A new characterization of $A_p$ with $p$ and $p - 2$ are twin primes

Seyed Sadegh Salehi Amiri and Alireza Khalili Asboei

Abstract: Let $G$ be a finite group and $\pi_e(G)$ be the set of element orders of $G$. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. Set $\text{nse}(G):=\{m_k | k \in \pi_e(G)\}$. Assume $p$ and $p - 2$ are twin primes. We prove that if $G$ is a group such that $\text{nse}(G)=\text{nse}(A_p)$ and $p \in \pi(G)$, then $G \cong A_p$. As a consequence of our results we prove that $A_p$ is uniquely determined by its nse and order.

Key Words: Element order, set of the numbers of elements of the same order, alternating group.

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1. Introduction

We denote by $\pi(G)$ the set of prime divisors of $|G|$ and by $\pi_e(G)$ the set of element orders of $G$. Set $m_i = m_i(G)=|\{g \in G| \text{the order of } g \text{ is } i\}|$. In fact, $m_i$ is the number of elements of order $i$ in $G$, and $\text{nse}(G):=\{m_i | i \in \pi_e(G)\}$, the set of sizes of elements with the same order.

For the set $\text{nse}(G)$, the most important problem is related to Thompson’s problem. In 1987, J. G. Thompson posed a very interesting problem as follows: Problem 1: For each finite group $G$ and each integer $d \geq 1$, let $G(d) = \{x \in G| x^d = 1\}$. Defining $G_1$ and $G_2$ is of the same order type if and only if, $|G_1(d)| = |G_2(d)|$, $d = 1, 2, 3, \ldots$. Suppose $G_1$ and $G_2$ are of the same order type. If $G_1$ is solvable, is $G_2$ necessarily solvable? ([20], Problem 12.37])

Unfortunately, as so far, no one can prove it completely, or even give a counterexample. However, if groups $G_1$ and $G_2$ are of the same order type, we see clearly that $|G_1| = |G_2|$ and $\text{nse}(G_1) = \text{nse}(G_2)$. So it is natural to investigate the Thompson’s Problem by $|G|$ and $\text{nse}(G)$. The influence of $\text{nse}(G)$ on the structure of finite groups was studied by some authors (see [2,3,4,6,19]).

In [4,19], it is proved that the groups $A_5$, $A_6$, $A_7$ and $A_8$ are uniquely determined only by $\text{nse}(G)$. In [19], the authors gave the following problem: Problem 2: Is a group $G$ isomorphic to $A_n$ $(n \geq 4)$ if and only if $\text{nse}(G) = \text{nse}(A_n)$?

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In this paper, we give a positive answer to this problem for some type of the alternating groups and show that the alternating groups $A_p$ with $p$ and $p - 2$ primes are characterizable by $\text{nse}(A_p)$ when $p \in \pi(G)$. In fact, main theorem of our paper is as follows:

**Main Theorem:** Let $G$ be a group such that $\text{nse}(G) = \text{nse}(A_p)$ with $p$ and $p - 2$ are twin primes. If $p \in \pi(G)$, then $G \cong A_p$.

We note that there are finite groups which are not characterizable by $\text{nse}(G)$ and $|G|$. In 1987, J. G. Thompson gave an example as follows: Let $G_1 = \langle C_2 \times C_2 \times C_2 \times C_2 \rangle \rtimes A_7$ and $G_2 = L_3(4) \times C_2$ be the maximal subgroups of $M_{23}$. Then $\text{nse}(G_1) = \text{nse}(G_2) = \{1, 435, 2240, 5040, 5760, 6300, 6720, 8064\}$ and $|G_1| = |G_2| = 40320$, but $G_1 \not\cong G_2$. Also there is another example as follow: Let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$, where $C_2$ and $C_4$ are cyclic groups of orders $2$ and $4$, respectively and $Q_8$ is a quaternion group of order $8$. It is easy to see that $\text{nse}(H_1) = \text{nse}(H_2) = \{1, 3, 12\}$ and $|H_1| = |H_2| = 16$, but $H_1$ is an abelian group and $H_2$ is a non-abelian group. Therefore $H_1 \not\cong H_2$.

