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More on the subconstituents of symplectic graphs

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ABSTRACT: In this paper, we are going to study the subconstituents of the subconstituents of symplectic graphs, in order to find some strongly regular and strictly Deza subgraphs of symplectic graphs.

Key Words: Symplectic graphs, strongly regular graphs, subconstituents of graphs, strictly Deza graphs, chromatic number, d-Deza graphs.

Contents

1. Introduction

Let $\mathbb{F}_q$ be a finite field and $\nu \geq 1$ an integer. Let

$$\mathbb{F}_q^{(2\nu)} = \{(a_1, \dots, a_{2\nu}) \ : \ a_1, \dots, a_{2\nu} \in \mathbb{F}_q\},$$

be the 2ν-dimensional row vector space over $\mathbb{F}_q$. If $0 \neq \alpha \in \mathbb{F}_q^{(2\nu)}$, then $[\alpha]$ denotes an one dimensional subspace of $\mathbb{F}_q^{(2\nu)}$. So obviously, for $k \in \mathbb{F}_q^{\times}$, $[\alpha] = [k\alpha]$. When $\alpha = (a_1, \dots, a_{2\nu})$, we also write $[\alpha] = [a_1, \dots, a_{2\nu}]$. Denote by ${}^tA$, the transpose of the matrix A. Let

$$K = \begin{pmatrix} \mathbf{0} & I^{(\nu)} \\ -I^{(\nu)} & \mathbf{0} \end{pmatrix}. \tag{1.1}$$

The symplectic graph $Sp(2\nu, q)$ relative to K over $\mathbb{F}_q$ is the graph with the set of one dimensional subspaces of $\mathbb{F}_q^{(2\nu)}$ as its vertex set and with the adjacency defined by

$$[\alpha] \sim [\beta] \text{ if and only if } \alpha K^t\beta \neq 0, \text{ for any } 1-\text{dimensional subspaces } [\alpha], [\beta].$$

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The symplectic graphs have been studied in [3,4,7,8,9], as one of strongly regular graphs constructed by Chevally groups.

Note that the diameter of $Sp(2\nu, q)$ is 2 when $\nu \geq 2$. For any $[\alpha] \in V(Sp(2\nu, q))$, the i-th subconstituent $\Gamma_i([\alpha])$ with respect to $[\alpha]$ is the induced subgraph of $Sp(2\nu, q)$ with vertices at distance i from $[\alpha]$, where $i = 1, 2$. In [6], authors show that the subconstituents of the symplectic graph $Sp(2\nu, q)$ are strictly Deza graphs except the case when $\nu = 2$.

In this paper, we are going to study the subconstituents of the subconstituents of the symplectic graph and their chromatic numbers in the case that they are regular.

2. Notations and preliminary results

Let G and H be two graphs. The *lexicographic product* $G[H]$ of G and H is a graph with the vertex set $V(G) \times V(H)$ and with the adjacency defined by

$$(u_1, u_2) \sim (v_1, v_2) \text{ if and only if } u_1 \sim v_1 \text{ or } u_1 = v_1 \text{ and } u_2 \sim v_2,$$

for any $u_1, v_1 \in V(G)$ and $u_2, v_2 \in V(H)$. A graph G is said to be n-partite if there are subsets $X_1, X_2, \ldots, X_n$ of the vertex set $V(G)$ such that there is no edge of G joining two vertices of the same subset and, $V(G) = X_1 \cup X_2 \cup \ldots \cup X_n$ and for all $i \neq j$, $X_i \cap X_j = \emptyset$. The chromatic number $\chi(G)$ of G is the minimal number n such that G is n-partite. For a graph G and $x \in G$, let $N_G(x)$ denote the set of neighbors of x in G. A simple connected graph G is called strongly regular graph with parameters (ν, k, λ, μ) if it consists ν vertices such that for every $x, y \in V(G)$,

$$|N_G(x) \cap N_G(y)| = \begin{cases} k & \text{if } x = y \\ \lambda & \text{if } x \sim y \\ \mu & \text{if } x \not\sim y \end{cases}.$$

In [9], the authors prove that:

Lemma 2.1. *The symplectic graph $Sp(2\nu, q)$ is strongly regular with parameters*

$$((q^{2\nu} - 1)/(q - 1), q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1))$$

and the chromatic number $q^\nu + 1$.

Let $0 \leq a \leq b \leq k \leq n$. A (n, k, b, a)-*Deza graph* G, which is introduced by Antoine and Michel Deza [1], is a graph with $|V(G)| = n$ such that for any $x, y \in V(G)$,

$$|N_G(x) \cap N_G(y)| = \begin{cases} a \text{ or } b & \text{if } x \neq y \\ k & \text{if } x = y \end{cases}.$$

Clearly, strongly regular graphs are Deza graphs. A *strictly Deza graph* is a Deza graph that is not strongly regular and has two diameters (see [2]). In [6], authors proved that the subconstituents of the symplectic graph $Sp(2\nu, q)$ are strictly Deza graph except in the case that $\nu = 2$. Recall that the symplectic group of degree

2ν over $\mathbb{F}_q$ with respect to K, which K is defined as in (1.1), denoted by $Sp_{2\nu}(\mathbb{F}_q)$ consists of all $2\nu \times 2\nu$ matrices T over $\mathbb{F}_q$ satisfying $TK^tT = K$. The proof of the following lemma is straightforward:

Lemma 2.2. (i) *$Sp_{2\nu}(\mathbb{F}_q) \leq \mathrm{Aut}(Sp(2\nu, q))$.*

(ii) *If $[v] \in V(Sp(2\nu, q))$ and $T \in \mathrm{Aut}(Sp(2\nu, q))$ such that $([v])T = [v]$, then $T \in \mathrm{Aut}(\Gamma_i[v])$.*

For $1 \leq i \leq 2\nu$, let e_i denote 2ν-dimensional row vector whose i-th entry is 1 and all other entries are zero. Since $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on $V(Sp(2\nu, q))$ (see [10]), we deduce by Lemma 2.2(i) that $\mathrm{Aut}(Sp(2\nu, q))$ acts transitively on $V(Sp(2\nu, q))$. Also, the diameter of $Sp(2\nu, q)$ is 2 when $\nu \geq 2$. Thus for studying the subconstituents of the symplectic graph, it is enough to study $\Gamma_i[e_1]$, where $i \in \{1, 2\}$. From [10], we obtain the following lemma:

Lemma 2.3. *For any two distinct $[\alpha], [\beta] \in V(Sp(2\nu, q))$, we have the following:*

