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A new characterization of A_p with p and $p-2$ are twin primes

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abstract: Let G be a finite group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):=$\{m_k | k \in \pi_e(G)\}$. Assume p and $p-2$ are twin primes. We prove that if G is a group such that nse(G)=nse(A_p) and $p \in \pi(G)$, then $G \cong A_p$. As a consequence of our results we prove that A_p is uniquely determined by its nse and order.

Key Words: Element order, set of the numbers of elements of the same order, alternating group.

Contents

1. Introduction

We denote by $\pi(G)$ the set of prime divisors of $|G|$ and by $\pi_e(G)$ the set of element orders of G. Set $m_i = m_i(G)$=$|\{g \in G|$ the order of g is $i\}|$. In fact, m_i is the number of elements of order i in G, and nse(G):=$\{m_i|\ i \in \pi_e(G)\}$, the set of sizes of elements with the same order.

For the set nse(G), the most important problem is related to Thompson's problem. In 1987, J. G. Thompson posed a very interesting problem as follows:
Problem 1: For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G|\ x^d = 1\}$. Defining G_1 and G_2 is of the same order type if and only if, $|G_1(d)| = |G_2(d)|$, $d = 1, 2, 3, \ldots$. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable? ([20, Problem 12.37])

Unfortunately, as so far, no one can prove it completely, or even give a counterexample. However, if groups G_1 and G_2 are of the same order type, we see clearly that $|G_1| = |G_2|$ and nse(G_1) = nse(G_2). So it is natural to investigate the Thompson's Problem by $|G|$ and nse(G). The influence of nse(G) on the structure of finite groups was studied by some authors (see [2,3,4,6,19]).

In [4,19], it is proved that the groups A_5, A_6, A_7 and A_8 are uniquely determined only by nse(G). In [19], the authors gave the following problem:
Problem 2: Is a group G isomorphic to A_n ($n \geq 4$) if and only if nse(G) = nse(A_n)?

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In this paper, we give a positive answer to this problem for some type of the alternating groups and show that the alternating groups A_p with p and $p-2$ primes are characterizable by nse(A_p) when $p \in \pi(G)$. In fact, main theorem of our paper is as follows:

Main Theorem: Let G be a group such that nse(G)=nse(A_p) with p and $p-2$ are twin primes. If $p \in \pi(G)$, then $G \cong A_p$.

We note that there are finite groups which are not characterizable by nse(G) and $|G|$. In 1987, J. G. Thompson gave an example as follows:
Let G_1= $(C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = L_3(4) \rtimes C_2$ be the maximal subgroups of M_{23}. Then nse(G_1) = nse(G_2)= $\{1, 435, 2240, 5040, 5760, 6300, 6720, 8064\}$ and $|G_1| = |G_2| = 40320$, but $G_1 \ncong G_2$. Also there is a another example as follow: Let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$, where C_2 and C_4 are cyclic groups of orders 2 and 4, respectively and Q_8 is a quaternion group of order 8. It is easy to see that nse(H_1) = nse(H_2)= $\{1, 3, 12\}$ and $|H_1|$=$|H_2|$ = 16, but H_1 is an abelian group and H_2 is a non-abelian group. Therefore $H_1 \ncong H_2$.

We construct the *prime graph* of G, denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and p' are joined by an edge if and only if G has an element of order pp' (we write $p \sim p'$). Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each i. These numbers m_i are called the order components of G. In particular, if m_i is odd, then we call it an odd component of G. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of G and $T(G)$ for the set of connected components of G (see [12]). According to the classification theorem of finite simple groups and [5,17,18], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-3 in [1].

Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroup of G, that is, $n_q(G)$=$|\mathrm{Syl}_q(G)|$. Also we say $p^k \parallel m$ if $p^k \mid m$ and $p^{k+1} \nmid m$. All other notations are standard and we refer to [16], for example.