We construct the **prime graph** of $G$, denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices $p$ and $p'$ are joined by an edge if and only if $G$ has an element of order $pp'$ (we write $p \sim p'$). Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi(t(G))$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

We can express $|G|$ as a product of integers $m_1, m_2, \ldots, m_{t(G)}$, where $\pi(m_i) = \pi_i$ for each $i$. These numbers $m_i$ are called the order components of $G$. In particular, if $m_i$ is odd, then we call it an odd component of $G$. Write $OC(G)$ for the set $\{m_1, m_2, \ldots, m_{t(G)}\}$ of order components of $G$ and $T(G)$ for the set of connected components of $G$ (see [12]). According to the classification theorem of finite simple groups and [5,17,18], we can list the order components of finite simple groups with disconnected prime graphs as in Tables 1-3 in [1].

Throughout this paper, we denote by $\phi$ the Euler totient function. If $G$ is a finite group, then we denote by $P_i$ a Sylow $q$-subgroup of $G$ and $n_q(G)$ is the number of Sylow $q$-subgroup of $G$, that is, $n_q(G) = |\text{Syl}_q(G)|$. Also we say $p^k \| m$ if $p^k | m$ and $p^{k+1} \nmid m$. All other notations are standard and we refer to [16], for example.

### 2. Preliminary Results

We first quote some lemmas that are used in deducing the main theorem of this paper.

Let $\alpha \in S_n$ be a permutation and let $\alpha$ have $t_i$ cycles of length $i$, $i = 1, 2, \ldots, l$, in its cycle decomposition. The cycle structure of $\alpha$ is denote by $1^{t_1}2^{t_2}\ldots l^{t_l}$ where $t_1 + 2t_2 + \ldots + lt_l = n$. One can easily show that two permutations in $S_n$ are conjugate if and only if they have the same cycle structure.

**Lemma 2.1.** [14] Let $\alpha \in S_n$ and assume that the cycle decomposition of $\alpha$ contains $t_1$ cycles of length $1$, $t_2$ cycles of length $2$, $\ldots$, $t_l$ cycles of length $l$. Then
Lemma 2.2. [9] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m | |L_m(G)|$.

Let $m_n$ be the number of elements of order $n$. We note that $m_n = k\phi(n)$ where $k$ is the number of cyclic subgroups of order $n$ in $G$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n | |G|$, then by Lemma 2.2 and the above notation we have:

$$\begin{cases}
\phi(n) | m_n \\
n | \sum_{d | n} m_d
\end{cases} \quad (\ast)$$

In the proof of the main theorem, we often apply $(\ast)$ and the above comments.

Lemma 2.3. [19] Let $G$ be a group containing more than two elements. Let $k \in \pi_c(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup\{m_k | k \in \pi_c(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

Lemma 2.4. [10] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^m$, where $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

Lemma 2.5. [15] Let $G$ be a finite group, $n \geq 4$ with $n \neq 8, 10$ and $r$ be the greatest prime not exceeding $n$. If $|G| = |A_n|$ and $|N_G(R)| = |N_{A_n}(S)|$ where $R \in \operatorname{Syl}_p(G)$ and $S \in \operatorname{Syl}_p(A_n)$, then $G \cong A_n$.

Lemma 2.6. [7] Let $G$ be a group and $P$ a cyclic Sylow $p$-subgroup of $G$ of order $p^n$. If there is a prime $r$ such that $p^r \in \pi_c(G)$, then $m_{p^r} = m_r(C_G(P))m_{p^r}$. In particular, $\phi(r)m_{p^r} | m_{p^r}$.

Lemma 2.7. [11] Let $G$ be a Frobenius group of even order with $H$ and $K$ its Frobenius kernel and Frobenius complement, respectively. Then $t(G) = 2$ and $T(G) = \{\pi(K), \pi(H)\}$.