(i) *If $[\alpha] \not\sim [\beta]$, then there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $[\alpha T] = [e_1]$ and $[\beta T] = [e_2]$;*

(ii) *if $[\alpha] \sim [\beta]$, then there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $[\alpha T] = [e_1]$ and $[\beta T] = [e_{\nu+1}]$.*

Let $i \in \{1, 2\}$ and $\nu \geq 2$. By Lemma 2.3, we can see at once that the stabilizer subgroup of $Sp_{2\nu}(\mathbb{F}_q)$ of $[e_1]$ acts transitively on $V(\Gamma_i[e_1])$. Thus Lemma 2.2(i) shows that $\mathrm{Aut}(\Gamma_i[e_1])$ acts transitively on $V(\Gamma_i[e_1])$. Also, the diameter of $V(\Gamma_i[e_1])$ is 2, $[e_{\nu+1}] \in \Gamma_1[e_1]$ and $[e_2] \in \Gamma_2[e_1]$. Thus for studying the subconstituents of $\Gamma_i[e_1]$, it is enough to study $(\Gamma_1[e_1])_j[e_{\nu+1}]$ and $(\Gamma_2[e_1])_j[e_2]$, where $j \in \{1, 2\}$ (the j-th subconstituent $(\Gamma_i[e_1])_j[\alpha]$ with respect to $[\alpha]$ is the induced subgraph of $\Gamma_i[e_1]$ with vertices in $V(\Gamma_i[e_1])$ at distance j from $[\alpha]$, where $j = 1, 2$.). For simplicity of notation, we write $\Gamma^{(1,j)}$ and $\Gamma^{(2,j)}$ instead of $(\Gamma_1[e_1])_j[e_{\nu+1}]$ and $(\Gamma_2[e_1])_j[e_2]$, respectively.

A trivial verification shows that $Sp(2, q)$ is a complete graph, so $\Gamma_1[e_1]$ is a clique and $V(\Gamma_2[e_1]) = \emptyset$. Therefore, in this paper we just consider the case when $\nu \geq 2$.

We collect here some basic properties of the natural action of the symplectic group $Sp_{2\nu}(\mathbb{F}_q)$ on the symplectic graph $Sp(2\nu, q)$:

Lemma 2.4. *If $[v], [w] \in V(Sp(2\nu, q))$ and $T \in \mathrm{Aut}(Sp(2\nu, q))$ such that $[w] \in \Gamma_i([v])$, $([v])T = [v]$ and $([w])T = [w]$, then $T \in \mathrm{Aut}(\Gamma^{(i,j)})$.*

Proof: The proof is straightforward. □

Lemma 2.5. [6, Propositions 2.2-2.5] *For any $[\alpha], [\beta], [\gamma] \in V(Sp(2\nu, q))$, we have the following:*

(i) *If* $[\alpha],[\beta]$ *and* $[\gamma]$ *are adjacent to each other, then there exists an element* $T\in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T]=[e_1]$, $[\beta T]=[e_{\nu+1}]$, *and* $[\gamma T]$ *is one of the following forms*

$$[e_1+a_{\nu+1}e_{\nu+1}],\ [e_1+e_2+a_{\nu+1}e_{\nu+1}],\ [e_1+a_{\nu+1}e_{\nu+1}+e_{\nu+2}], \tag{2.1}$$

where $a_{\nu+1}\in\mathbb{F}_q^{\times}$*;*

(ii) *if* $[\alpha]\sim[\beta]$, $[\alpha]\sim[\gamma]$ *and* $[\beta]\not\sim[\gamma]$, *then there exists an element* $T\in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T]=[e_1]$, $[\beta T]=[e_{\nu+1}]$, *and* $[\gamma T]$ *is* $[e_2+e_{\nu+1}]$ *or* $[e_{\nu+1}+e_{\nu+2}]$*;*

(iii) *if* $[\alpha]\not\sim[\beta]$, $[\alpha]\not\sim[\gamma]$ *and* $[\beta]\sim[\gamma]$, *then there exists an element* $T\in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T]=[e_1]$, $[\beta T]=[e_2]$, *and* $[\gamma T]$ *is* $[e_{\nu+2}]$*;*

(iv) *if* $[\alpha],[\beta]$ *and* $[\gamma]$ *are nonadjacent to each other, then there exists an element* $T\in Sp_{2\nu}(\mathbb{F}_q)$ *such that* $[\alpha T]=[e_1]$, $[\beta T]=[e_2]$, *and* $[\gamma T]$ *is one of the following forms*

$$[e_1+e_2],\ [e_3],\ [e_{\nu+3}], \tag{2.2}$$

in which the latter two cases occur only when $\nu\geq 3$.

3. The subconstituent $\Gamma^{(1,1)}$

Let $[\theta]=[\theta_1,\ldots,\theta_{2\nu}]$. If $[\theta]\in V(\Gamma^{(1,1)})$, then $[\theta]\sim[e_1]$ and $[\theta]\sim[e_{\nu+1}]$. Thus $\theta K^t e_1\neq 0$ and $\theta K^t e_{\nu+1}\neq 0$. This shows that $\theta_{\nu+1}\neq 0$ and $\theta_1\neq 0$. Therefore, we can assume that $[\theta]=[1,\theta_2,\ldots,\theta_{2\nu}]$ such that $\theta_{\nu+1}\in\mathbb{F}_q^{\times}$.

Theorem 3.1. $\Gamma^{(1,1)}$ *is not regular.*

Proof: Let $[\gamma]\in V(\Gamma^{(1,1)})$. Then $[\gamma]\sim[e_1]$ and $[\gamma]\sim[e_{\nu+1}]$. Since $[e_1]\sim[e_{\nu+1}]$, Lemma 2.5(i) shows that there exists $T\in Sp_{2\nu}(\mathbb{F}_q)$ such that $[e_1]T=[e_1]$, $[e_{\nu+1}]T=[e_{\nu+1}]$ and $[\gamma]T$ is one of the forms in (2.1). Thus by Lemma 2.4, $T\in \mathrm{Aut}(\Gamma^{(1,1)})$ and hence, $\deg_{\Gamma^{(1,1)}}([\gamma])=\deg_{\Gamma^{(1,1)}}([\gamma T])$. We are going to consider the different forms in (2.1), in the following cases:

(i) Let $[\gamma T]=[e_1+a_{\nu+1}e_{\nu+1}]$, where $a_{\nu+1}\in\mathbb{F}_q^{\times}$. Then since

$$\begin{aligned}
N_{\Gamma^{(1,1)}}([e_1+a_{\nu+1}e_{\nu+1}]) &= \{[\theta]\in\Gamma^{(1,1)}\ :\ [\theta]\sim[e_1+a_{\nu+1}e_{\nu+1}]\}\\
&= \{[\theta]\in\Gamma^{(1,1)}\ :\ \theta K^t(e_1+a_{\nu+1}e_{\nu+1})\neq 0\}\\
&= \{[1,\theta_2,\ldots,\theta_{2\nu}]\ :\ \theta_{\nu+1}\in\mathbb{F}_q^{\times},\ \theta_{\nu+1}-a_{\nu+1}\neq 0\},
\end{aligned}$$

$\deg_{\Gamma^{(1,1)}}([\gamma])=q^{2(\nu-1)}(q-2)$.