2. Preliminary Results

We first quote some lemmas that are used in deducing the main theorem of this paper.
Let $\alpha \in S_n$ be a permutation and let α have t_i cycles of length i, i = 1, 2, ..., l, in its cycle decomposition. The cycle structure of α is denote by $1^{t_1}2^{t_2}...l^{t_l}$ where $1t_1 + 2t_2... + lt_l = n$. One can easily show that two permutations in S_n are conjugate if and only if they have the same cycle structure.

Lemma 2.1. [14] *Let $\alpha \in S_n$ and assume that the cycle decomposition of α contains t_1 cycles of length 1, t_2 cycles of length 2, ..., t_l cycles of length l. Then*

$|cl_{S_n}(\alpha)| = n!/1^{t_1}2^{t_2}\cdots l^{t_l}t_1!t_2!\cdots t_l!$.

Lemma 2.2. [9] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.*

Let m_n be the number of elements of order n. We note that $m_n = k\phi(n)$ where k is the number of cyclic subgroups of order n in G. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \mid |G|$, then by Lemma 2.2 and the above notation we have:

$$\begin{cases} \phi(n) \mid m_n \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we often apply $(*)$ and the above comments.

Lemma 2.3. [19] *Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = sup\{m_k | k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leq s(s^2-1)$.*

Lemma 2.4. [10] *Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow $p-$subgroup of G and $n = p^s m$, where $(p,m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

Lemma 2.5. [15] *Let G be a finite group, $n \geq 4$ with $n \neq 8$, 10 and r be the greatest prime not exceeding n. If $|G| = |A_n|$ and $|N_G(R)| = |N_{A_n}(S)|$ where $R \in Syl_r(G)$ and $S \in Syl_r(A_n)$, then $G \cong A_n$.*

Lemma 2.6. [7] *Let G be a group and P a cyclic Sylow p-subgroup of G of order p^a. If there is a prime r such that $p^a r \in \pi_e(G)$, then $m_{p^a r} = m_r(C_G(P))m_{p^a}$. In particular, $\phi(r)m_{p^a} \mid m_{p^a r}$.*

Lemma 2.7. [11] *Let G be a Frobenius group of even order with H and K its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.*

Lemma 2.8. [11] *Let G be a 2-Frobenius group of even order which has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$.*

Lemma 2.9. [17, Theorem A] *Let G be a finite group with more than one prime graph component. Then either G is a Frobenius or a 2-Frobenius group, or G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that such that H and G/K are π_1-groups, K/H is a non-abelian simple group and H is a nilpotent group with $\pi_i \subseteq \pi(K)$ for every $i > 1$ and H is a nilpotent group, especially, $K/H \trianglelefteq G/H \trianglelefteq Aut(K/H)$.*

Lemma 2.10. [13, Lemma 8] *Let G be a finite group with $t(G) \geq 2$ and N a normal subgroup of G. If N is a π_i-group for some prime graph component of G, and μ_1, μ_2, . . . , μ_r are some of order components of G but not a π_i-number then $\mu_1\mu_2...\mu_r$ is a divisor of $|N| - 1$.*

Now we bring the following Lemma which is proved in [5, Lemma 6], with some differences and classify the simple groups of Lie type with prime odd order component by θ function which is introduced later.

Lemma 2.11. *If L is a simple group of Lie type and has prime odd order component $p \geq 17$ and $\pi(L)$ has at most $\theta(L)$ prime numbers t, where $\frac{p+1}{2} < t < p$. Then $\theta(L) \leq 3$.*

Throughout the proof of the above Lemma, we can divide simple groups of Lie type, L, with prime odd order component $p \geq 17$, into the following cases:

(1) $\theta(L) = 0$ *if* L *is isomorphic to* $A_{p'-1}(q)$, $A_{p'}(q)$,
where $q - 1 \mid p^{'} + 1$, $A_2(2)$, ${}^2A_{p'-1}(q)$, ${}^2A_{p'}(q)$,
where $q + 1 \mid p^{'} + 1$, ${}^2A_3(2)$, $B_n(q)$,
where $n = 2^{m^{'}}$ *and* q *is odd,* $B_{p'}(3)$, $C_n(q)$,
where $n = 2^{m^{'}}$ *or* $(n, q) = (p^{'}, 3)$, $D_{p'+1}(3)$, $D_{p'}(q)$, *for* $q = 3, 5$,
${}^2D_n(q)$, *for* $(n, q) = (2^{m^{'}}, q)$, $(p^{'}, 3)$, *where* $5 \leq p^{'} \neq 2^{m^{'}} + 1$ *or* $(2^{m^{'}} + 1, 3)$,
where $5 \leq p^{'} \neq 2^{m^{'}} + 1$, $G_2(q)$,
where $q \equiv \epsilon$ *(mod 3), for* $\epsilon = \pm 1$, ${}^3D_4(q)$, $E_6(q)$ *or* ${}^2E_6(q)$;

(2) $\theta(L) = 1$ *if* L *is isomorphic to one of the simple groups* $A_1(q)$,
where $2 \mid q$, $A_2(4)$, ${}^2A_5(2)$, $C_{p'}(2)$, $D_n(2)$,
where $n = p^{'}$ *or* $p^{'} + 1$, ${}^2D_n(2)$, *where* $(n, q) = (2^{m^{'}} + 1, 2)$ *or* $(p^{'} = 2^{m^{'}} + 1, 3)$,
where $m^{'} \geq 2$, $E_7(2)$, $E_7(3)$, $F_4(q)$, ${}^2F_4(q)$,
where $q = 2^{2n+1} > 2$, *or* $G_2(q)$, *where* $3 \mid q$;

(3) $\theta(L) = 2$ *if* L *is isomorphic to the simple groups* $A_1(q)$,
where $q \equiv \epsilon$ *(mod 4) for* $\epsilon = \pm 1$, ${}^2B_2(q)$,
where $q = 2^{2m^{'}+1} > 2$, *or* ${}^2G_2(q)$, *where* $q = 3^{2m^{'}+1} > 3$;

(4) $\theta(L) = 3$ *if* L *is isomorphic to the simple groups* $E_8(q)$ *or* ${}^2E_6(2)$.

Lemma 2.12. [5, Lemma 1] *If $n \geq 6$ is a natural number, then there are at least $s(n)$ prime numbers p_i such that $\frac{n+1}{2} < p_i < n$. Here*

$$\begin{gathered}
s(n) = 6 \text{ for } n \geq 48; \\
s(n) = 5 \text{ for } 42 \leq n \leq 47; \\
s(n) = 4 \text{ for } 38 \leq n \leq 41; \\
s(n) = 3 \text{ for } 18 \leq n \leq 37; \\
s(n) = 2 \text{ for } 14 \leq n \leq 17; \\
s(n) = 1 \text{ for } 6 \leq n \leq 13.
\end{gathered}$$

In particular, for every natural number $n \geq 6$, there exists a prime p such that $\frac{n+1}{2} < p < n$, and for every natural number $n > 3$, there exists an odd prime number p such that $n - p < p < n$.

3. Proof of the Main Theorem

Let G be a group such that nse(G)=nse(A_p) where p and $p-2$ are twin primes and $p \in \pi(G)$. By Lemma 2.3, we can assume that G is finite. Let $p' = p-2$. The following lemmas reduce the problem to a study of groups with the same order with A_p.

Lemma 3.1. *If $i \in \pi_e(A_p)$ and $i \neq 1$, p, then $p \parallel m_i(A_p)$.*

Proof: We have $m_i(A_p) = \sum |cl_{A_p}(x_k)|$ such that $|x_k| = i$. Since $i \neq 1$, p, the cyclic structure of x_k for any k is $1^{r_1}2^{r_2}...l^{r_l}$ where $r_1, r_2, ..., r_l$ and 1, 2, ..., l are not equal to p. On the other hand, by Lemma 2.1, $|cl_{A_p}(x_k)| = p!/1^{r_1}2^{r_2}...l^{r_l}r_1!r_2!\cdots r_l!$. Then $p \mid |cl_{A_p}(x_k)|$ for any k. Hence $p \mid m_i(A_p)$ and $m_i(A_p) = p! \cdot \alpha$ where $\alpha < 1$. Therefore, $p \parallel m_i(A_p)$. $\square$

Lemma 3.2. $m_p(G) = m_p(A_p)$.