Lemma 2.8. [11] Let $G$ be a 2-Frobenius group of even order which has a normal series $1 \leq H \triangleleft K \triangleleft G$ such that $K$ and $G/H$ are Frobenius groups with kernels $H$ and $K/H$, respectively. Then $t(G) = 2$ and $T(G) = \{\pi_1(G) = \pi(H) \cup \pi(G/K), \pi_2(G) = \pi(K/H)\}$.

Lemma 2.9. [17, Theorem A] Let $G$ be a finite group with more than one prime graph component. Then either $G$ is a Frobenius or a 2-Frobenius group, or $G$ has a normal series $1 \leq H \triangleleft K \triangleleft G$ such that such that $H$ and $G/K$ are $\pi_1$-groups, $K/H$ is a non-abelian simple group and $H$ is a nilpotent group with $\pi_i \subseteq \pi(K)$ for every $i > 1$ and $H$ is a nilpotent group, especially, $K/H \triangleleft G/H \triangleleft \operatorname{Aut}(K/H)$.
Lemma 2.10. [13, Lemma 8] Let $G$ be a finite group with $t(G) \geq 2$ and $N$ a normal subgroup of $G$. If $N$ is a $\pi_i$-group for some prime graph component of $G$, and $\mu_1, \mu_2, \ldots, \mu_r$ are some order components of $G$ but not a $\pi_i$-number then $\mu_1\mu_2\ldots\mu_r$ is a divisor of $|N| - 1$.

Now we bring the following Lemma which is proved in [5, Lemma 6], with some differences and classify the simple groups of Lie type with prime odd order component by $\theta$ function which is introduced later.

Lemma 2.11. If $L$ is a simple group of Lie type and has prime odd order component $p \geq 17$ and $\pi(L)$ has at most $\theta(L)$ prime numbers $t$, where $\frac{p+1}{2} < t < p$. Then $\theta(L) \leq 3$.

Throughout the proof of the above Lemma, we can divide simple groups of Lie type, $L$, with prime odd order component $p \geq 17$, into the following cases:

(1) $\theta(L) = 0$ if $L$ is isomorphic to $A_{p-1}(q)$, $A_p'(q)$, where $q - 1 \mid p' + 1$, $A_2(2)$, $2A_{p-1}(q)$, $2A_p'(q)$, where $q + 1 \mid p' + 1$, $2A_4(2)$, $B_n(q)$, where $n = 2^m$ and $q$ is odd, $B_p'(3)$, $C_n(q)$, where $n = 2^m$ or $(n, q) = (p', 3)$, $B_{p+1}(3)$, $D_p'(q)$, for $q = 3, 5$.

(2) $\theta(L) = 1$ if $L$ is isomorphic to one of the simple groups $A_1(q)$, where $2 \mid q$, $A_2(4)$, $2A_3(2)$, $C_p'(2)$, $D_n(2)$, where $n = p'$ or $p' + 1$, $2D_n(q)$, where $(n, q) = (2^{m'} + 1, 2)$ or $(p' = 2^{m'} + 1, 3)$.

(3) $\theta(L) = 2$ if $L$ is isomorphic to the simple groups $A_1(q)$, where $q \equiv \epsilon \pmod{4}$ for $\epsilon = \pm 1$, $2B_2(q)$, where $q = 2^{2m'+1} > 2$, or $2G_2(q)$, where $q = 3^{2m'+1} > 3$.

(4) $\theta(L) = 3$ if $L$ is isomorphic to the simple groups $E_6(q)$ or $2E_6(2)$.

