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Multiplicity Results for Nonlocal Elliptic Transmission Problem with Variable Exponent

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abstract: In this paper, a transmission problem given by a system of two nonlinear equations of p(x)-Kirchhoff type with nonstandard growth conditions are studied. Using the mountain pass theorem combined with the Ekeland's variational principle, we obtain at least two distinct, non-trivial weak solutions.

Key Words: Variational method; multiple solutions; nonlocal elliptic transmission problem; Mountain Pass Theorem; Ekeland's principle.

Contents

1. Introduction

Problems related to PDEs involving variable exponents became popular a recently due to their applications and research developments in the modelling of electrorheological fluids, elasticity problems, image processing, mathematical description of the processes filtration of an ideal barotropic gas through a porous medium, etc.; see for example [3,4,9,14,22] and references therein. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field, see [22,23]. The functional analytical tools needed for the analysis have been extensively developed, see [10,11,15] and references therein.

Transmission problems arise in several applications in physics and biology. For instance, one of the important problems of the electrodynamics of solid media is the electromagnetic process research in ferromagnetic media with different dielectric constants. These problems appear as well as in solid mechanics if a body consists of composite materials. We refer the reader to [21] for nonlinear elliptic transmission problems, to [16] for a nonlinear nonlocal elliptic transmission problem. Furthermore, uniqueness and regularity of the solutions to the thermoelastic transmission problem were investigated in [17].

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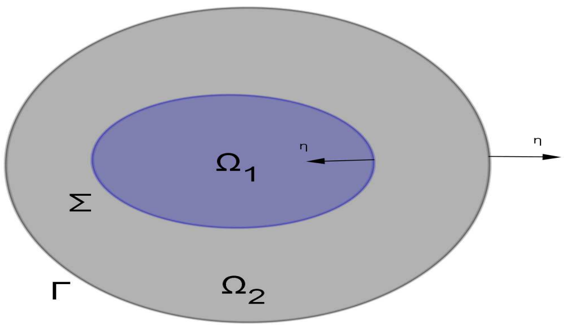
As shown the figure below, let Ω be smooth bounded domain of $\mathbb{R}^n$, $n \geq 2$, and let $\Omega_1 \subset \Omega$ be a subdomain with smooth boundary Σ satisfying $\overline{\Omega}_1 \subset \Omega$. Writing $\Gamma = \partial\Omega$ and $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ we have $\Omega = \overline{\Omega}_1 \cup \Omega_2$ and $\partial\Omega_2 = \Sigma \cup \Gamma$.



In this paper, we are interested in the multiplicity of solutions for nonlocal elliptic systems of gradient type with nonstandard growth conditions. More precisely, we consider the following system

$$(\mathcal{P}) \begin{cases} -M\Big(\int_{\Omega_1} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big) div(|\nabla u|^{p(x)-2}\nabla u) & = & f(x,u) & \text{in} & \Omega_1, \\ -N\Big(\int_{\Omega_2} \frac{1}{p(x)}|\nabla v|^{p(x)}dx\Big) div(|\nabla v|^{p(x)-2}\nabla v) & = & g(x,v) & \text{in} & \Omega_2, \\ v & = & 0 & \text{on} & \Gamma, \end{cases}$$

with the transmission conditions

$$u = v \quad \text{and} \quad M\Big(\int_{\Omega_1} \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\frac{\partial u}{\partial \eta} = N\Big(\int_{\Omega_2} \frac{1}{p(x)}|\nabla v|^{p(x)}dx\Big)\frac{\partial v}{\partial \eta} \quad \text{on } \Sigma. \tag{1.1}$$

where $p \in C(\overline{\Omega})$, η is outward normal to Ω_2 and is inward to Ω_2. M and $N : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions. We confine ourselves to the case where $M = N$ for simplicity. Notice that the results of this paper remain valid for $M \neq N$ by adding some slight changes in the hypothesis $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$ below.

We note that problem $(\mathcal{P})$ with the transmission condition is a generalization of the stationary problem of two wave equations of Kirchhoff type,

$$\begin{cases} u_{tt} - M(\int_{\Omega_1} |\nabla u|^2 dx)\Delta u & = & f(x,u) & \text{in} & \Omega_1, \\ v_{tt} - N(\int_{\Omega_2} |\nabla v|^2 dx)\Delta v & = & g(x,v) & \text{in} & \Omega_2, \end{cases}$$

which models the transverse vibrations of the membrane composed by two different materials in Ω_1 and Ω_2. Controllability and stabilization of transmission problems for the wave equations can be found in [19,20]. We refer the reader to [1] for the stationary problems of Kirchhoff type, to [6] for elliptic equation p-Kirchhoff type, and to [7,8] for $p(x)$-Kirchhoff type equation.