(ii) Let $[\gamma T]=[e_1+e_2+a_{\nu+1}e_{\nu+1}]$, where $a_{\nu+1}\in\mathbb{F}_q^{\times}$. Then since

$$\begin{aligned}
N_{\Gamma^{(1,1)}}([e_1+e_2+a_{\nu+1}e_{\nu+1}]) &= \{[\theta]\in\Gamma^{(1,1)}\ :\ [\theta]\sim[e_1+e_2+a_{\nu+1}e_{\nu+1}]\}\\
&= \{[\theta]\in\Gamma^{(1,1)}\ :\ \theta K^t(e_1+e_2+a_{\nu+1}e_{\nu+1})\neq 0\}\\
&= \{[1,\theta_2,\ldots,\theta_{2\nu}]\ :\\
&\qquad \theta_{\nu+1}\in\mathbb{F}_q^{\times},\ \theta_{\nu+1}+\theta_{\nu+2}-a_{\nu+1}\neq 0\},
\end{aligned}$$

$\deg_{\Gamma^{(1,1)}}([\gamma]) = q^{2\nu-3}(q-1)^2$.

(iii) Let $[\gamma T] = [e_1 + a_{\nu+1}e_{\nu+1} + e_{\nu+2}]$, where $a_{\nu+1} \in \mathbb{F}_q{}^{\times}$. Then since $N_{\Gamma^{(1,1)}}([\gamma T]) = \{[1, \theta_2, \ldots, \theta_{2\nu}] \ : \ \theta_{\nu+1} \in \mathbb{F}_q{}^{\times}, \ \theta_{\nu+1} - a_{\nu+1} - \theta_2 \neq 0\}$, $\deg_{\Gamma^{(1,1)}}([\gamma]) = q^{2\nu-3}(q-1)^2$.

This shows that $\Gamma^{(1,1)}$ is not a regular graph. □

4. The subconstituent $\Gamma^{(1,2)}$

Let $[\theta] = [\theta_1, \ldots, \theta_{2\nu}]$. If $[\theta] \in V(\Gamma^{(1,2)})$, then $[\theta] \sim [e_1]$ and $[\theta] \not\sim [e_{\nu+1}]$. Thus $\theta K^t e_1 \neq 0$ and $\theta K^t e_{\nu+1} = 0$. This shows that $\theta_{\nu+1} \neq 0$ and $\theta_1 = 0$. Also, $e_{\nu+1} \notin V(\Gamma^{(1,2)})$. Therefore,

$$V(\Gamma^{(1,2)}) = \{[0, \theta_2, \ldots, \theta_\nu, 1, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \ : \ [\theta_2, \ldots, \theta_\nu, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \neq 0\}. \quad (4.1)$$

Proposition 4.1. $|V(\Gamma^{(1,2)})| = q^{2(\nu-1)} - 1$.

Proof: It follows immediately from (4.1). □

Proposition 4.2. $\Gamma^{(1,2)}$ *is a* $q^{2\nu-3}(q-1)$*-regular graph.*

Proof: Let $[\gamma] \in V(\Gamma^{(1,2)})$. By (4.1), $[\gamma] = [0, \gamma_2, \ldots, \gamma_\nu, 1, \gamma_{\nu+1}, \ldots, \gamma_{2\nu}]$ and

$$\begin{aligned} N_{\Gamma^{(1,2)}}([\gamma]) &= \{[\theta] \in \Gamma^{(1,2)} \ : \ [\theta] \sim [\gamma]\} \\ &= \{[\theta] = [0, \theta_2, \ldots, \theta_\nu, 1, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \ : \ \theta K^t \gamma \neq 0\} \\ &= \{[0, \theta_2, \ldots, \theta_\nu, 1, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \ : \\ &\qquad (\theta_2, \ldots, \theta_\nu, \theta_{\nu+2}, \ldots, \theta_{2\nu}) K'^t \gamma' \neq 0\} \\ &= \{[0, \theta_2, \ldots, \theta_\nu, 1, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \ : \\ &\qquad [\theta_2, \ldots, \theta_\nu, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \in N_{Sp(2(\nu-1),q)}([\gamma'])\}, \end{aligned}$$

where $\gamma' = (\gamma_2, \ldots, \gamma_\nu, \gamma_{\nu+2}, \ldots, \gamma_{2\nu})$ and $K' = \begin{pmatrix} 0 & I^{(\nu-1)} \\ -I^{(\nu-1)} & 0 \end{pmatrix}$. Thus by Lemma 2.1,

$$\deg_{\Gamma^{(1,2)}}([\gamma]) = (q-1)\deg_{Sp(2(\nu-1),q)}([\gamma_2, \ldots, \gamma_\nu, \gamma_{\nu+2}, \ldots, \gamma_{2\nu}]) = (q-1)q^{2\nu-3}.$$

□

Proposition 4.3. *If* $[\alpha], [\beta] \in V(\Gamma^{(1,2)})$ *such that* $[\alpha] \sim [\beta]$*, then*

$$|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = (q-1)^2 q^{2(\nu-2)}.$$

Proof: By (4.1), $[\alpha] = [0, \alpha_2, \ldots, \alpha_\nu, 1, \alpha_{\nu+2}, \ldots, \alpha_{2\nu}]$ and $[\beta] = [0, \beta_2, \ldots, \beta_\nu, 1, \beta_{\nu+2}, \ldots, \beta_{2\nu}]$. Put $[\alpha'] = [\alpha_2, \ldots, \alpha_\nu, \alpha_{\nu+2}, \ldots, \alpha_{2\nu}]$ and $[\beta'] = [\beta_2, \ldots, \beta_\nu, \beta_{\nu+2}, \ldots, \beta_{2\nu}]$. Obviously $[\alpha] \sim [\beta]$ if and only if $[\alpha']$ and $[\beta']$ are adjacent in $Sp(2(\nu -$