Lemma 2.12. [5, Lemma 1] If $n \geq 6$ is a natural number, then there are at least $\frac{n}{2}$ prime numbers $p$, such that $\frac{n+1}{2} < p < n$. Here

$$
\begin{align*}
\text{s}(n) &= 6 \text{ for } n \geq 48; \\
\text{s}(n) &= 5 \text{ for } 42 \leq n \leq 47; \\
\text{s}(n) &= 4 \text{ for } 38 \leq n \leq 41; \\
\text{s}(n) &= 3 \text{ for } 18 \leq n \leq 37; \\
\text{s}(n) &= 2 \text{ for } 14 \leq n \leq 17; \\
\text{s}(n) &= 1 \text{ for } 6 \leq n \leq 13.
\end{align*}
$$

In particular, for every natural number $n \geq 6$, there exists a prime $p$ such that $\frac{n+1}{2} < p < n$, and for every natural number $n > 3$, there exists an odd prime number $p$ such that $n - p < p < n$. 

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3. Proof of the Main Theorem

Let \( G \) be a group such that \( \text{nse}(G) = \text{nse}(A_p) \) where \( p \) and \( p - 2 \) are twin primes and \( p \in \pi(G) \). By Lemma 2.3, we can assume that \( G \) is finite. Let \( p' = p - 2 \). The following lemmas reduce the problem to a study of groups with the same order with \( A_p \).

Lemma 3.1. If \( i \in \pi_{\epsilon}(A_p) \) and \( i \neq 1, p \), then \( p \parallel m_i(A_p) \).

Proof: We have \( m_i(A_p) = \sum |c_{A_p}(x_k)| \) such that \( |x_k| = i \). Since \( i \neq 1, p \), the cyclic structure of \( x_k \) for any \( k \) is \( 1^{r_1}2^{r_2}...l^{r_l} \) where \( r_1, r_2, ..., r_l \) and \( 1, 2, ..., l \) are not equal to \( p \). On the other hand, by Lemma 2.1, \( |c_{A_p}(x_k)| = p!1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \). Then \( p \parallel |c_{A_p}(x_k)| \) for any \( k \). Hence \( p \parallel m_i(A_p) \) and \( m_i(A_p) = p! \cdot \alpha \) where \( \alpha < 1 \). Therefore, \( p \parallel m_i(A_p) \). \( \square \)

Lemma 3.2. \( m_p(G) = m_p(A_p) \).

Proof: Let \( m_p(G) \neq m_p(A_p) \). Since \( \text{nse}(G) = \text{nse}(A_p) \), \( m_p(G) \in \text{nse}(A_p) \). Suppose that there exists \( k \neq p \in \pi_{\epsilon}(G) \) such that \( m_p(G) = m_k(A_p) \). Thus \( m_k(A_p) \equiv -1 \) (mod \( p \)). We know that \( m_k(A_p) = \sum |c_{A_p}(x_i)| \) such that \( |x_i| = k \) for every \( i \). Since \( m_k(A_p) \equiv -1 \) (mod \( p \)), \( (p, m_k(A_p)) = 1 \).

If for every \( i \), the cyclic structure of \( x_i \) is \( 1^{r_1}2^{r_2}...l^{r_l} \) such that \( p \parallel 1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \), then \( |c_{A_p}(x_i)| = 1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \equiv 0 \) (mod \( p \)). Hence \( (p, m_k(A_p)) \neq 1 \), which is a contradiction. Hence \( p \parallel 1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \). Therefore, there exists at least a \( x_i \) with cyclic structure \( 1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \) such that \( r_j = p + t \) for some \( 1 \leq j \leq l \), where \( t \) is a non-negative integer, or one of the numbers \( 1, 2, ..., l \) is equal to \( p \). Thus there exists \( x_i \) such that the cyclic structure of \( x_i \) is \( 1^p \) or \( p! \).

If the cyclic structure of \( x_i \) is \( 1^p \), then \( |x_i| = 1 \), which is a contradiction.

If the cyclic structure of \( x_i \) is \( p! \), then \( |x_i| = k = p \). Therefore, \( m_p(G) = m_p(A_p) \). \( \square \)

Lemma 3.3. If \( i \in \pi_{\epsilon}(A_p) \), \( i \neq 1 \) and \( i \neq p' \), then \( p' \parallel m_i(A_p) \).