In the sequel, we will assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following assumptions:

(M_1) There exist real numbers $m_1, m_2 > 0$ and $\theta > 0$ such that

$$m_1 t^{\theta-1} \le M(t) \le m_2 t^{\theta-1}, \text{ for all } t \ge 0.$$

(M_2) For all $t \in \mathbb{R}^+$, it holds that

$$\widehat{M}(t) \ge M(t)t,$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$.

Although a natural extension of the theory, the problem addressed here is a natural continuation of recent papers. In [5], the authors treat and show the existence of nontrivial solution in the case $f(x,u) = \lambda_1 |u|^{q(x)-2}u$ and $g(x,v) = \lambda_1 |v|^{q(x)-2}v$ when, $\lambda_1, \lambda_2 > 0$, $q \in C(\overline{\Omega})$ and $1 < q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = \frac{np(x)}{n-p(x)}$ if $p(x) < n$ or $p^*(x) = \infty$ otherwise.

Motivated by above and the ideas introduced in [13], in this work, we will study the existence of multiple solutions for problem ($\mathcal{P}$) in a more general case when the nonlinear terms f and g are defined by

$$f(x,u) = \lambda_1 a(x)|u|^{\alpha(x)-1}u \quad \text{ and } g(x,v) = \lambda_2 b(x)|v|^{\beta(x)-1}v,$$

with $\alpha, \beta \in C_+(\overline{\Omega})$ such that

$$\alpha^- \le \alpha^+ < \theta p^- < \theta p^+ < \beta^- \le \beta^+ < \min(N, \frac{Np^-}{N-p^-}) \tag{1.2}$$

and the following conditions hold:

(A) $a : \overline{\Omega}_1 \to \mathbb{R}$, satisfies $a \in L^{\alpha_0(x)}(\Omega_1)$ and $\alpha_0 \in C_+(\overline{\Omega}_1)$ such that

$$\frac{Np(x)}{Np(x) - \alpha(x)(N-p(x))} < \alpha_0(x) < \frac{p(x)}{p(x)-\alpha(x)} \quad \text{ for all } x \in \overline{\Omega}_1.$$

(B) $b : \overline{\Omega}_2 \to \mathbb{R}$, satisfies $b \in L^{\beta_0(x)}(\Omega_2)$ and $\beta_0 \in C_+(\overline{\Omega}_2)$ such that

$$\frac{p(x)}{p(x)-\beta(x)} < \beta_0(x) < \frac{Np(x)}{Np(x) - \beta(x)(N-p(x))} \quad \text{ for all } x \in \overline{\Omega}_2.$$

where

$$C_+(\overline{\Omega}) := \Big\{h;\ h \in C(\overline{\Omega}) \text{ and } h(x) > 1 \text{ for all } x \in \overline{\Omega}\Big\},$$

and $h^+ := \max_{\overline{\Omega}} h(x)$, $h^- := \min_{\overline{\Omega}} h(x)$, for any $h \in C_+(\overline{\Omega})$.

More precisely, using the mountain pass theorem and the Ekeland's variational principle, we establish that the problem ($\mathcal{P}$) has at least two distinct, non trivial weak solutions provided that $\lambda_1 + \lambda_2 \in (0, \lambda^*)$, $\lambda^* > 0$ is small enough.

Here, any solution of problem $(\mathcal{P})$ with the transmission conditions will belong to the framework of the generalized Sobolev space, which will be briefly described in the following section,

$$E := \Big\{(u,v) \in W^{1,p(x)}(\Omega_1) \times W_\Gamma^{1,p(x)}(\Omega_2) : u = v \text{ on } \Sigma\Big\},$$

where

$$W_\Gamma^{1,p(x)}(\Omega_2) = \Big\{v \in W^{1,p(x)}(\Omega_2) : v = 0 \text{ on } \Gamma\Big\}.$$

Definition 1.1. *We say that $(u,v) \in E$ is a weak solution of $(\mathcal{P})$, if*

$$\begin{aligned} &M\Big(\int_{\Omega_1} \frac{1}{p(x)} |\nabla u|^{p(x)} dx\Big) \int_{\Omega_1} |\nabla u|^{p(x)} \nabla u \nabla \varphi dx \\ &+M\Big(\int_{\Omega_1} \frac{1}{p(x)} |\nabla v|^{p(x)} dx\Big) \int_{\Omega_2} |\nabla v|^{p(x)} \nabla v \nabla \psi dx \\ &= \lambda_1 \int_{\Omega_1} a(x)|u|^{\alpha(x)-1} u\varphi dx + \lambda_2 \int_{\Omega_2} b(x)|v|^{\beta(x)-1} v\psi dx, \end{aligned}$$

for any $(\varphi,\psi) \in E$.

