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Existence of Solution for Dirichlet Problem with $p(x)$-Laplacian

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abstract: In this paper we study an elliptic equation involving the $p(x)$-laplacian operator, for that equation we prove the existence of a non trivial weak solution. The proof relies on simple variational arguments based on the Mountain-Pass theorem.

Key Words: p(x)-laplacian; generalized Lebesgue (Sobolev) spaces; critical points.

Contents

1. Introduction

We consider the following problem:

$$(1.1)\quad \begin{cases} -\Delta_{p(x)}u &= f(x,u) \quad \text{in}\,\Omega \\ u &= 0 \quad \text{on}\,\partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\Delta_{p(x)}(u) = div(|\nabla u(x)|^{p(x)-2}\nabla u(x))$, $p \in C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega});\, h(x) > 1\,\text{for any}\, x \in \overline{\Omega}\}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The study of equation involving $p(x)$-growth conditions has captured a special attention, since there are some physical phenomena which can be modelled by such kind of equation (see [5], [10], [11] [8]).
Existence results for $p(x)$-laplacian Dirichlet problems on bounded domains were studied in [6], [7], [9] .... In [7] the authors established the existence of weak solution in the case $f(x,s) = \lambda v(x)|s|^{q(x)-2}s$ where $q(x) < p(x)$.Fan, Zhang and Zhao [4] proved the existence of weak solutions under assumption of type Ambrosetti-Rabinowitz (AR) [1]: there exists $\theta > p^+$ such that $0 < \theta F(x,s) \le sf(x,s) \quad \forall x \in \Omega$ and $s \in \mathbb{R}$ where $p^+ = max_{x\in\overline{\Omega}}p(x)$. Petre Sorin Ilias [6] proved the existence of weak solution for the Dirichlet problem (1.1) in the case $f(x,-s) = -f(x,s)$. Maria Mag [9] studied problem (1.1) where $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$.

In this paper we study the problem (1.1) under the assumption:
(H_1) $\quad |f(x,s)| \le a|s|^{\alpha(x)-1} + \text{b}$ where $a > 0$, $\alpha(x) \in C_+(\overline{\Omega})$, $b \in \mathbb{R}$ and $\alpha(x) < p^*(x)\,\forall x \in \Omega$ with:

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if} \quad p(x) < N, \\ +\infty & \text{if} \quad p(x) \ge N. \end{cases}$$

(H_2) $F(x,s) - sf(x,s) \geq B(x) - \beta|s|^{\eta}$ where $B(.) \in L^1(\Omega)$, η, $\beta \in \mathbb{R}$ and $\eta < p^-$ where $p^- = min_{x\in\overline{\Omega}} p(x)$

(H_3) $f(x,s) = o(|s|^{P^+-1})$ as $s \to 0$ and uniformly for $x \in \Omega$.

and

(H_4) $F(x,s) \geq \gamma|s|^{\theta} - b_1|s|^r + B_1(x)$ in a subset $\Omega_1 \subset \Omega$, with $|\Omega_1| > 0$, $\gamma > 0$, $s \in \mathbb{R}$, $r \geq 0$, $\theta > \sup(p^+; r)$, $B_1(.) \in L^1(\Omega)$ and $b_1 \in \mathbb{R}$.

Remark 1.1. *1. It is known that (H_4) is weaker than the condition (AR), moreover we assume the condition on measuring portion of the set Ω.*

2. Similar result can be obtained, if we replace in (H_4) the assumption $s \in \mathbb{R}$ by $s \in \mathbb{R}^+$ (or $s \in \mathbb{R}^-$).

This paper is divided into three sections. In the second section, we introduce some basic properties of the generalized Lebesgue-Sobolev spaces and several important properties of $p(x)$-Laplace operator. In the third section, we give some existence results of weak solutions of problem (1.1).

2. Preliminary results

In this section we recall some results on variable exponent Sobolev space, the reader is referred to [2], [6], [3] and the references therein for more details.

Set

$M = \{u : \Omega \to \mathbb{R};\ u \text{ is a measurable real- valued function}\}$.
$L^{p(x)}(\Omega) = \{u \in M;\ \int_\Omega |u(x)|^{p(x)} dx < +\infty\}$.