Proof: We can prove this lemma as the proof of the Lemma 3.1. \( \square \)

Lemma 3.4. If \( p' \in \pi(G) \), then \( m_{p'}(G) = m_{p'}(A_p) \).

Proof: There exists \( k \in \pi_{\epsilon}(G) \) such that \( p' \parallel (1 + m_k(A_p)) \). We know that \( m_k(A_p) = \sum |c_{A_p}(x_i)| \) such that \( |x_i| = k \). Since \( p' \parallel (1 + m_k(A_p)) \), \( (p', m_k(A_p)) = 1 \). If the cyclic structure of \( x_i \) for any \( i \) is \( 1^{r_1}2^{r_2}...l^{r_l} \) such that \( r_1, r_2, ..., r_l \) and \( 1, 2, ..., l \) are not equal to \( p' \), then \( p' \parallel 1^{r_1}2^{r_2}...l^{r_1}r_1!r_2!...r_l! \). Hence \( p' \parallel |c_{A_p}(x_i)| \) for any \( i \) and therefore \( (p', m_k(A_p)) \neq 1 \), which is a contradiction. Thus there exists \( i \in N \) such that \( r_i = p' \) or one of the numbers \( 1 \) or 2 \( ... \) or \( l \) is equal to \( p' \).

If there exists \( i \in N \) such that \( r_i = p' \), then the cyclic structure of \( x_i \) is \( 1^p 2^1 \). Hence \( x_i \) is an odd permutation, which is a contradiction.
If one of the numbers 1 or 2 ... or \( t \) are equal to \( p' \), then the cyclic structure of \( x_i \) is \( 1^2p'^1 \). Hence \( |x_i| = p' \) and \( k = p' \). Therefore, \( m_{p'}(G) = m_{p'}(A_p) \). 

**Lemma 3.5.** \( |P_p| = p \).

**Proof:** By Lemma 3.1, \( p^2 \nmid m_i(G) \) for any \( i \in \pi_e(G) \). If \( p^3 \in \pi_e(G) \), then by (\( \ast \)), \( \phi(p^3) \mid m_{p^3}(G) \). Thus \( p^2 \mid m_{p^3}(G) \), which is a contradiction. Therefore \( p^3 \not\in \pi_e(G) \). Hence \( \exp(P_p) = p \) or \( p^2 \). We claim that \( \exp(P_p) = p \). Suppose that \( \exp(P_p) = p^2 \). There exists an element of order \( p^2 \) in \( G \) such that \( \phi(p^2) \mid m_{p^2}(G) \). Thus \( p(p - 1) \mid m_{p^2}(G) \).

If \( |P_p| = p^2 \), then \( P_p \) is a cyclic group and \( n_p(G) = m_{p^2}(G)/\phi(p^2) = p(p - 1)/p(p - 1) = 1 \). On the other hand, we know two of the Sylow \( p \)-subgroups might intersect in a subgroup of order \( p \). Therefore, the number of cyclic subgroups of order \( p \) is something between 1 and the number of Sylow \( p \)-subgroups. Since \( m_p(G) = (p - 1)!, (p - 1)! \leq (p - 1)n_p(G) = (p - 1)t. \) Therefore \( n_p(G) = t \geq (p - 2)! \) and \( m_{p^2}(G) \geq p(p - 1)(p - 2)! = p! \), which is a contradiction.

If \( |P_p| = p^s \) where \( s \geq 3 \), then by Lemma 2.4, \( m_{p^2}(G) = p^s \) for some \( l \), which is a contradiction by Lemma 3.1. Thus \( \exp(P_p) = p \).

We obtain that \( |P_p| = 1 + m_p = 1 + (p - 1)! \), which implies that \( |P_p| = p \) or \( |P_p| = p^s \) where \( s \geq 2 \). Consider the case \( |P_p| = p^s \) where \( s \geq 2 \).