Now, we are ready to state our main result.

Theorem 1.2. *Let us assume that the conditions 1.2, $(\boldsymbol{A}),(\boldsymbol{B}),(\boldsymbol{M}_1)$ and $(\boldsymbol{M}_2)$ are satisfied. Then, there exists $\lambda^* > 0$ such that for any $\lambda_1 + \lambda_2 \in (0,\lambda^*)$, problem $(\mathcal{P})$ has at least two distinct non trivial weak solutions.*

The paper consists of three sections. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Section 3, we give the proof of our main result.

2. Preliminary results and notations

In order to guarantee the integrity of the paper, we recall some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces. For details, we refer to [10,11,15] for the fundamental properties of these spaces.

Let $\sigma : \Omega \to \mathbb{R}$ be a measurable real function such that $\sigma(x) > 0$ for a.e. $x \in \Omega$. For $p \in C_+(\Omega)$, define the weighted variable exponent Lebesgue space

$$L^{p(x)}_{\sigma(x)}(\Omega) = \Big\{u;\ \text{measurable real-valued function and } \int_\Omega \sigma(x)\big|u(x)\big|^{p(x)} dx < \infty\Big\}.$$

Equipped with the so-called Luxemburg norm

$$\big|u\big|_{p(x),\sigma(x),\Omega} := \inf\Big\{\mu > 0 : \quad \int_\Omega \sigma(x)\Big|\frac{u(x)}{\mu}\Big|^{p(x)} dx \le 1\Big\},$$

$L^{p(x)}_{\sigma(x)}(\Omega)$ becomes a separable, reflexive and Banach space. In particular, when $\sigma(x) \equiv 1$ on Ω, $L^{p(x)}_{\sigma(x)}(\Omega)$ is the usual variable exponent Lebesgue space $L^{p(x)}(\Omega)$.

Proposition 2.1. *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{p(x)}+\frac{1}{q(x)}=1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have*

$$|\int_\Omega uvdx| \leq (\frac{1}{p^-}+\frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

An important role in manipulating the generalized Lebesgue spaces is played by the mapping $\rho_{p(x),\sigma(x),\Omega} : L^{p(x)}_{\sigma(x)}(\Omega) \to \mathbb{R}$, called the *modular* of the $L^{p(x)}_{\sigma(x)}(\Omega)$ space, defined by

$$\rho_{p(x),\sigma(x),\Omega}(u) = \int_\Omega \sigma(x)|u|^{p(x)}dx.$$

The following proposition illuminates the close relation between the $|u|_{p(x),\sigma(x),\Omega}$ and the convex modular $\rho_{p(x),\sigma(x),\Omega}$.

Proposition 2.2. *For all* $u_n, u \in L^{p(x)}(\Omega)$, *we have*

1. $|u|_{p(x),\sigma(x),\Omega} = a \iff \rho_{p(x),\sigma(x),\Omega}(\frac{u}{a}) = 1$, *pour* $u \neq 0$ *and* $a > 0$.
2. $|u|_{p(x),\sigma(x),\Omega} > 1(=1;<1) \iff \rho_{p(x),\sigma(x),\Omega}(u) > 1(=1;<1)$.
3. $|u|_{p(x),\sigma(x),\Omega} \to 0($ *resp.* $\to +\infty) \iff \rho_{p(x),\sigma(x),\Omega}(u) \to 0($ *resp.* $\to +\infty)$.
4. $|u|_{p(x),\sigma(x),\Omega} > 1 \Longrightarrow |u|^{p^-}_{p(x),\sigma(x),\Omega} \leq \rho_{p(x),\sigma(x),\Omega}(u) \leq |u|^{p^+}_{p(x),\sigma(x),\Omega}$.
5. $|u|_{p(x),\sigma(x),\Omega} < 1 \Longrightarrow |u|^{p^+}_{p(x),\sigma(x),\Omega} \leq \rho_{p(x),\sigma(x),\Omega}(u) \leq |u|^{p^-}_{p(x),\sigma(x),\Omega}$.