Proof: Let $m_p(G) \neq m_p(A_p)$. Since nse(G)=nse(A_p), $m_p(G) \in$ nse(A_p). Suppose that there exists $k \neq p \in \pi_e(G)$ such that $m_p(G) = m_k(A_p)$. Thus $m_k(A_p) \equiv -1 \pmod p$. We know that $m_k(A_p) = \sum |cl_{A_p}(x_i)|$ such that $|x_i| = k$ for every i. Since $m_k(A_p) \equiv -1 \pmod p$, $(p, m_k(A_p)) = 1$.

If for every i, the cyclic structure of x_i is $1^{r_1}2^{r_2}...l^{r_l}$ such that $p \nmid 1^{r_1}2^{r_2}...l^{r_l}r_1!$ $r_2!...r_l!$, then $|cl_{A_p}(x_i)| = \frac{p!}{1^{r_1}2^{r_2}...l^{r_l}r_1!r_2!...r_l!} \equiv 0 \pmod p$. Hence $(p, m_k(A_p)) \neq 1$, which is a contradiction. Hence $p \mid 1^{r_1}2^{r_2}...l^{r_l}r_1!r_2!...r_l!$. Therefore, there exists at least a x_i with cyclic structure $1^{r_1}2^{r_2}...l^{r_l}$such that $r_j = p+t$ for some $1 \leq j \leq l$, where t is a non-negative integer, or one of the numbers 1, 2, ..., l is equal to p. Thus there exists x_i such that the cyclic structure of x_i is 1^p or p^1.

If the cyclic structure of x_i is 1^p, then $|x_i| = 1$, which is a contradiction.

If the cyclic structure of x_i is p^1, then $|x_i| = k = p$. Therefore, $m_p(G) = m_p(A_p)$. $\square$

Lemma 3.3. *If $i \in \pi_e(A_p)$, $i \neq 1$ and $i \neq p'$, then $p' \parallel m_i(A_p)$.*

Proof: We can prove this lemma as the proof of the Lemma 3.1. $\square$

Lemma 3.4. *If $p' \in \pi(G)$, then $m_{p'}(G) = m_{p'}(A_p)$.*

Proof: There exists $k \in \pi_e(G)$ such that $p' \mid (1 + m_k(A_p))$. We know that $m_k(A_p) = \sum |cl_{A_p}(x_i)|$ such that $|x_i| = k$. Since $p' \mid (1 + m_k(A_p))$, $(p', m_k(A_p)) = 1$. If the cyclic structure of x_i for any i is $1^{t_1}2^{t_2}\cdots l^{t_l}$ such that $r_1, r_2, ..., r_l$ and 1, 2, ..., l are not equal to p', then $p' \mid p!/1^{r_1}2^{r_2}...l^{r_l}r_1!r_2!\cdots r_l!$. Hence $p' \mid |cl_{A_p}(x_i)|$ for any i and therefore $(p', m_k(A_p)) \neq 1$, which is a contradiction. Thus there exists $i \in \mathbb{N}$ such that $r_i = p'$ or one of the numbers 1 or 2 ... or l are equal to p'.

If there exists $i \in \mathbb{N}$ such that $r_i = p'$, then the cyclic structure of x_i is $1^{p'}2^1$. Hence x_i is an odd permutation, which is a contradiction.

If one of the numbers 1 or 2 ... or l are equal to p', then the cyclic structure of x_i is $1^2 p'^1$. Hence $|x_i| = p'$ and $k = p'$. Therefore, $m_{p'}(G) = m_{p'}(A_p)$. □

Lemma 3.5. $|P_p| = p$.