We prove that \( 2p \in \pi_e(G) \). Assume that this is false. Then the group \( P_p \) acts fixed-point-freely on the set of elements of order 2, which implies that \( p^s = |P_p| \mid m_2 \). By Lemma 3.1, we get a contradiction.

Therefore \( 2p \in \pi_e(G) \), as required. Since \( \exp(P_p) = p \), \( 2p^2 \not\in \pi_e(G) \), so by Lemma 2.2, \( L_{2p^2}(G) = L_{2p}(G) \), and so \( 2p^2 \mid (1 + m_2 + m_p + m_{2p}) \). Since \( p^2 \mid (1 + m_p) \), \( p^2 \mid (m_2 + m_{2p}) \). But \( m_2 + m_{2p} = p!k \), where \( 0 < k < 1 \), which is a contradiction. Therefore, \( |P_p| = p \). 

**Lemma 3.6.** \( \pi(G) = \pi(A_p) \).

**Proof:** By Lemma 3.5, \( |P_p| = p \). Hence \( (p - 2)! = m_p(G)/\phi(p) = n_p(G) \mid |G| \). Thus \( \pi(A_p) \subseteq \pi(G) \). Now we show that \( \pi(A_p) = \pi(G) \). Let \( r \mid p \) be a prime such that \( r \in \pi(G) \). If \( pr \in \pi_e(G) \), then by Lemma 2.6, \( (r - 1)(p - 1)! \mid m_{pr} \). But \( (r - 1)(p - 1)! > p! \), which is a contradiction. Thus \( pr \not\in \pi_e(G) \). Then the group \( P_r \) acts fixed point freely on the set of elements of order \( p \), and so \( |P_r| \mid (p - 1)! \), a contradiction. Therefore \( r \not\in \pi(G) \) and \( \pi(G) = \pi(A_p) \). 

**Lemma 3.7.** \( G \) has not any element of order \( 2p \) and \( 2p' \).

**Proof:** Suppose that \( G \) has an element of order \( 2p \). By Lemma 2.6, \( (p - 1)! \mid m_{2p}(G) \). On the other hand, by (\( \ast \)) \( 2p \mid (1 + m_2 + m_p + m_{2p}) \). Since \( p \mid (1 + m_p) \) and \( p \mid m_2, p \mid m_{2p}(G) \). Therefore \( p! \mid m_{2p}(G) \), which is a contradiction. Now suppose that \( G \) has an element of order \( 2p' \). By Lemma 2.6, \( p!/2p' \mid m_{2p}(G) \). On the other hand, by (\( \ast \)) \( 2p' \mid (1 + m_2 + m_{p'} + m_{2p'}) \). Since \( p' \mid (1 + m_{p'}) \) and by Lemma 3.3, \( p' \mid m_2, p' \mid m_{2p'}(G) \). Therefore \( p!/2 \mid m_{2p'}(G) \), which is a contradiction.
Lemma 3.8. \( rp, sp' \notin \pi_e(G) \) for every \( r \in \pi(G) \) and \( s \in \pi(G) \{p'\} \).

Proof: The proof of this lemma is completely similar to Lemma 3.7. □

Lemma 3.9. \(|G| = |A_p|\).

Proof: Suppose that \(|A_p| = 2^{k_2}3^{k_3}5^{k_5} \cdots p'p\) where \(k_2, k_3, k_5, \ldots\) are non-negative integers. By Lemma 3.7, \(2^{n_p} \notin \pi_e(G)\), so the group \(P_2\) acts fixed point freely on the set of elements of order \(p'\), and so \(|P_2| \| p'/2^{k_2} = m_{p'}(G)\). Thus \(|P_2| \| 2^{k_2}\). By Lemma 3.8 and arguing as above, \(|P_3| \| 3^{k_3}, |P_5| \| 5^{k_5}, \ldots\). Therefore \(|G| \| p'/2\). On the other hand, \((p - 2)! = n_{p'}(G) \| |G|\) and \(p!/2p'(p' - 1) = n_{p'}(G) \| |G|\). Then the least common multiple of \((p - 2)\) and \(p'/2p'(p' - 1)\) divide the order of \(G\).