Proposition 2.3. *If* $u, u_n \in L^{p(x)}(\Omega), n = 1, 2, ...,$ *then the following statements are equivalent each other:*

1. $\lim_{n\to\infty} |u_n - u|_{p(x),\sigma(x),\Omega} = 0$,
2. $\lim_{n\to\infty} \rho_{p(x),\sigma(x),\Omega}(u_n - u) = 0$,
3. $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(x),\sigma(x),\Omega}(u_n) = \rho_{p(x),\sigma(x),\Omega}(u)$.

Proposition 2.4. *(see [18]) Let* p *and* r *be measurable functions such that* $p \in L^\infty(\Omega)$ *checking* $1 \leq p(x)r(x) \leq \infty$ *for a.e.* $x \in \Omega$. *Then, for* $u \in L^{r(x)}(\Omega)$ *with* $u \neq 0$, *the following relations hold.*

1. $|u|_{p(x)} > 1 \Longrightarrow |u|^{p^-}_{p(x)r(x)} \leq \left||u|^{p(x)}\right|_{r(x)} \leq |u|^{p^+}_{p(x)r(x)}$.
2. $|u|_{p(x)} < 1 \Longrightarrow |u|^{p^+}_{p(x)r(x)} \leq \left||u|^{p(x)}\right|_{r(x)} \leq |u|^{p^-}_{p(x)r(x)}$.

In particular, if $p(x) = p$ *is a constant, then* $\left||u|^p\right|_{r(x)} = |u|^p_{pr(x)}$.

As in the constant exponent case, for any positive integer k, set

$$W^{k,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):D^{\alpha}u\in L^{p(x)}(\Omega),\ |\alpha|\leq k\}.$$

Endowed with the norm

$$\|u\|_{k,p(x),\Omega}=\sum_{|\alpha|\leq k}|D^{\alpha}u|_{p(x)},$$

the space $W^{k,p(x)}(\Omega)$ becomes a separable, reflexive and Banach space. In $W_0^{1,p(x)}(\Omega)$, which denote the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(x)}(\Omega)$, the Poincaré inequality holds ([10]), that is, there exists a positive constant C such that

$$\|u\|_{1,p(x),\Omega}\leq C|\nabla u|_{p(x),\Omega},\qquad \forall u\in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{p(x),\Omega}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discusion and write $\|u\|_{p(x),\Omega}=|\nabla u|_{p(x),\Omega}$ for simplicity.

Proposition 2.5. *Assume that Ω is bounded, the boundary of Ω possesses the cone property and $p\in C_+(\bar{\Omega})$. If $q\in C+(\bar{\Omega})$ and $q(x)\leq p^*(x)$ $\Big(q(x)<p^*(x)\Big)$ for all $x\in\bar{\Omega}$ then there is a continuous $\big($compact$\big)$ embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.*

Recall the following lemma, see [5, Lemma 2.8], which will permit the variational setting of the problem $(\mathcal{P})$.

Lemma 2.6. *E is a closed subspace of $W^{1,p(x)}(\Omega_1)\times W^{1,p(x)}(\Omega_2)$, and*

$$\|(u,v)\|=\|u\|_{p(x),\Omega_1}+\|v\|_{p(x),\Omega_2}\tag{2.1}$$

defines a norm in E, equivalent to the standard norm of $W^{1,p(x)}(\Omega_1)\times W^{1,p(x)}(\Omega_2)$.

3. Proof of the main result

For simplicity, we use the letters $c, c_i, i=1,2,...$ to denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process. $\langle .,.\rangle$ denote the dual pair.

The energy functional associated to problem $(\mathcal{P})$ is defined as $J:E\to\mathbb{R}$,

$$J(u,v)=\Phi(u,v)-\Psi(u,v),\tag{3.1}$$

where

$$\Phi(u,v)=\widehat{M}\Big(\int_{\Omega_1}\frac{|\nabla u|^{p(x)}}{p(x)}\,dx\Big)+\widehat{M}\Big(\int_{\Omega_2}\frac{|\nabla v|^{p(x)}}{p(x)}\,dx\Big),$$

and

$$\Psi(u,v)=\lambda_1\int_{\Omega_1}\frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx+\lambda_2\int_{\Omega_2}\frac{b(x)}{\beta(x)}|v|^{\beta(x)}dx.$$