We define on $L^{p(x)}$ the so- called Luxemburg norm by the formula:

$$|u|_{p(x)} = \inf\{\mu > 0; \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\, dx \leq 1\}.$$

Variable exponent Lebesgue spaces $(L^{p(x)}(\Omega), |.|_{p(x)})$ resemble to classical Lebesgue spaces in many respects; they are reflexive and Banach space.
On $L^{p(x)}(\Omega)$ we also consider the function $\varphi_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by:

$$\varphi_{p(x)}(u) = \int_\Omega |u|^{p(x)} dx.$$

Proposition 2.1. *([3])*

1. *We have the equivalence:*
   $|u|_{p(x)} < (>,=) 1 \Leftrightarrow \varphi_{p(x)}(u) < (>,=) 1.$

2. $|u|_{p(x)} > 1 \Longrightarrow |u|^{p^-}_{p(x)} \leq \varphi_{p(x)}(u) \leq |u|^{p^+}_{p(x)}.$

3. $|u|_{p(x)} < 1 \Longrightarrow |u|^{p^+}_{p(x)} \leq \varphi_{p(x)}(u) \leq |u|^{p^-}_{p(x)}.$

4. *$A \subseteq L^{p(x)}(\Omega)$ is bounded if and only if $\varphi_{p(x)}(A) \subseteq \mathbb{R}$ is bounded.*

5. *For a sequence $(u_n) \subset L^{p(x)}(\Omega)$ and an element $u \in L^{p(x)}(\Omega)$, the following statements are equivalent:*
   - $\lim\limits_{n\to+\infty} u_n = u$ *in* $L^{p(x)}(\Omega)$.
   - $\lim\limits_{n\to+\infty} \varphi_{p(x)}(u_n - u) = 0$.
   - $u_n \to u$ *in measure in* Ω *and* $\lim\limits_{n\to+\infty} \varphi_{p(x)}(u_n) = \varphi_{p(x)}(u)$.

6. $\lim\limits_{n\to+\infty} |u_n|_{p(x)} = +\infty$ *if and only if* $\lim\limits_{n\to+\infty} \varphi_{p(x)}(u_n) = +\infty$.

We define the variable Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega); \quad \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega) \quad \text{for all} \quad 1 \leq i \leq N \right\}$$

and equipp it with the norm

$$||u||_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

denote by $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.2. *(see [2])*

1. *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

2. *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

3. *There is a constant $C > 0$, such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)} \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

By the assertion 3 of Proposition 2.2, we know that $|\nabla u|_{p(x)}$ and $||u||_{1,p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.
Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$ equipped with the norm $||u|| = |\nabla u|_{p(x)}$, the p(x)-laplacian operator is defined by:

$$-\Delta_{p(x)} : E \longrightarrow E^*$$

$$< -\Delta_{p(x)} u, v >= \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx; \quad u, v \in E.$$

Proposition 2.3. *(see [4])*

1. *$-\Delta_{p(x)} : E \longrightarrow E^*$ is a homeomorphisme from E into E^*.*
2. *$-\Delta_{p(x)} : E \longrightarrow E^*$ is a strictly monotone operator, that is*

$$< -\Delta_{p(x)}u - (-\Delta_{p(x)})v, u - v >> 0, \quad \forall u \neq v$$

3. *$-\Delta_{p(x)} : E \longrightarrow E^*$ is a mapping of type S_+, that is, if $u_n \rightharpoonup u$ in E and $\limsup_{n\to+\infty} < -\Delta_{p(x)}u_n - (-\Delta_{p(x)})u, u_n - u >\leq 0$ then $u_n \longrightarrow u$ in E*

Proposition 2.4. *([4]) The functional $H : E \longrightarrow \mathbb{R}$ defined by:*

$$H(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$$

is continuously Fréchet differentiable and $H'(u) = -\Delta_{p(x)}u$, for all $u \in E$.

In the last part of this section we recall the basic results of the Nemytskii operator.
Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ a Carathéodory function and $u \in M$, the function $N_f(u) : \Omega \to \mathbb{R}$ defined by $N_f(u)(x) = f(x, u(x))$ is mesurable in Ω, thus the Carathéodory function f defines an operator $N_f : M \to M$, which is called the Nemytskii operator.

Proposition 2.5. *([12]) Suppose $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)| \leq c|t|^{\frac{\alpha(x)}{\beta(x)}} + h(x), \text{ for any } x \in \Omega,\ t \in \mathbb{R},$$

where $\alpha(.), \beta(.) \in C_+(\bar{\Omega})$, $c \geq 0$ is constant and $h \in L^{\beta(x)}(\Omega)$. Then $N_f(L^{\alpha(x)}(\Omega)) \subset L^{\beta(x)}(\Omega)$. Moreover, N_f is continuous from $L^{\alpha(x)}(\Omega)$ to $L^{\beta(x)}(\Omega)$ and maps bounded set into bounded set.