$1),q)$. Thus by Lemma 2.3(ii), there exists $T' \in Sp_{2(\nu-1)}(\mathbb{F}_q)$ such that $[\alpha' T'] = [1, 0, \dots, 0]$ and $[\beta' T'] = [0, \dots, 0, \underbrace{1}_{\text{entry } \nu}, 0, \dots, 0]$. Let $T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_{\nu-1}(\mathbb{F}_q)$, the set of $(\nu-1) \times (\nu-1)$ matrices over $\mathbb{F}_q$. Then

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in Sp_{2\nu}(\mathbb{F}_q)$$

and, $[e_1]T = [e_1]$, $[e_{\nu+1}]T = [e_{\nu+1}]$, $[\alpha]T = [\lambda e_{\nu+1} + e_2]$ and $[\beta]T = [\lambda' e_{\nu+1} + e_{\nu+2}]$, where $\lambda, \lambda' \in \mathbb{F}_q^{\times}$. Thus $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = |N_{\Gamma^{(1,2)}}([\lambda e_{\nu+1} + e_2]) \cap N_{\Gamma^{(1,2)}}([\lambda' e_{\nu+1} + e_{\nu+2}])|$. Also, by (4.1), $N_{\Gamma^{(1,2)}}([\lambda e_{\nu+1} + e_2]) \cap N_{\Gamma^{(1,2)}}([\lambda' e_{\nu+1} + e_{\nu+2}])$ is

$$\begin{aligned} &\{[\theta] \in \Gamma^{(1,2)} \ : \ [\theta] \sim [\lambda e_{\nu+1} + e_2],\ [\theta] \sim [\lambda' e_{\nu+1} + e_{\nu+2}]\} = \\ &\{[\theta] \in \Gamma^{(1,2)} \ : \ \theta K^t(\lambda e_{\nu+1} + e_2) \neq 0,\ \theta K^t(\lambda' e_{\nu+1} + e_{\nu+2}) \neq 0\} = \\ &\{[0, \theta_2, \dots, \theta_\nu, 1, \theta_{\nu+2}, \dots, \theta_{2\nu}] \ : \ \theta_{\nu+2} \neq 0,\ \theta_2 \neq 0\}. \end{aligned}$$

This gives that $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = (q-1)^2 q^{2(\nu-2)}$. □

Proposition 4.4. *If $(q, \nu) \neq (2, 2)$ and $[\alpha], [\beta] \in V(\Gamma^{(1,2)})$ such that $[\alpha] \not\sim [\beta]$, then*

$$|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = (q-1)^2 q^{2(\nu-2)} \text{ or } (q-1) q^{2(\nu-3)}.$$

In particular, if $\nu = 2$, then $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = (q-1) q^{2(\nu-3)}$ and if $q = 2$, then $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = 2^{2(\nu-2)}$.

Proof: By (4.1), $[\alpha] = [0, \alpha_2, \dots, \alpha_\nu, 1, \alpha_{\nu+2}, \dots, \alpha_{2\nu}]$ and $[\beta] = [0, \beta_2, \dots, \beta_\nu, 1, \beta_{\nu+2}, \dots, \beta_{2\nu}]$. Put $\alpha' = (\alpha_2, \dots, \alpha_\nu, \alpha_{\nu+2}, \dots, \alpha_{2\nu})$ and $\beta' = (\beta_2, \dots, \beta_\nu, \beta_{\nu+2}, \dots, \beta_{2\nu})$. Obviously $[\alpha] \not\sim [\beta]$ if and only if either $\nu \geq 3$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-1), q)$ or $q \neq 2$ and $[\alpha'] = [\beta']$ such that $\alpha' \neq \beta'$. We continue the proof in the following cases:

(i) If $\nu \geq 3$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-1), q)$, then by Lemma 2.3(i), there exists $T' \in Sp_{2(\nu-1)}(\mathbb{F}_q)$ such that $[\alpha' T'] = [1, \dots, 0]$ and $[\beta' T'] = [0, 1, 0, \dots, 0]$. Let $T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_{\nu-1}(\mathbb{F}_q)$. Then $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in Sp_{2\nu}(\mathbb{F}_q)$ and, $[e_1]T = [e_1]$, $[e_{\nu+1}]T = [e_{\nu+1}]$, $[\alpha]T = [\lambda e_2 + e_{\nu+1}]$ and $[\beta]T = [\lambda' e_3 + e_{\nu+1}]$, where $\lambda, \lambda' \in \mathbb{F}_q^{\times}$. Thus $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = |N_{\Gamma^{(1,2)}}([\lambda e_2 + e_{\nu+1}]) \cap N_{\Gamma^{(1,2)}}([\lambda' e_3 + e_{\nu+1}])|$. Also

$$\begin{aligned} &N_{\Gamma^{(1,2)}}([\lambda e_2 + e_{\nu+1}]) \cap N_{\Gamma^{(1,2)}}([\lambda' e_3 + e_{\nu+1}]) = \\ &\{[\theta] \in \Gamma^{(1,2)} \ : \ [\theta] \sim [\lambda e_2 + e_{\nu+1}], [\theta] \sim [\lambda' e_3 + e_{\nu+1}]\} = \\ &\{[0, \theta_2, \dots, \theta_\nu, 1, \theta_{\nu+2}, \dots, \theta_{2\nu}] \ : \ \theta_{\nu+2} \neq 0, \theta_{\nu+3} \neq 0\}. \end{aligned}$$

Thus $|N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta])| = (q-1)^2 q^{2(\nu-2)}$.

(ii) If $q \neq 2$ and $[\alpha'] = [\beta']$ such that $\alpha' \neq \beta'$, then it is easy to see that $N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta]) = N_{\Gamma^{(1,2)}}([\alpha])$ and hence $N_{\Gamma^{(1,2)}}([\alpha]) \cap N_{\Gamma^{(1,2)}}([\beta]) = (q-1)q^{2\nu-3}$.

□

Theorem 4.5. (i) *If $\nu \geq 3$ and $q \neq 2$, then $\Gamma^{(1,2)}$ is a strictly Deza graph with parameters*

$$(q^{2(\nu-1)} - 1, q^{2\nu-3}(q-1), q^{2\nu-3}(q-1), q^{2(\nu-2)}(q-1)^2).$$

(ii) *If $\nu = 2$ and $q \neq 2$, then $\Gamma^{(1,2)}$ is a strongly regular graph with parameters*

$$(q^2 - 1, q(q-1), (q-1)^2, q(q-1)).$$

(iii) *If $\nu \geq 3$ and $q = 2$, then $\Gamma^{(1,2)}$ is a strongly regular graph with parameters*

$$(2^{2(\nu-1)} - 1, 2^{2\nu-3}, 2^{2(\nu-2)}, 2^{2(\nu-2)}).$$

More precisely, in this case, $\Gamma^{(1,2)}$ is isomorphic to $Sp_{2(\nu-1)}(2)$.