In a standard way, it can be proved that $J \in C^1(E, \mathbb{R})$. Moreover, we have, for all $(\varphi, \psi) \in E$,

$$\langle J^{'}(u, v), (\varphi, \psi)\rangle = \langle \Phi^{'}(u, v), (\varphi, \psi)\rangle - \langle \Psi^{'}(u, v), (\varphi, \psi)\rangle,$$

where

$$\begin{aligned}\langle \Phi^{'}(u, v), (\varphi, \psi)\rangle &= M\Big(\int_{\Omega_1} \frac{|\nabla u|^{p(x)}}{p(x)}dx\Big)\int_{\Omega_1} |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx \\ &\quad +M\Big(\int_{\Omega_1} \frac{|\nabla v|^{p(x)}}{p(x)}dx\Big)\int_{\Omega_2} |\nabla v|^{p(x)-2}\nabla v\nabla\psi dx\end{aligned}$$

and

$$\langle \Psi^{'}(u, v), (\varphi, \psi)\rangle = \lambda_1 \int_{\Omega_1} a(x)|u|^{\alpha(x)-1}u\varphi dx + \lambda_2 \int_{\Omega_2} b(x)|v|^{\beta(x)-1}v\psi dx.$$

Thus, weak solutions of problem $(\mathcal{P})$ are exactly the critical points of the functional J. Due to the conditions $(\mathbf{M}_1)$ and (1.2), we can show that J is weakly lower semicontinuous in E.

In order to establish Theorem 1.2, we need the following lemmas which play an important role in our arguments.

Lemma 3.1. *There exist $\lambda^* > 0$ and $\rho, r > 0$ such that for any $\lambda_1 + \lambda_2 \in (0, \lambda^*)$, we have*

$$J(u, v) \geq r, \quad \forall (u, v) \in E, \text{ with } \|(u, v)\| = \rho.$$

Proof. By the lemma 2.6, we have

$$\|u\|_{1,\alpha(x),\Omega_1} + \|v\|_{1,\beta(x),\Omega_2} \leq C_1\|(u, v)\|, \quad \forall (u, v) \in E. \tag{3.2}$$

We fix $\rho \in (0, 1)$ such that $\rho < \frac{1}{C_1}$. Then, the above relation implies

$$\|u\|_{1,\alpha(x),\Omega_1} + \|v\|_{1,\beta(x),\Omega_2} < 1, \quad \forall (u, v) \in E. \tag{3.3}$$

From 1.2 and the conditions $(\mathbf{A})$ and $(\mathbf{B})$, the embeddings from E to the weighted spaces $L^{\alpha(x)}\big(\Omega_1, a(x)\big)$ and $L^{\beta(x)}\big(\Omega_2, b(x)\big)$ are compact, see [18, Theorems 2.7, 2.8]. Moreover, there exist two positive constants c_1 and c_2 such that

$$\int_{\Omega_1} a(x)|u|^{\alpha(x)}dx \leq c_1\Big(\|u\|^{\alpha^-}_{1,\alpha(x),\Omega_1} + \|u\|^{\alpha^+}_{1,\alpha(x),\Omega_1}\Big), \quad \forall u \in W^{1,p(x)}(\Omega_1),$$

and

$$\int_{\Omega_2} b(x)|v|^{\beta(x)}dx \leq c_2\Big(\|v\|^{\beta^-}_{1,\beta(x),\Omega_2} + \|v\|^{\beta^+}_{1,\beta(x),\Omega_2}\Big), \quad \forall v \in W^{1,p(x)}(\Omega_2).$$

Then, by (3.3), for any $(u,v)\in E$, we get

$$\int_{\Omega_1} a(x)|u|^{\alpha(x)}dx+\int_{\Omega_2} b(x)|v|^{\beta(x)}dx\le C_4\Big(\|u\|_{1,\alpha(x),\Omega_1}+\|v\|_{1,\beta(x),\Omega_2}\Big).$$

Hence, from (3.2), we deduce

$$\int_{\Omega_1} a(x)|u|^{\alpha(x)}dx+\int_{\Omega_2} b(x)|v|^{\beta(x)}dx\le C_5\|(u,v)\|. \tag{3.4}$$

