \mathbb{N}$. Since $M = (M_k)$ is non decreasing, we have

$$\begin{aligned} \sum_{k=k_0}^{\infty}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{q_k} &\leq \sum_{k=k_0}^{\infty}\left[M_k\left(\left\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\right\|\right)\right]^{p_k} \\ &< \infty. \end{aligned}$$

Hence, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, q, \|\cdot, \cdots, \cdot\|\big]$. This completes the proof. □

Theorem 4.4. *(i) If $0 < \inf p_k \leq p_k < 1$ for each k, then $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$.*
(ii) If $1 \leq p_k \leq \sup p_k < \infty$ for each k, then $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big] \subseteq l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$.

Proof: (i) Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Since $0 < \inf p_k < 1$, we have

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \le \sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}$$

and hence, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$.

(ii) Suppose p_k for each k $\sup p_k < \infty$ and let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, \|\cdot, \cdots, \cdot\|\big]$. Then for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big] \le \epsilon < 1, \text{ for all } k \in \mathbb{N}.$$

This implies that

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \le \sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big].$$

Thus, $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. This completes the proof. □

Theorem 4.5. *The sequence space* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ *is solid.*

Proof: Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Then

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty.$$

Let (α_k) be a sequence of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m \alpha_k x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \le \sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}.$$

This completes the proof. □

Corollary 4.6. *The sequence space* $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ *is monotone.*

Proof: It is obvious so we omit the proof. □

Theorem 4.7. *Let* $\mathcal{M} = (M_k)$ *and* $\mathcal{M}' = (M_k')$ *be two sequences of Orlicz functions. Then, we have*

$$l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \cap l_\lambda^{\mathcal{M}'}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subseteq l_\lambda^{\mathcal{M}+\mathcal{M}'}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big].$$

Proof: Let $x = (x_k) \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \cap l_\lambda^{\mathcal{M}'}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$.
Then

$$\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_1 > 0$$

and

$$\sum_{k\geq 1}\Big[M_k'\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the inequality

$$\begin{aligned}
&\sum_{k\geq 1}\Big[(M_k + M_k')\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&= \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big) + M_k'\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\leq D\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}\\
&\quad + D\sum_{k\geq 1}\Big[M_k'\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k}.
\end{aligned}$$

This completes the proof. □

Theorem 4.8. *If $\mathcal{Z}$ is a normal sequence space containing λ, then $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ is a proper subspace of $\mathcal{Z}$. In addition, if $\mathcal{Z}$ is equipped with the monotone norm (quasi-norm) $\|.\|_{\mathcal{Z}}$. The inclusion $R : l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \to \mathcal{Z}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ is continuous with $\|R\| \leq \|\{\lambda_k\}\|_{\mathcal{Z}}$.*

Proof: Let $x \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$, then

$$\sum_{k=1}^{\infty}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0.$$

So there exists a constant $K > 0$ such that

$$\Big\|\frac{\Delta_n^m x_k}{\lambda_k\rho}, z_1, \cdots, z_{n-1}\Big\| \leq K \text{ for all } k \in \mathbb{N}.$$

Since $\mathcal{Z}$ is a normal sequence space containing λ, we have $\big[\|\Delta_n^m x_k, z_1, \cdots, z_{n-1}\|\big]^{p_k} \in \mathcal{Z}$ and so that $x \in \mathcal{Z}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Hence, $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subseteq \mathcal{Z}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Further, since $M_k(1) = 1$ for all $k \in \mathbb{N}$ then

$$\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k\|x\|_\lambda^{\mathcal{M}}}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \leq 1$$

so that $\|\Delta_n^m x_k, z_1, \cdots, z_{n-1}\| \leq \lambda_k \|x\|_\lambda^{\mathcal{M}}$. As $\|.\|_{\mathcal{Z}}$ is monotone, $\|Rx\|_{\mathcal{Z}} = \|\Delta_n^m x_k, z_1, \cdots, z_{n-1}\|_{\mathcal{Z}} \leq \|\{\lambda_k\}\|_{\mathcal{Z}} \|x\|_\lambda^{\mathcal{M}}$ and hence, $\|R\| \leq \|\{\lambda_k\}\|_{\mathcal{Z}}$. This completes the proof. □

Theorem 4.9. *If $\mathcal{Y}$ is a normal sequence space containing $\lambda^{-1} \equiv \{\frac{1}{\lambda_k}\}$, then $l_{\mathcal{N}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ is a proper subspace of $\mathcal{Y}$. In addition, if $\mathcal{Y}$ is equipped with the monotone norm (quasi-norm) $\|.\|_{\mathcal{Y}}$. The inclusion $S : l_{\mathcal{N}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] \to \mathcal{Y}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ is continuous with $\|S\| \leq \|\{\lambda_k^{-1}\}\|_{\mathcal{Y}}$.*