(iii) *If $\nu = 2$ and $q = 2$, then $\Gamma^{(1,2)}$ is a complete graph with 3 vertices.*

Proof: It follows immediately from Propositions 4.1, 4.2, 4.3 and 4.4. □

For a natural number m define the function

$$f_m : (\mathbb{F}_q)^{2m} - \{0\} \longrightarrow \mathbb{F}_q^{\times} \tag{4.2}$$

which maps $(a_1, \ldots, a_{2m})$ to a_j, where $1 \leq j \leq 2m$, $a_j \neq 0$ and for every natural number $i < j$, $a_i = 0$. Note that $\mathbb{F}_q^{\times}$ is a clique with $\mathbb{F}_q^{\times}$ as its vertex set and the adjacency defined by $x \sim y$ if and only if $x \neq y$. Then the complement $\overline{\mathbb{F}_q^{\times}}$ of $\mathbb{F}_q^{\times}$ is a coclique.

Theorem 4.6. *$\Gamma^{(1,2)}$ is isomorphic to $Sp(2(\nu-1), q)[\overline{\mathbb{F}_q^{\times}}]$.*

Proof: Define

$$\begin{aligned} \phi : \Gamma^{(1,2)} &\longrightarrow Sp(2(\nu-1), q)[\overline{\mathbb{F}_q^{\times}}] \\ [0, \alpha_2, \ldots, \alpha_\nu, 1, \alpha_{\nu+2}, \ldots, \alpha_{2\nu}] &\longmapsto ([\alpha_2, \ldots, \alpha_\nu, \alpha_{\nu+2}, \ldots, \alpha_{2\nu}], \\ &\qquad f_{2(\nu-1)}(\alpha_2, \ldots, \alpha_\nu, \alpha_{\nu+2}, \ldots, \alpha_{2\nu})). \end{aligned}$$

We can check at once that ϕ is a graph isomorphism, so theorem follows. □

Theorem 4.7. $\chi(\Gamma^{(1,2)}) = q^{\nu-1} + 1$.

Proof: Theorem 4.6 shows that $\chi(\Gamma^{(1,2)}) = \chi(Sp(2(\nu-1), q))$, so Lemma 2.1 completes the proof. □

5. The subconstituent $\Gamma^{(2,1)}$

Let $[\theta] = [\theta_1, \ldots, \theta_{2\nu}]$. If $[\theta] \in V(\Gamma^{(2,1)})$, then $[\theta] \not\sim [e_1]$ and $[\theta] \sim [e_2]$. Thus $\theta K^t e_1 = 0$ and $\theta K^t e_2 \neq 0$. This forces $\theta_{\nu+1} = 0$ and $\theta_{\nu+2} \neq 0$ and hence,

$$V(\Gamma^{(2,1)}) = \{[\theta_1, \theta_2, \ldots, \theta_\nu, 0, 1, \theta_{\nu+3}, \ldots, \theta_{2\nu}] \ : \ \theta_1, \theta_2, \ldots, \theta_\nu, \theta_{\nu+3}, \ldots, \theta_{2\nu} \in \mathbb{F}_q\}. \tag{5.1}$$

Proposition 5.1. $|V(\Gamma^{(2,1)})| = q^{2(\nu-1)}$.

Proof: It follows immediately from (5.1). □

Proposition 5.2. $\Gamma^{(2,1)}$ *is a* $q^{2\nu-3}(q-1)$*-regular graph.*

Proof: Let $[\gamma] \in V(\Gamma^{(2,1)})$. By (5.1), $[\gamma] = [\gamma_1, \gamma_2, \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}]$. Put $[\gamma'] = [\gamma_2, \ldots, \gamma_\nu, 1, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}]$. It is obvious that $[\gamma']$ and $[1, 0, \ldots, 0]$ are adjacent in $Sp(2(\nu-1), q)$. Thus Lemma 2.3(ii) shows that there exists $T' \in Sp_{2(\nu-1)}(\mathbb{F}_q)$ such that $[1, 0, \ldots, 0]T' = [1, 0, \ldots, 0]$ and $[\gamma']T' = [0, \ldots, 0, \underbrace{1}_{\text{entry } \nu}, 0, \ldots, 0]$. Let $T' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, B, C, D \in M_{\nu-1}(\mathbb{F}_q)$. Then $T = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{smallmatrix}\right) \in Sp_{2\nu}(\mathbb{F}_q)$ and, $[e_1]T = [e_1]$, $[e_2]T = [e_2]$ and $[\gamma]T = [\lambda e_1 + e_{\nu+2}]$, for some $\lambda \in \mathbb{F}_q$. Thus $T \in \mathrm{Aut}(\Gamma^{(2,1)})$, so $\deg_{\Gamma^{(2,1)}}([\gamma]) = \deg_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}])$. But

$$\begin{aligned} N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) &= \{[\theta] \in \Gamma^{(2,1)} \ : \ [\theta] \sim [\lambda e_1 + e_{\nu+2}]\} \\ &= \{[\theta] = [\theta_1, \ldots, \theta_\nu, 0, 1, \theta_{\nu+2}, \ldots, \theta_{2\nu}] \ : \\ &\quad\ \theta K^t(\lambda e_1 + e_{\nu+2}) \neq 0\} \\ &= \{[\theta_1, \ldots, \theta_\nu, 0, 1, \theta_{\nu+3}, \ldots, \theta_{2\nu}] \ : \ \theta_2 \neq 0\}. \end{aligned}$$

Thus $\deg_{\Gamma^{(2,1)}}([\gamma]) = q^{2\nu-3}(q-1)$. □

Proposition 5.3. *If* $[\alpha], [\beta] \in V(\Gamma^{(2,1)})$ *such that* $[\alpha] \sim [\beta]$*, then*

$$|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-1)^2 q^{2(\nu-2)} \text{ or } (q-2)q^{2\nu-3}.$$

In particular, if $\nu = 2$*, then* $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = q(q-2)$.