Therefore, using $(\mathbf{M}_1)$, $(\mathbf{M}_2)$, (3.4) and by help of the following elementary inequality

$$|a+b|^s\le 2^{s-1}\big(|a|^s+|b|^s\big) \qquad \text{for } a,b\in\mathbb{R}^N, \tag{3.5}$$

the following inequalities hold true:

$$\begin{aligned}
J(u,v) &= \widehat{M}\Big(\int_{\Omega_1}\frac{|\nabla u|^{p(x)}}{p(x)}\,dx\Big)+\widehat{M}\Big(\int_{\Omega_2}\frac{|\nabla v|^{p(x)}}{p(x)}\,dx\Big)\\
&\quad -\lambda_1\int_{\Omega_1}\frac{a(x)}{\alpha(x)}|u|^{\alpha(x)}dx-\lambda_2\int_{\Omega_2}\frac{b(x)}{\beta(x)}|v|^{\beta(x)}dx\\
&\ge m_1\Big(\int_0^{\frac{1}{p^+}\rho_{p(x),\Omega_1}(\nabla u)} t^{\theta-1}dt+\int_0^{\frac{1}{p^+}\rho_{p(x),\Omega_2}(\nabla v)} t^{\theta-1}dt\Big)\\
&\quad -\frac{\lambda_1}{\alpha^-}\int_{\Omega_1} a(x)|u|^{\alpha(x)}dx-\frac{\lambda_2}{\beta^-}\int_{\Omega_2} b(x)|v|^{\beta(x)}dx\\
&\ge \frac{m_1}{\theta(p^+)^\theta}\Big[\big(\int_{\Omega_1}|\nabla u|^{p(x)}dx\big)^\theta+\big(\int_{\Omega_2}|\nabla v|^{p(x)}dx\big)^\theta\Big]-\frac{C_5(\lambda_1+\lambda_2)}{\alpha^-}\|(u,v)\|\\
&\ge \frac{m_1}{\theta(p^+)^\theta}\Big(\|u\|^{\theta p^+}_{p(x),\Omega_1}+\|v\|^{\theta p^+}_{p(x),\Omega_2}\Big)-\frac{C_5(\lambda_1+\lambda_2)}{\alpha^-}\|(u,v)\|\\
&\ge \frac{m_1 2^{1-\theta p^+}}{\theta(p^+)^\theta}\Big(\|u\|_{p(x),\Omega_1}+\|v\|_{p(x),\Omega_2}\Big)^{\theta p^+}-\frac{C_5(\lambda_1+\lambda_2)}{\alpha^-}\|(u,v)\|\\
&\ge \frac{m_1 2^{1-\theta p^+}}{\theta(p^+)^\theta}\|(u,v)\|^{\theta p^+}-\frac{C_5(\lambda_1+\lambda_2)}{\alpha^-}\|(u,v)\|.
\end{aligned}$$

Define

$$\lambda^*=\frac{m_1 2^{1-\theta p^+}\alpha^-\rho^{\theta p^+-1}}{C_5\theta(p^+)^\theta}.$$

Then, by the above inequality, for any $\lambda_1+\lambda_2\in(0,\lambda^*)$ and $(u,v)\in E$ with $\|(u,v)\|=\rho$ there exists $r>0$ such that $J(u,v)\ge r$. The proof of Lemma 3.1 is complete.

Lemma 3.2. *There exist $(\tilde\varphi,\tilde\psi)\in E$, $\tilde\varphi,\tilde\psi\ge 0$ such that*

$$\lim_{t\to\infty} J(t\tilde\varphi,t\tilde\psi)=-\infty.$$

Proof. Let $\tilde{\varphi}, \tilde{\psi} \in C_0^\infty(\Omega), \tilde{\varphi}, \tilde{\psi} \neq 0$ and $t > 1$. By $(\mathbf{M}_1)$, we have

$$\begin{aligned} J(t\tilde{\varphi}, t\tilde{\psi}) &= \widehat{M}\Big(\int_{\Omega_1} \frac{|t\nabla\tilde{\varphi}|^{p(x)}}{p(x)}\,dx\Big) + \widehat{M}\Big(\int_{\Omega_2} \frac{|t\nabla\tilde{\psi}|^{p(x)}}{p(x)}\,dx\Big) \\ &\quad -\lambda_1 \int_{\Omega_1} \frac{a(x)}{\alpha(x)}|t\tilde{\varphi}|^{\alpha(x)}dx - \lambda_2 \int_{\Omega_2} \frac{b(x)}{\beta(x)}|t\tilde{\psi}|^{\beta(x)}dx \\ &\leq \frac{m_2 t^{\theta p^+}}{\theta(p^-)^\theta}\Big(\rho^\theta_{p(x),\Omega_1}(\nabla\tilde{\varphi}) + \rho^\theta_{p(x),\Omega_2}(\nabla\tilde{\psi})\Big) \\ &\quad -\frac{\lambda_1 t^{\alpha^-}}{\alpha^+}\rho_{p(x),\Omega_1}(\tilde{\varphi}) - \frac{\lambda_2 t^{\beta^-}}{\beta^+}\rho_{p(x),\Omega_2}(\tilde{\psi}) \end{aligned}$$