Therefore \(p'/2 \| |G|\) and so \(|G| = |A_p|\) or \(|G| = p!/4\). We will show that \(|G| = |A_p|\).

Let \(|G| = p!/4\). By Lemma 3.8, \(t(G) \geq 3\). Hence by Lemma 2.7 and 2.8, \(G\) is neither Frobenius nor 2-Frobenius.

By Lemma 2.9, \(G\) has a normal series \(1 \leq H \leq K \leq G\) such that \(H\) and \(G/K\) are \(\pi_1\)-groups, \(K/H\) is a non-abelian simple group and \(H\) is a nilpotent group. If \(K/H\) has an element of order \(r\) where \(r\) and \(q\) are primes, then \(G\) has also such element. Hence by definition of order components, an odd order component of \(G\) must be an odd order component of \(K/H\). Now we consider the following claims.

Claim 1.

(a) If \(t \in \pi(H)\), then \(t \leq \frac{p+1}{2}\);

(b) If \(K/H\) is isomorphic to a sporadic simple group. Moreover, if \(K/H\) is isomorphic to an alternating simple group, then we show that it is impossible.

(a) if \(t\) divides \(|H|\) where \(\frac{p+1}{2} < t < p\), then since \(H\) is nilpotent subgroup of \(G\) and the order of \(T\), the Sylow \(t\)-subgroup of \(H\), is equal to \(t\). By Lemma 2.10, we must have \(p \| t - 1\), which is impossible. Thus \(|H|\) is not divisible by the primes \(t\) with \(\frac{p+1}{2} < t < p\).

For the subcase (b), we note that if \(H \neq 1\), by nilpotency of \(H\), we may assume that \(H\) is a \(t\)-group for \(t \in \pi_1(G)\).

If \(K/H \cong J_4\), then \(p = 43\). Since \(19 \in \pi(G) \setminus \pi(\text{Aut}(J_4))\), then \(19 \in \pi(H)\). By Lemma 2.10, \(43 \| 19^i - 1\) for \(i = 1\) or 2, which is impossible.

If \(K/H \cong M_{22}\), then \(p = 11\). Since \(5^2 \| |G|\) and \(5 \| |\text{Aut}(M_{22})|\), \(5 \in \pi(H)\). So by Lemma 2.10, we get a contradiction.

If \(K/H\) is isomorphic to other sporadic simple groups we can view a contradiction similarly.

Now let \(K/H\) be isomorphic to an alternating group. By Tables 1 and 2 in [1], \(K/H\) must be isomorphic to \(A_p, p, p - 2\) both primes, since \(|G| = p!/4\), we get a contradiction.

Claim 2. Let \(K/H\) is a simple group of Lie type over finite field GF(q). Then the order of factor group \(G/K\) cannot be divided by primes \(t\), with \(\frac{p+1}{2} < t < p\). Therefore \(t \in \pi(K/H)\). It follows from the proof of Lemma 6 in [5].
Claim 3. $K/H$ cannot be isomorphic to a simple group of Lie type.

By the Claim 2 and Lemma 2.12, we must have $17 \leq p \leq 37$ and $\theta(K/H) \geq 2$. Therefore $K/H$ is isomorphic to one of the following simple groups by Lemma 2.11.

1. $L_2(q)$, where $q \equiv \epsilon \pmod{4}$ for $\epsilon = \pm 1$;
2. $2B_2(q)$, where $q = 2^{2m}+1 > 2$;
3. $2G_2(q)$, where $q = 3^{2m}+1 > 3$;
4. $E_6(q)$ or $E_8(2)$.