Proof: By (5.1), $[\alpha] = [\alpha_1 \ldots, \alpha_\nu, 0, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta] = [\beta_1, \ldots, \beta_\nu, 0, 1, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Put $[\alpha'] = [\alpha_2, \ldots, \alpha_\nu, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta'] = [\beta_2, \ldots, \beta_\nu, 1, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Obviously $[\alpha] \sim [\beta]$ if and only if $[\alpha']$ and $[\beta']$ are adjacent in $Sp(2(\nu-1), q)$. Thus since $[\alpha'] \sim [1, 0, \ldots, 0]$ and $[\beta'] \sim [1, 0, \ldots, 0]$, Lemma 2.5(i) shows that there exists $T' \in Sp_{2(\nu-1)}(\mathbb{F}_q)$ such that $[1,0,\ldots,0]T' = [1,0,\ldots,0]$, $[\alpha']T' = [0,\ldots,0,\underbrace{1}_{\text{entry }\nu},0,\ldots,0]$ and

$$[\beta']T' = [1,0,\ldots,0,\underbrace{a}_{\text{entry }\nu},0,\ldots,0],$$

$$[1,1,0\ldots,0,\underbrace{a}_{\text{entry }\nu},0,\ldots,0] \quad \text{or} \quad [1,0\ldots,0,\underbrace{a}_{\text{entry }\nu},1,0,\ldots,0],$$

where $a \in \mathbb{F}_q^\times$ and the latter two cases occur only when $\nu \geq 3$. Let $T' = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, B, C, D \in M_{\nu-1}(\mathbb{F}_q)$. Then $T = \left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{smallmatrix}\right) \in Sp_{2\nu}(\mathbb{F}_q)$ and, $[e_1]T = [e_1]$, $[e_2]T = [e_2]$, $[\alpha]T = [\lambda e_1 + e_{\nu+2}]$ and

$$[\beta]T = [\lambda' e_1 + e_2 + ae_{\nu+2}],\ [\lambda' e_1 + e_2 + e_3 + ae_{\nu+2}] \text{ or } [\lambda' e_1 + e_2 + ae_{\nu+2} + e_{\nu+3}],$$

where $\lambda, \lambda' \in \mathbb{F}_q$. Thus $T \in \mathrm{Aut}(\Gamma^{(2,1)})$, so $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = |N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + ae_{\nu+2}])|$, $|N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + e_3 + ae_{\nu+2}])|$ or $|N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + ae_{\nu+2} + e_{\nu+3}])|$. But

$$\begin{aligned} &N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + ae_{\nu+2}]) = \\ &\{[\gamma] \in \Gamma^{(2,1)} \ : \ [\gamma] \sim [\lambda e_1 + e_{\nu+2}],\ [\gamma] \sim [\lambda' e_1 + e_2 + ae_{\nu+2}]\} = \\ &\{[\gamma_1, \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, v_{2\nu}] : \gamma_2 \neq 0,\ a\gamma_2 - 1 \neq 0\}, \end{aligned}$$

$$\begin{aligned} &N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + e_3 + ae_{\nu+2}]) = \\ &\{[\gamma] \in \Gamma^{(2,1)} \ : \ [\gamma] \sim [\lambda e_1 + e_{\nu+2}],\ [\gamma] \sim [\lambda' e_1 + e_2 + e_3 + ae_{\nu+2}]\} = \\ &\{[\gamma_1, \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, v_{2\nu}] : \gamma_2 \neq 0,\ a\gamma_2 - 1 - \gamma_{\nu+3} \neq 0\} \end{aligned}$$

and

$$\begin{aligned} &N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_2 + ae_{\nu+2} + e_{\nu+3}]) = \\ &\{[\gamma] \in \Gamma^{(2,1)} \ : \ [\gamma] \sim [\lambda e_1 + e_{\nu+2}],\ [\gamma] \sim [\lambda' e_1 + e_2 + ae_{\nu+2} + e_{\nu+3}]\} = \\ &\{[\gamma_1, \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, v_{2\nu}] : \gamma_2 \neq 0,\ a\gamma_2 + \gamma_3 - 1 \neq 0\}. \end{aligned}$$

Thus $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-2)q^{2\nu-3}$ or $(q-1)^2 q^{2(\nu-2)}$. Also, if $\nu = 2$, then $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-2)q$. $\square$

Proposition 5.4. *If $[\alpha], [\beta] \in V(\Gamma^{(2,1)})$ such that $[\alpha] \not\sim [\beta]$, then*

$$|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-1)^2 q^{2(\nu-2)} \text{ or } (q-1)q^{2\nu-3}.$$

In particular, if $\nu = 2$, then $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = q(q-1)$.

Proof: By (5.1), $[\alpha] = [\alpha_1 \ldots, \alpha_\nu, 0, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta] = [\beta_1, \ldots, \beta_\nu, 0, 1, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Put $[\alpha'] = [\alpha_2, \ldots, \alpha_\nu, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta'] = [\beta_2, \ldots, \beta_\nu, 1, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Obviously $[\alpha] \not\sim [\beta]$ if and only if $[\alpha'] = [\beta']$ or $\nu \geq 3$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-1), q)$. We continue the proof in the following cases:

(i) If $[\alpha'] = [\beta']$, then $N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta]) = N_{\Gamma^{(2,1)}}([\alpha])$. Therefore, $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-1)q^{2\nu-3}$.

(ii) If $\nu \geq 3$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-1), q)$, then since $[\alpha'] \sim [1, 0, \ldots, 0]$ and $[\beta'] \sim [1, 0, \ldots, 0]$, Lemma 2.5(ii) shows that there exists $T' \in Sp_{2(\nu-1)}(\mathbb{F}_q)$ such that $[1, 0, \ldots, 0]T' = [1, 0, \ldots, 0]$, $[\alpha']T' = [0, \ldots, 0, \underbrace{1}_{\text{entry } \nu}, 0, \ldots, 0]$ and

$$[\beta']T' = [0, 1, 0 \ldots, 0, \underbrace{1}_{\text{entry } \nu}, 0, \ldots, 0] \text{ or } [0 \ldots, 0, \underbrace{1}_{\text{entry } \nu}, 1, 0, \ldots, 0].$$