Since $\beta^- > \theta p^+$, we get $J(t\tilde{\varphi}, t\tilde{\psi}) \to -\infty$ as $t \to \infty$. This ends the proof of Lemma 3.3.

Lemma 3.3. *There exist $(\varphi, \psi) \in E$ such that $\varphi, \psi \geq 0$, $\varphi, \psi \neq 0$ and $J(t\varphi, t\psi) < 0$ for all $t > 0$ small enough.*

Proof. Let choose $\varphi, \psi \in C_0^\infty(\Omega)$, $0 \leq \varphi \leq 1$ in Ω_1 and $0 \leq \psi \leq 1$ in Ω_2. Then, for any $t \in (0, 1)$, by $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$ it follows

$$\begin{aligned} J(t\varphi, t\psi) &= \widehat{M}\Big(\int_{\Omega_1} \frac{|t\nabla\varphi|^{p(x)}}{p(x)}\,dx\Big) + \widehat{M}\Big(\int_{\Omega_2} \frac{|t\nabla\psi|^{p(x)}}{p(x)}\,dx\Big) \\ &\quad -\lambda_1 \int_{\Omega_1} \frac{a(x)}{\alpha(x)}|t\varphi|^{\alpha(x)}dx - \lambda_2 \int_{\Omega_2} \frac{b(x)}{\beta(x)}|t\psi|^{\beta(x)}dx \\ &\leq m_2\Big(\int_0^{\frac{1}{p^+}\rho_{p(x),\Omega_1}(\nabla t\varphi)} s^{\theta-1}dt + \int_0^{\frac{1}{p^+}\rho_{p(x),\Omega_2}(\nabla t\psi)} s^{\theta-1}dt\Big) \\ &\quad -\frac{\lambda_1}{\alpha^+}\int_{\Omega_1} a(x)|t\varphi|^{\alpha(x)}dx - \frac{\lambda_2}{\beta^+}\int_{\Omega_2} b(x)|t\psi|^{\beta(x)}dx \\ &\leq \frac{m_2 t^{\theta p^-}}{\theta(p^-)^\theta}\Big(\rho^\theta_{p(x),\Omega_1}(\nabla\varphi) + \rho^\theta_{p(x),\Omega_2}(\nabla\psi)\Big) \\ &\quad -\frac{\lambda_1 t^{\beta^+}}{\beta^+}\rho_{p(x),\Omega_1}(\varphi) - \frac{\lambda_2 t^{\beta^+}}{\beta^+}\rho_{p(x),\Omega_2}(\psi) \end{aligned}$$

Let

$$R_\theta = m_2\Big(\rho^\theta_{p(x),\Omega_1}(\nabla\varphi) + \rho^\theta_{p(x),\Omega_2}(\nabla\psi)\Big). \tag{3.6}$$

and

$$S_{\lambda_1,\lambda_2} = \lambda_1\rho_{p(x),\Omega_1}(\varphi) + \lambda_2\rho_{p(x),\Omega_2}(\psi). \tag{3.7}$$

Then,

$$J(t\varphi, t\psi) \leq \frac{t^{\theta p^-}}{\theta(p^-)^\theta}R_\theta - \frac{t^{\beta^+}}{\beta^+}S_{\lambda_1,\lambda_2}.$$

Therefore, we conclude

$$J(t\varphi, t\psi) < 0,$$

for $0 < t < \sigma^{\frac{1}{\theta p^- - \beta^+}}$ providing that

$$0 < \sigma < \min\Big\{1, \frac{\theta (p^-)^\theta S_{\lambda_1,\lambda_2}}{\beta^+ R_\theta}\Big\}.$$

The proof of Lemma 3.2 is complete.

Lemma 3.4. *The functional J satisfies the Palais-Smale condition in E.*

Proof. Let $\{(u_n, v_n)\} \subset E$ be a sequence such that

$$J(u_n, v_n) \to \overline{c} > 0, \quad J^{'}(u_n, v_n) \to 0 \quad \text{in } E^*. \tag{3.8}$$

where E^* is the dual space of E.