Proof: By Lemma 3.1, $p^2 \nmid m_i(G)$ for any $i \in \pi_e(G)$. If $p^3 \in \pi_e(G)$, then by $(*)$, $\phi(p^3) \mid m_{p^3}(G)$. Thus $p^2 \mid m_{p^3}(G)$, which is a contradiction. Therefore $p^3 \notin \pi_e(G)$. Hence $\exp(P_p) = p$ or p^2. We claim that exp $(P_p) = p$. Suppose that $\exp(P_p) = p^2$. There exists an element of order p^2 in G such that $\phi(p^2) \mid m_{p^2}(G)$. Thus $p(p-1) \mid m_{p^2}(G)$.

If $|P_p| = p^2$, then P_p is a cyclic group and $n_p(G) = m_{p^2}(G)/\phi(p^2) = p(p-1)t/p(p-1) = t$. On the other hand, we know two of the Sylow p-subgroups might intersect in a subgroup of order p. So the number of cyclic subgroups of order p is something between 1 and the number of Sylow p -subgroups. Since $m_p(G) = (p-1)!$, $(p-1)! \leq (p-1)n_p(G) = (p-1)t$. Therefore $n_p(G) = t \geq (p-2)!$ and $m_{p^2}(G) \geq p(p-1)(p-2)! = p!$, which is a contradiction.

If $|P_p| = p^s$ where $s \geq 3$, then by Lemma 2.4, $m_{p^2}(G) = p^2 l$ for some l, which is a contradiction by Lemma 3.1. Thus $\exp(P_p) = p$.

We obtain that $|P_p| \mid (1 + m_p) = 1 + (p-1)!$, which implies that $|P_p| = p$ or $|P_p| = p^s$ where $s \geq 2$. Consider the case $|P_p| = p^s$ where $s \geq 2$.

We prove that $2p \in \pi_e(G)$. Assume that this is false. Then the group P_p acts fixed-point-freely on the set of elements of order 2, which implies that $p^s = |P_p| \mid m_2$. By Lemma 3.1, we get a contradiction.

Therefore $2p \in \pi_e(G)$, as required. Since $\exp(P_p) = p$, $2p^2 \notin \pi_e(G)$, so by Lemma 2.2, $L_{2p^2}(G) = L_{2p}(G)$, and so $2p^2 \mid (1+m_2+m_p+m_{2p})$. Since $p^2 \mid (1+m_p)$, $p^2 \mid (m_2 + m_{2p})$. But $m_2 + m_{2p} = p!k$, where $0 < k < 1$, which is a contradiction. Therefore, $|P_p| = p$. □

Lemma 3.6. $\pi(G) = \pi(A_p)$.

Proof: By Lemma 3.5, $|P_p| = p$. Hence $(p-2)! = m_p(G)/\phi(p) = n_p(G) \mid |G|$. Thus $\pi(A_p) \subseteq \pi(G)$. Now we show that $\pi(A_p) = \pi(G)$. Let $r > p$ be a prime such that $r \in \pi(G)$. If $pr \in \pi_e(G)$, then by Lemma 2.6, $(r-1)(p-1)! \mid m_{pr}$. But $(r-1)(p-1)! > p!$, which is a contradiction. Thus $pr \notin \pi_e(G)$. Then the group P_r acts fixed point freely on the set of elements of order p, and so $|P_r| \mid (p-1)!$, a contradiction. Therefore $r \notin \pi(G)$ and $\pi(G) = \pi(A_p)$. □

Lemma 3.7. *G has not any element of order $2p$ and $2p'$.*

Proof: Suppose that G has an element of order $2p$. By Lemma 2.6, $(p-1)! \mid m_{2p}(G)$. On the other hand, by $(*)$ $2p \mid (1 + m_2 + m_p + m_{2p})$. Since $p \mid (1 + m_p)$ and $p \mid m_2$, $p \mid m_{2p}(G)$. Therefore $p! \mid m_{2p}(G)$, which is a contradiction. Now suppose that G has an element of order $2p'$. By Lemma 2.6, $p!/2p' \mid m_{2p}(G)$. On the other hand, by $(*)$ $2p' \mid (1 + m_2 + m_{p'} + m_{2p'})$. Since $p' \mid (1 + m_{p'})$ and by Lemma 3.3, $p' \mid m_2$, $p' \mid m_{2p'}(G)$. Therefore $p!/2 \mid m_{2p'}(G)$, which is a contradiction. □