Let $T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in M_{\nu-1}(\mathbb{F}_q)$. Then $T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \in Sp_{2\nu}(\mathbb{F}_q)$ and, $[e_1]T = [e_1]$, $[e_2]T = [e_2]$, $[\alpha]T = [\lambda e_1 + e_{\nu+2}]$ and

$$[\beta]T = [\lambda' e_1 + e_3 + e_{\nu+2}] \text{ or } [\lambda' e_1 + e_{\nu+2} + e_{\nu+3}],$$

where $\lambda, \lambda' \in \mathbb{F}_q$. Thus $T \in \mathrm{Aut}(\Gamma^{(2,1)})$, so $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = |N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_3 + e_{\nu+2}])|$ or $|N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_{\nu+2} + e_{\nu+3}])|$. But

$$\begin{aligned}
&N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_3 + e_{\nu+2}]) = \\
&\{[\gamma] \in \Gamma^{(2,1)} \ : \ [\gamma] \sim [\lambda e_1 + e_{\nu+2}],\ [\gamma] \sim [\lambda' e_1 + e_3 + e_{\nu+2}]\} = \\
&\{[\gamma] \in \Gamma^{(2,1)} \ : \ \gamma K^t(\lambda e_1 + e_{\nu+2}) \neq 0,\ \gamma K^t(\lambda' e_1 + e_3 + e_{\nu+2}) \neq 0\} = \\
&\{[\gamma_1 \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, v_{2\nu}] \ : \ \gamma_2 \neq 0,\ \gamma_2 - \gamma_{\nu+3} \neq 0\}
\end{aligned}$$

and

$$\begin{aligned}
&N_{\Gamma^{(2,1)}}([\lambda e_1 + e_{\nu+2}]) \cap N_{\Gamma^{(2,1)}}([\lambda' e_1 + e_{\nu+2} + e_{\nu+3}]) = \\
&\{[\gamma] \in \Gamma^{(2,1)} \ : \ [\gamma] \sim [\lambda e_1 + e_{\nu+2}],\ [\gamma] \sim [\lambda' e_1 + e_{\nu+2} + e_{\nu+3}]\} = \\
&\{[\gamma] \in \Gamma^{(2,1)} \ : \ \gamma K^t(\lambda e_1 + e_{\nu+2}) \neq 0,\ \gamma K^t(\lambda' e_1 + e_{\nu+2} + e_{\nu+3}) \neq 0\} = \\
&\{[\gamma_1 \ldots, \gamma_\nu, 0, 1, \gamma_{\nu+3}, \ldots, v_{2\nu}] : \gamma_2 \neq 0,\ \gamma_2 + \gamma_3 \neq 0\}.
\end{aligned}$$

Thus $|N_{\Gamma^{(2,1)}}([\alpha]) \cap N_{\Gamma^{(2,1)}}([\beta])| = (q-1)^2 q^{2(\nu-2)}$.

□

As a generalization of Deza graphs, the author in [5] gives the definition of d-Deza graphs. A k-regular graph G on n vertices is called a d-Deza graph with parameters $(n, k, \{c_1, \ldots, c_d\})$ if every two distinct vertices of G have $c_1, c_2, \ldots, c_d$ common adjacent vertices. In particular, the 2-Deza graph is just the ordinary Deza graph.

Theorem 5.5. (i) *If $\nu \geq 3$, then $\Gamma^{(2,1)}$ is a 3-Deza graph with parameters*

$$(q^{2(\nu-1)}, q^{2\nu-3}(q-1), \{q^{2\nu-3}(q-1), q^{2\nu-3}(q-2), (q-1)^2 q^{2(\nu-2)}\}).$$

(ii) *If $\nu = 2$, then $\Gamma^{(2,1)}$ is a strongly regular graph with parameters*

$$(q^2, q(q-1), q(q-2), q(q-1)).$$

Proof: It follows immediately from Propositions 5.1, 5.2, 5.3 and 5.4. □

Theorem 5.6. $\Gamma^{(2,1)}$ *is isomorphic to $\Gamma'_1[\overline{\mathbb{F}}_q]$, where Γ'_1 denotes the first subconstituent of $Sp(2(\nu-1), q)$ with respect to $[1, 0, \ldots, 0]$.*

Proof: Define

$$\begin{array}{rl} \phi : \Gamma^{(2,1)} & \longrightarrow \Gamma'_1[\overline{\mathbb{F}}_q] \\ [\alpha_1, \alpha_2, \ldots, \alpha_\nu, 0, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}] & \longmapsto ([\alpha_2, \ldots, \alpha_\nu, 1, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}], \alpha_1). \end{array}$$

We can check at once that ϕ is a graph isomorphism, so theorem follows. □

Theorem 5.7. $\chi(\Gamma^{(2,1)}) = q^{\nu-1}$.

Proof: Theorem 5.6 shows that $\chi(\Gamma^{(2,1)}) = \chi(\Gamma'_1)$, where Γ'_1 denotes the first subconstituent of $Sp(2(\nu-1), q)$ with respect to $[1, 0, \ldots, 0]$ and [6, Theorem 3.7] implies that $\chi(\Gamma'_1) = q^{\nu-1}$, so theorem follows. □

6. The subconstituent $\Gamma^{(2,2)}$

Let $[\theta] = [\theta_1, \ldots, \theta_{2\nu}]$. If $[\theta] \in V(\Gamma^{(2,2)})$, then $[\theta] \not\sim [e_1]$ and $[\theta] \not\sim [e_2]$. Thus $\theta K^t e_1 = 0$ and $\theta K^t e_2 = 0$. This forces $\theta_{\nu+1} = 0$ and $\theta_{\nu+2} = 0$ and hence,

$$V(\Gamma^{(2,2)}) = \{[\theta_1, \theta_2, \ldots, \theta_\nu, 0, 0, \theta_{\nu+3}, \ldots, \theta_{2\nu}] \ : \ \theta_1, \theta_2, \ldots, \theta_\nu, \theta_{\nu+3}, \ldots, \theta_{2\nu} \in \mathbb{F}_q\},$$

such that there exists $i \in \{1, \ldots, \nu, \nu+3, \ldots, 2\nu\}$ with $\theta_i \neq 0$. For $\nu \geq 3$, let $\Delta^{(2,2)}$ denote the induced subgraph on

$$\{[\theta_1, \theta_2, \ldots, \theta_\nu, 0, 0, \theta_{\nu+3}, \ldots, \theta_{2\nu}] \ : \ (\theta_3, \ldots, \theta_\nu, \theta_{\nu+3}, \ldots, \theta_{2\nu}) \neq 0\}. \tag{6.1}$$

Proposition 6.1. $|V(\Delta^{(2,2)})| = q^2(q^{2(\nu-2)} - 1)/(q-1)$.