Since one of the odd order components of $K/H$ is equal to $p$, by Tables 2 and 3 in [1], we must have:

1. $K/H \cong L_2(17)$, for $p = 17$;
2. $K/H \cong L_2(19)$, $2G_2(27)$ or $2E_6(2)$, for $p = 19$;
3. $K/H \cong L_2(q)$, where $p = q$ is equal to 23, 29 or 31;
4. $K/H \cong L_2(37)$, $2G_2(27)$ or $2E_6(2)$, for $p = 37$.

If $K/H \cong L_2(17)$, then since $13 \notin \pi(L_2(17))$, we get a contradiction by the Claim 2.

If $K/H \cong 2G_2(27)$ for $p = 37$ or $K/H \cong L_2(2)$, where $p = q$ is equal to 23, 29, 31, or 37, we view a contradiction similarly. Let $p = 19$. If $K/H \cong L_2(19)$, again by the Claim 2, we get a contradiction because $17 \notin \pi(L_2(19))$. Since $37 \mid |2G_2(27)|$, we have $K/H \not\cong 2G_2(27)$.

For the case $K/H \cong 2E_6(2)$, we view a contradiction by $2^{36} | 2^2E_6(2)$.

Also if $5 \leq p \leq 13$ we know that $G$ has a normal series $1 \triangleleft H \triangleleft K \triangleleft G$ such that $H$ and $G/K$ are $\pi_1$-groups, $K/H$ is a non-abelian simple group, $G/H \leq \text{Aut}(K/H)$ and $H$ is a nilpotent group. By the Claim 1(b), $K/H$ cannot be isomorphic to a sporadic simple group, and if $K/H$ is isomorphic to an alternating simple group, then we get a contradiction.

Let $K/H$ be isomorphic to the simple group of Lie type $G(q)$ where $q = s^m$ and $s$ is a prime number. Let $p = 5$. We know, 5 is one of the odd order components of $K/H$. So $s = 5$ and $K/H$ be isomorphic to $L_2(5) \cong A_5$, a contradiction.

Let $p = 7$. We know, 7 is one of the odd order components of $K/H$. So $s = 2$, 3 or 7. Then the order of all Sylow $t$-subgroups of $G$ are less than or equal to 64 or 81, respectively. Therefore for both of these cases, $K/H$ is isomorphic to $L_2(7)$ or $L_2(8)$. Let $K/H \cong L_2(7)$ or $L_2(8)$. Since $5 \in \pi(G) \setminus \pi(\text{Aut}(K/H))$, $5 \in \pi(H)$, which it contradicts Claim 1(a). For the cases $p = 11, 13$, we do similarly. Therefore, $|G| = |A_p|$.

Now, we are prepared to prove our main theorem.

**Proof of the main theorem.** We have $|G| = |A_p|$ and $m_p(G) = m_p(A_p)$. Hence the number of Sylow $p$-subgroups of $G$ is equal to the number of Sylow $p$-subgroups of $A_p$. Since $|G| = |A_p|$ and $n_p(G) = n_p(A_p)$, $|N_G(R)| = |N_{A_p}(S)|$ where $R \leq \text{Syl}_p(G)$ and $S \leq \text{Syl}_p(A_p)$. Now by Lemma 2.5, $G \cong A_p$ and the proof is completed.

**Corollary.** Let $G$ be a finite group. If $\text{nse}(G) = \text{nse}(A_p)$ and $|G| = |A_p|$, where $p$ and $p - 2$ are twin primes, then $G \cong A_p$. \hfill \square
A new characterization of $A_p$ with $p$ and $p-2$ are twin primes

**Proof:** This is an immediate consequence of the main theorem.

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**References**


Seyed Sadegh Salehi Amiri (Corresponding Author)  
Department of Mathematics, Babol Branch,  
Islamic Azad University, Babol, Iran  
E-mail address: salehisss@baboliau.ac.ir

and

Alireza Khalili Asboei  
Department of Mathematics, Farhangian University,  
Shariati Mazandaran, Iran