Lemma 3.8. *rp, $sp' \notin \pi_e(G)$ for every $r \in \pi(G)$ and $s \in \pi(G)\backslash\{p'\}$.*

Proof: The proof of this lemma is completely similar to Lemma 3.7. □

Lemma 3.9. $|G| = |A_p|$.

Proof: Suppose that $|A_p| = 2^{k_2}3^{k_3}5^{k_5}\cdots p'p$ where k_2, k_3, k_5, ...are non-negative integers. By Lemma 3.7, $2p' \notin \pi_e(G)$, so the group P_2 acts fixed point freely on the set of elements of order p', and so $|P_2| \mid p!/2p' = m_{p'}(G)$. Thus $|P_2| \mid 2^{k_2}$. By Lemma 3.8 and arguing as above, $|P_3| \mid 3^{k_3}$, $|P_5| \mid 5^{k_5}$, ... Therefore $|G| \mid p!/2$. On the other hand, $(p-2)! = n_p(G) \mid |G|$ and $p!/2p'(p'-1) = n_{p'}(G) \mid |G|$. Then the least common multiple of $(p-2)!$ and $p!/2p'(p'-1)$ divide the order of G. Therefore $p!/2 \mid |G|$ and so $|G| = |A_p|$ or $|G| = p!/4$. We will show that $|G| = |A_p|$.

Let $|G| = p!/4$. By Lemma 3.8, $t(G) \geq 3$. Hence by Lemma 2.7 and 2.8, G is neither Frobenius nor 2-Frobenius.

By Lemma 2.9, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group and H is a nilpotent group. If K/H has an element of order rq where r and q are primes, then G has also such element. Hence by definition of order components, an odd order component of G must be an odd order component of K/H. Note that $t(K/H) \geq 3$. Now we consider the following claims.

Claim 1.

(a) If $t \in \pi(H)$, then $t \leq \frac{p+1}{2}$;

(b) K/H cannot be isomorphic to a sporadic simple group. Moreover, if K/H is isomorphic to an alternating simple group, then we show that it is impossible.

(a) if t divides $|H|$ where $\frac{p+1}{2} < t < p$, then since H is nilpotent subgroup of G and the order of T, the Sylow t-subgroup of H, is equal to t. By Lemma 2.10, we must have $p \mid t-1$, which is impossible. Thus $|H|$ is not divisible by the primes t with $\frac{p+1}{2} < t < p$.

For the subcase (b), we note that if $H \neq 1$, by nilpotency of H, we may assume that H is a t-group for $t \in \pi_1(G)$.

If $K/H \cong J_4$, then $p = 43$. Since $19 \in \pi(G)\backslash\pi(\mathrm{Aut}(J_4))$, then $19 \in \pi(H)$. By Lemma 2.10, $43 \mid 19^i - 1$ for $i = 1$ or 2, which is impossible.

If $K/H \cong M_{22}$, then $p = 11$. Since $5^2 \mid |G|$ and $5 \parallel |\mathrm{Aut}(M_{22})|$, $5 \in \pi(H)$. So by Lemma 2.10, we get a contradiction.

If K/H is isomorphic to other sporadic simple groups we can view a contradiction similarly.

Now let K/H be isomorphic to an alternating group. By Tables 1 and 2 in [1], K/H must be isomorphic to A_p, p, $p-2$ both primes, since $|G| = p!/4$, we get a contradiction.