Proof: It follows immediately from (6.1). □

Proposition 6.2. $\Delta^{(2,2)}$ *is a $q^{2\nu-3}$-regular graph.*

Proof: Let $[\gamma] \in V(\Delta^{(2,2)})$. By (6.1), $[\gamma] = [\gamma_1, \gamma_2, \ldots, \gamma_\nu, 0, 0, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}]$ such that

$$(\gamma_3, \ldots, \gamma_\nu, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}) \neq 0.$$

Put $[\gamma'] = [\gamma_3, \ldots, \gamma_\nu, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}]$. We can check at once that $N_{\Delta^{(2,2)}}([\gamma]) = \{[\theta] \in V(\Delta^{(2,2)}) : [\theta] \sim [\gamma]\} = \{[\theta_1, \theta_2, \ldots, \theta_\nu, 0, 0, \theta_{\nu+3}, \ldots, \theta_{2\nu}] : [\theta_3, \ldots, \theta_\nu, \theta_{\nu+3}, \ldots, \theta_{2\nu}] \in N_{Sp(2(\nu-2),q)}([\gamma'])\}$. Thus Lemma 2.1 shows that $\deg_{\Delta^{(2,2)}}([\gamma]) = q^2 q^{2\nu-5} = q^{2\nu-3}$. □

Proposition 6.3. *If $[\alpha], [\beta] \in V(\Delta^{(2,2)})$ such that $[\alpha] \sim [\beta]$, then $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^{2(\nu-2)}(q-1)$.*

Proof: By (6.1), $[\alpha] = [\alpha_1 \ldots, \alpha_\nu, 0, 0, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta] = [\beta_1, \ldots, \beta_\nu, 0, 0, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Put $[\alpha'] = [\alpha_3, \ldots, \alpha_\nu, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta'] = [\beta_3, \ldots, \beta_\nu, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Obviously $[\alpha] \sim [\beta]$ if and only if $[\alpha']$ and $[\beta']$ are adjacent in $Sp(2(\nu-2), q)$ and we can check at once that

$$\begin{aligned} N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta]) = \{&[\gamma_1, \gamma_2, \ldots, \gamma_\nu, 0, 0, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}] : \\ &[\gamma_3, \ldots, \gamma_\nu, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}] \in N_{Sp(2(\nu-2),q)}([\alpha']) \cap N_{Sp(2(\nu-2),q)}([\beta'])\}. \end{aligned}$$

Thus $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^2 |N_{Sp(2(\nu-2),q)}([\alpha']) \cap N_{Sp(2(\nu-2),q)}([\beta'])|$ and hence, Lemma 2.1 shows that $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^{2(\nu-2)}(q-1)$. □

Proposition 6.4. *If $[\alpha], [\beta] \in V(\Delta^{(2,2)})$ such that $[\alpha] \not\sim [\beta]$, then $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^{2\nu-3}$ or $q^{2(\nu-2)}(q-1)$. In particular, if $\nu = 3$, then $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^3$*

Proof: By (5.1), $[\alpha] = [\alpha_1 \ldots, \alpha_\nu, 0, 0, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta] = [\beta_1, \ldots, \beta_\nu, 0, 0, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Put $[\alpha'] = [\alpha_3, \ldots, \alpha_\nu, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}]$ and $[\beta'] = [\beta_3, \ldots, \beta_\nu, \beta_{\nu+3}, \ldots, \beta_{2\nu}]$. Obviously $[\alpha] \not\sim [\beta]$ if and only if $[\alpha'] = [\beta']$ or $\nu \geq 4$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-2), q)$. We continue the proof in the following cases:

(i) If $[\alpha'] = [\beta']$, then $N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta]) = N_{\Delta^{(2,2)}}([\alpha])$. Therefore, $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^{2\nu-3}$.

(ii) If $\nu \geq 4$ and $[\alpha']$ and $[\beta']$ are not adjacent in $Sp(2(\nu-2), q)$, then we can see at once that

$$\begin{aligned} N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta]) = \{&[\gamma_1, \gamma_2, \ldots, \gamma_\nu, 0, 0, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}] : \\ &[\gamma_3, \ldots, \gamma_\nu, \gamma_{\nu+3}, \ldots, \gamma_{2\nu}] \in N_{Sp(2(\nu-2),q)}([\alpha']) \cap N_{Sp(2(\nu-2),q)}([\beta'])\}. \end{aligned}$$

Thus $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^2 |N_{Sp(2(\nu-2),q)}([\alpha']) \cap N_{Sp(2(\nu-2),q)}([\beta'])|$ and hence, Lemma 2.1 shows that $|N_{\Delta^{(2,2)}}([\alpha]) \cap N_{\Delta^{(2,2)}}([\beta])| = q^{2(\nu-2)}(q-1)$.

□

Theorem 6.5. (i) *If $\nu \geq 4$, then $\Delta^{(2,2)}$ is a strictly Deza graph with parameters*

$$(q^2(q^{2(\nu-2)}-1)/(q-1), q^{2\nu-3}, q^{2\nu-3}, q^{2(\nu-2)}(q-1)).$$

(ii) *If $\nu = 3$, then $\Delta^{(2,2)}$ is a strongly regular graph with parameters*

$$(q^2(q+1), q^3, q^2(q-1), q^3).$$

Proof: It follows immediately from Propositions 6.1, 6.2, 6.3 and 6.4. □

Let $\mathbb{F}_q \times \mathbb{F}_q$ be a clique with $\mathbb{F}_q \times \mathbb{F}_q$ as its vertex set and the adjacency defined by $x \sim y$ if and only if $x \neq y$. Then the complement $\overline{\mathbb{F}_q \times \mathbb{F}_q}$ of $\mathbb{F}_q \times \mathbb{F}_q$ is a coclique.

Theorem 6.6. *$\Delta^{(2,2)}$ is isomorphic to $Sp(2(\nu-2), q)[\overline{\mathbb{F}_q \times \mathbb{F}_q}]$.*

Proof: Define

$$\begin{aligned} \phi : \Delta^{(2,2)} &\longrightarrow Sp(2(\nu-2), q)[\overline{\mathbb{F}_q \times \mathbb{F}_q}] \\ [\alpha_1, \alpha_2, \ldots, \alpha_\nu, 0, 0, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}] &\longmapsto ([\alpha_3, \ldots, \alpha_\nu, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}], \alpha_1, \alpha_2), \end{aligned}$$

under conditions that $f_{2(\nu-1)}(\alpha_3, \ldots, \alpha_\nu, \alpha_{\nu+3}, \ldots, \alpha_{2\nu}) = 1$, where $f_{2(\nu-1)}$ is defined as (4.2). We can check at once that ϕ is a graph isomorphism, so theorem follows. □

Theorem 6.7. $\chi(\Gamma^{(2,2)}) = q^{\nu-2} + 1$.

Proof: Theorem 6.6 shows that $\chi(\Delta^{(2,2)}) = \chi(Sp(2(\nu-2), q))$, so Lemma 2.1 completes the proof. □

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