Proof: The proof of the theorem is similar to that of Theorem 4.8 and so is omitted. □

Theorem 4.10. *If $\lambda = (\lambda_k)$ is a bounded sequence such that $\inf \lambda_k > 0$ (i.e both λ and λ^{-1} are in l_∞). Then $l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] = l_\lambda^{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] = l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$.*

Proof: Let $x = (x_k) \in l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$. Then

$$\sum_{k \geq 1} \Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty, \text{ for some } \rho > 0.$$

Since $\lambda = (\lambda_k)$ is bounded, we can write $a \leq \lambda_k \leq b$ for some $b > a \geq 0$. Define $\rho_1 = \rho b$. Also since $M_k's$ are increasing, it follows that

$$\begin{aligned}\sum_{k \geq 1} \Big[M_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} &\leq \sum_{k \geq 1} \Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \\ &< \infty.\end{aligned}$$

Hence, $l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] \subseteq l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$. The other inclusion $l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] \subseteq l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ follows from the inequality

$$\begin{aligned}\sum_{k \geq 1} \Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\frac{\rho}{a}}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} &\leq \sum_{k \geq 1} \Big[M_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \\ &< \infty.\end{aligned}$$

Therefore, $l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] = l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$. Similarly one can prove that
$l_\lambda^{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] = l_{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$. This completes the proof. □

Theorem 4.11. *If $\{\lambda_k\} \in l_\infty$ with $c = \sup_{k \geq 1} \lambda_k \geq 1$ and $\{\lambda_k^{-1}\}$ is unbounded, then $l_\lambda^{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ is properly contained in $l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ and the inclusion map $U : l_\lambda^{\mathcal{M}}[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|] \to l_{\mathcal{M}}^\lambda[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|]$ is continuous with $\|U\| \leq c^2$.*

Proof: For any $\rho > 0$ and $\rho' = \rho c^2$, we have

$$\sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho'}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \leq \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} < \infty,$$

for $x = \{x_k\}$. Hence, $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subset l_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. We now show that the containment $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subset l_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ is proper. From the unboundedness of the sequence $\{\lambda_k^{-1}\}$, choose a sequence $\{k_l\}$ of positive integers such that $\lambda_{k_l}^{-1} \geq l$. Now define

$$\Delta_n^m x_k = \begin{cases} \frac{1}{l}, & k = k_l, \ \ l = 1, 2, ...; \\ 0, & otherwise. \end{cases}$$

Then $x \in l_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ but $x \notin l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. To prove the continuity of the inclusion map U, let us first consider the case obtained for $c = 1$. For $x \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$, we write

$$A_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \left\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\Delta_n^m x_k}{\lambda_k \rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \leq 1\right\}$$

and

$$B_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \left\{\rho > 0 : \sum_{k\geq 1}\Big[M_k\Big(\Big\|\frac{\lambda_k \Delta_n^m x_k}{\rho}, z_1, \cdots, z_{n-1}\Big\|\Big)\Big]^{p_k} \leq 1\right\}.$$

Since $M_k's$ are increasing and $c = 1$, we get $A_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \subset B_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$. Hence,

$$\|x\|_{\mathcal{M}}^\lambda = \inf B_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \leq \inf A_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] = \|x\|_\lambda^{\mathcal{M}} \tag{4.1}$$

i.e $\|U(x)\|_{\mathcal{M}}^\lambda \leq \|x\|_\lambda^{\mathcal{M}}$. Thus, U is continuous with $\|U\| \leq 1 = c^2$. If $c = 1$, define $\delta_k = \frac{\lambda_k}{c}$, $k \in \mathbb{N}$. Then $\delta_k \leq 1$ and from (4.1), it follows that

$$\|x\|_{\mathcal{M}}^\delta \leq \|x\|_\delta^{\mathcal{M}} \ \text{ for } \ x \in l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]. \tag{4.2}$$

Hence, from (4.2)

$$\|U(x)\|_{\mathcal{M}}^\lambda = \|x\|_{\mathcal{M}}^\lambda \leq e^2 \|x\|_\lambda^{\mathcal{M}},$$

thus, U is continuous with $\|U\| \leq c^2$. This completes the proof. □

Theorem 4.12. *If $\{\lambda_k\}$ is unbounded with $\sup_{k\geq 1} \lambda_k^{-1} = d \geq 1$, $\lambda_k > 0$ for all k, then $l_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ is properly contained in $l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ and the inclusion map $V : l_{\mathcal{M}}^\lambda\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big] \to l_\lambda^{\mathcal{M}}\big[\Delta_n^m, p, \|\cdot, \cdots, \cdot\|\big]$ is continuous with $\|V\| \leq d^2$.*

Proof: The proof of the theorem is similar to that of Theorem 4.11 and so is omitted. □

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