Claim 2. *Let K/H is a simple group of Lie type over finite field GF(q). Then the order of factor group G/K cannot be divided by primes t, with $\frac{p+1}{2} < t < p$. Therefore $t \in \pi(K/H)$.*

It follows from the proof of Lemma 6 in [5].

Claim 3. *K/H can not be isomorphic to a simple group of Lie type.*
By the Claim 2 and Lemma 2.12, we must have $17 \leq p \leq 37$ and $\theta(K/H) \geq 2$. Therefore K/H is isomorphic to one of the following simple groups by Lemma 2.11.

(1) $L_2(q)$, where $q \equiv \epsilon \pmod 4$ for $\epsilon = \pm 1$;
(2) ${}^2B_2(q)$, where $q = 2^{2m'+1} > 2$;
(3) ${}^2G_2(q)$, where $q = 3^{2m'+1} > 3$;
(4) $E_8(q)$ or ${}^2E_6(2)$.

Since one of the odd order components of K/H is equal to p, by Tables 2 and 3 in [1], we must have:

(1) $K/H \cong L_2(17)$, for $p = 17$;
(2) $K/H \cong L_2(19)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 19$;
(3) $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29 or 31;
(4) $K/H \cong L_2(37)$, ${}^2G_2(27)$ or ${}^2E_6(2)$, for $p = 37$.

If $K/H \cong L_2(17)$, then Since $13 \notin \pi(L_2(17))$, we get a contradiction by the Claim 2.

If $K/H \cong {}^2G_2(27)$ for $p = 37$ or $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29, 31, or 37, we view a contradiction similarly. Let $p = 19$. If $K/H \cong L_2(19)$, again by the Claim 2, we get a contradiction because $17 \notin \pi(L_2(19))$. Since $37 \mid |{}^2G_2(27)|$, we have $K/H \ncong {}^2G_2(27)$.

For the case $K/H \cong {}^2E_6(2)$, we view a contradiction by $2^{36} \mid |{}^2E_6(2)|$.
Also if $5 \leq p \leq 13$ we know that G has a normal series $1 \lhd H \lhd K \lhd G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, $G/H \leq \mathrm{Aut}(K/H)$ and H is a nilpotent group. By the Claim 1(b), K/H cannot be isomorphic to a sporadic simple group, and if K/H is isomorphic to an alternating simple group, then we get a contradiction.

Let K/H be isomorphic to the simple group of Lie type $G(q)$ where $q = s^m$ and s is a prime number. Let $p = 5$. We know, 5 is one of the odd order components of K/H. So $s = 5$ and K/H be isomorphic to $L_2(5) \cong A_5$, a contradiction.

Let $p = 7$. We know, 7 is one of the odd order components of K/H. So $s = 2$, 3 or 7. Then the order of all Sylow t-subgroups of G are less than or equal to 64 or 81, respectively. Therefore for both of these cases, K/H is isomorphic to $L_2(7)$ or $L_2(8)$. Let $K/H \cong L_2(7)$ or $L_2(8)$. Since $5 \in \pi(G) \backslash \pi(\mathrm{Aut}(K/H))$, $5 \in \pi(H)$, which it contradicts Claim 1(a). For the cases $p = 11$, 13, we do similarly. Therefore, $|G| = |A_p|$. □

Now, we are prepared to prove our main theorem.

Proof of the main theorem. We have $|G| = |A_p|$ and $m_p(G) = m_p(A_p)$. Hence the number of Sylow $p-$subgroups of G is equal to the number of Sylow $p-$subgroups of A_p. Since $|G| = |A_p|$ and $n_p(G) = n_p(A_p)$, $|N_G(R)| = |N_{A_p}(S)|$ where $R \in \mathrm{Syl}_p(G)$ and $S \in \mathrm{Syl}_p(A_p)$. Now by Lemma 2.5, $G \cong A_p$ and the proof is completed. □

Corollary. *Let G be a finite group. If nse(G)=nse(A_p) and $|G| = |A_p|$, where p and $p-2$ are twin primes, then $G \cong A_p$.*

Proof: This is an immediate consequence of the main theorem.

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