For a function $\alpha(.) \in C_+(\overline{\Omega})$, we recall that $\beta(.) \in C_+(\overline{\Omega})$ is its conjugate function if $\frac{1}{\alpha(x)} + \frac{1}{\beta(x)} = 1$ for all $x \in \overline{\Omega}$.

Proposition 2.6. *([6], [3]) Suppose $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function and satisfies the growth condition*

$$|f(x,t)| \leq c|t|^{\alpha(x)-1} + h(x), \text{ for any } x \in \Omega,\ t \in \mathbb{R}$$

where $c \geq 0$ is constant, $\alpha \in C_+(\overline{\Omega})$, $h \in L^{\beta(x)}(\overline{\Omega})$ and $\beta \in C_+(\overline{\Omega})$ is the conjugate function of α.
Let $F : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, defined by $F(x,t) = \int_0^t f(x,s)ds$, then:

1. *F is a Carathéodory function and there exist a constant $c_1 \geq 0$ and $\sigma \in L^1(\Omega)$ such that:*
$$|F(x,t)| \leq c_1|t|^{\alpha(x)} + \sigma(x); \quad x \in \Omega, t \in \mathbb{R}.$$
2. *The functional $\bar{J} : L^{\alpha(x)}(\Omega) \longrightarrow \mathbb{R}$ defined by $\bar{J}(u) = \int_\Omega F(x, u(x))dx$ is continuously Fréchet differentiable and $< \bar{J}'(u), v >= \int_\Omega f(x, u(x))v(x)\,dx$ for all u, $v \in L^{\alpha(x)}(\Omega)$.*

Lemma 2.7. *Suppose $f:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ is a Carathéodory function and satisfies the growth condition as in proposition 2.6 above and $\alpha(x)<p^*(x)$, then $\bar{N}_f:E\longrightarrow E^*$, where $\bar{N}_f(u)v=\int_\Omega f(x,u(x))v(x)dx$ is strongly continuous.*

Proof: The embedding $E\hookrightarrow L^{\alpha(x)}(\Omega)$ is compact, hence the diagram

$$E\overset{I}{\hookrightarrow}L^{\alpha(x)}(\Omega)\overset{N_f}{\longrightarrow}L^{\beta(x)}(\Omega)\overset{I^*}{\hookrightarrow}E^*$$

shows that $\bar{N}_f:E\longrightarrow E^*$ is strongly continuous. □

3. The main results

Let the functional Φ defined by:

$$\Phi:\quad E\longrightarrow\mathbb{R}$$

$$\Phi(u)=\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx-\int_\Omega F(x,u(x))\,dx.$$

Under assymption (H_1), the result from proposition 2.4 and proposition 2.6, show that Φ is a C^1 functional on E and

$$\Phi^{'}(u)=-\Delta_{p(x)}u-\bar{N}_f(u),\quad\forall u\in E.$$

It is obvious that $u\in E$ is a weak solution for problem (1.1) if and only if $\Phi^{'}(u)=0$. For that we will apply a mountain pass type argument to find nonzero critical point of Φ.
Our main result is given by the following theorem.

Theorem 3.1. *Assume (H_1), (H_2), (H_3) and (H_4) hold, then the problem (1.1) has a non trivial weak solution.*

Definition 3.2. *We say that a C^1 functional $I:E\longrightarrow\mathbb{R}$ satisfies the Palais-Smale condition (PS) if any sequence $(u_n)\subset E$ such that $(I(u_n))$ is bounded and $I^{'}(u_n)\longrightarrow 0$ has a convergent subsequence.*

Lemma 3.3. *Assume (H_1) and (H_2) hold, then the functional $\Phi:E\longrightarrow\mathbb{R}$ satisfies the (PS) condition.*

Proof: Let $(u_n)\subset E$ such that

$$|\Phi(u_n)|\le d\ \text{ for some } d\in\mathbb{R}\text{ and }\Phi^{'}(u_n)\longrightarrow 0.\tag{3.1}$$

We will show that $(u_n)_{n\in\mathbb{N}}$ is bounded in E.
Arguing by contradiction and passing to a subsequence, we have $||u_n||\to+\infty$. Using (3.1) it follows that for n large enough, we have

$$|\Phi'(u_n)u_n-\Phi(u_n)|\le d+||u_n||\quad(d\in\mathbb{R}).$$

So, we obtain

$$\int_\Omega |\nabla u_n|^{p(x)}dx - \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_\Omega F(x,u_n(x)) - u_n f(x,u_n(x))dx \leq d + ||u_n||,$$

hence

$$\int_\Omega (1-\frac{1}{p(x)})|\nabla u_n|^{p(x)}dx + \int_\Omega F(x,u_n(x)) - u_n f(x,u_n(x))dx \leq d + ||u_n||.$$

The above inequalities combined with (H_2) and proposition 2.1, yields:

$$(1-\frac{1}{p^-})||u_n||^{p^-} - B - \beta||u_n||^\eta \leq d + ||u_n|| \quad (B = \int_\Omega b(x)\,dx \in \mathbb{R}). \qquad (3.2)$$

passing to the limit as $n \to +\infty$, taking account that, $1 < p^-$ and $\eta < p^-$, we obtain a contradiction, so (u_n) is bounded, hence, up to a subsequence we may assume that $u_n \rightharpoonup u$.
Let $J = \bar{J}/E :\ J(u) = \int_\Omega F(x,u(x))dx$, $J' :\ E \to E^*$ is completely continuous (see [4]), since $u_n \rightharpoonup u$, we have

$$J'(u_n) \to J'(u).$$

In other hand

$$\Phi'(u_n) = -\Delta_{p(x)}(u_n) - J'(u_n) \to 0.$$

So

$$-\Delta_{p(x)}(u_n) \to J'(u).$$

Since $-\Delta_{p(x)}$ is of type (S_+), we deduce that $u_n \to u$, and so Φ satisfies (PS) condition. □

We will show that Φ satisfies conditions of Mountain Pass lemma.

Lemma 3.4. *Assume (H_1) and (H_3), then there exist $\rho > 0$ and $\delta > 0$ such that $\Phi(u) \geq \delta > 0$ for every $u \in E$ and $||u|| = \rho$.*

Proof: From the embedding $E \hookrightarrow L^{p^+}(\Omega)$, there exists $C_0 > 0$ such that:

$$|u|_{p^+} \leq C_0||u|| \quad \forall u \in E.$$

Let $\epsilon > 0$ be small enough such that $\epsilon C_0^{p^+} \leq \frac{1}{2p^+}$.
The assumptions (H_1) and (H_3) gives:

$$F(x,s) \leq \epsilon|s|^{p^+} + C(\epsilon)|s|^{\alpha(x)} \quad \forall (x,s) \in \Omega \times \mathbb{R}.$$

Without loss of generality, we assume $p^+ < \alpha^-$ (via (H_1)), hence for $||u|| < 1$ we have:

$$\begin{aligned}
\Phi(u) &\geq \int_\Omega \frac{1}{p^+}|\nabla u|^{p(x)}dx - \epsilon\int_\Omega |u|^{p^+}dx - C(\epsilon)\int_\Omega |u|^{\alpha(x)}dx \\
&\geq \frac{1}{p^+}||u||^{p^+} - \epsilon C_0^{P^+}||u||^{p^+} - kC(\epsilon)||u||^{\alpha_-} \quad (||u||_{\alpha(x)} \leq k||u||) \\
&\geq \frac{1}{2p^+}||u||^{p^+} - kC(\epsilon)||u||^{\alpha_-} \\
&\geq [\frac{1}{2p^+} - kC(\epsilon)||u||^{\alpha_- - p^+}]||u||^{p^+}.
\end{aligned}$$

So the proof is complete. □

Lemma 3.5. *Assume* (H_1) *and* (H_4)*, then, there exist* $e \in E$ *such that* $||e|| > 0$ *and* $\Phi(e) < 0$.

Proof: Let $\varphi \in C_0^\infty(\Omega)$ such that $supp\,\varphi \subset \Omega_1$, and $||\varphi|| > 0$.
For $t > 1$, we have:

$$\begin{aligned}
\Phi(t\varphi) &= \int_\Omega \frac{1}{p(x)}|\nabla t\varphi|^{p(x)}dx - \int_\Omega F(x, t\varphi(x))dx \\
&\leq t^{p^+}\int_\Omega \frac{1}{p(x)}|\nabla\varphi|^{p(x)}dx - \gamma t^\theta\int_{\Omega_1}|\varphi|^\theta dx + b^{'}t^r\int_{\Omega_1}|\varphi|^r dx - C^{'} \quad (C^{'} \in \mathbb{R}).
\end{aligned}$$

Since $\sup(r, p^+) < \theta$, $\int_{\Omega_1}|\varphi|^\theta dx > 0$ and $\gamma > 0$, the inequality above implies $\Phi(t\varphi) \to -\infty$ as $t \to +\infty$, hence the proof is complete. □

Proof of theorem 3.1. To prove theorem 3.1, we will apply the Mountain Pass theorem of Ambrosetti- Rabinowitz, taking e as given in lemma 3.5, and ρ as follow in lemma 3.4.

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