First, we show that $\{(u_n, v_n)\}$ is bounded in E. Assume by contradiction the contrary. Then, passing eventually to a subsequence, still denoted by (u_n, v_n), we may assume that $\|(u_n, v_n)\| \to \infty$. Thus, we may consider that $\|u_n\|_{p(x),\Omega_1}$, $\|v_n\|_{q(x),\Omega_2} > 1$ for any integer n. Using $(\mathbf{M}_1)$, $(\mathbf{M}_2)$ and (3.5), we deduce from

(3.8) that

$$
\begin{aligned}
&\bar{c}+1+\|(u_n,v_n)\| \geq J_\lambda(u_n,v_n)-\frac{1}{\beta^-}\langle J_\lambda^{'}(u_n,v_n),(u_n,v_n)\rangle\\
= \;& \widehat{M}\Big(\int_{\Omega_1}\frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\Big)+\widehat{M}\Big(\int_{\Omega_2}\frac{|\nabla v_n|^{p(x)}}{p(x)}\,dx\Big)-\lambda_1\int_{\Omega_1}\frac{a(x)}{\alpha(x)}|u_n|^{\alpha(x)}dx\\
&-\lambda_2\int_{\Omega_2}\frac{b(x)}{\beta(x)}|v_n|^{\beta(x)}dx-\frac{1}{\beta^-}M\Big(\int_{\Omega_1}\frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\Big)\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx\\
&-\frac{1}{\beta^-}M\Big(\int_{\Omega_2}\frac{|\nabla v_n|^{p(x)}}{p(x)}\,dx\Big)\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx\\
&+\frac{\lambda_1}{\beta^-}\int_{\Omega_1}a(x)|u_n|^{\alpha(x)}dx+\frac{\lambda_2}{\beta^-}\int_{\Omega_2}b(x)|v_n|^{\beta(x)}dx\\
\geq \;& (\frac{1}{p^+}-\frac{1}{\beta^-})M\Big(\int_{\Omega_1}\frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\Big)\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx\\
&+(\frac{1}{p^+}-\frac{1}{\beta^-})M\Big(\int_{\Omega_2}\frac{|\nabla v_n|^{p(x)}}{p(x)}\,dx\Big)\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx\\
&+\lambda_1(\frac{1}{\beta^-}-\frac{1}{\alpha^-})\int_{\Omega_1}a(x)|u_n|^{\alpha(x)}dx\\
\geq \;& \frac{\beta^- - p^+}{(p^+)^\theta\beta^-}\Big[\Big(\int_{\Omega_1}|\nabla u_n|^{p(x)}\,dx\Big)^\theta+\Big(\int_{\Omega_2}|\nabla v_n|^{p(x)}\,dx\Big)^\theta\Big]\\
&-2\lambda_1(\frac{1}{\alpha^-}-\frac{1}{\beta^-})\|u_n\|^{\alpha^+}_{1,p(x),\Omega_1}\\
\geq \;& \frac{\beta^- - p^+}{(p^+)^\theta\beta^-}\Big(\|\nabla u_n\|^{\theta p^-}_{p(x),\Omega_1}+\|\nabla v_n\|^{\theta p^-}_{p(x),\Omega_2}\Big)\\
&-2\lambda_1(\frac{1}{\alpha^-}-\frac{1}{\beta^-})\Big(\|u_n\|_{1,p(x),\Omega_1}+\|v_n\|_{1,p(x),\Omega_2}\Big)^{\alpha^+}\\
\geq \;& 2^{1-\theta p^-}\frac{\beta^- - p^+}{(p^+)^\theta\beta^-}\Big(\|\nabla u_n\|_{p(x),\Omega_1}+\|\nabla v_n\|_{p(x),\Omega_2}\Big)^{\theta p^-}\\
&-2C\lambda_1(\frac{1}{\alpha^-}-\frac{1}{\beta^-})\|(u_n,v_n)\|^{\alpha^+}\\
\geq \;& 2^{1-\theta p^-}\frac{\beta^- - p^+}{(p^+)^\theta\beta^-}\|(u_n,v_n)\|^{\theta p^-}-2C\lambda_1(\frac{1}{\alpha^-}-\frac{1}{\beta^-})\|(u_n,v_n)\|^{\alpha^+}
\end{aligned}
$$

Taking into account of 2.1, since $\alpha^+ < \theta p^-$ and $p^+ < \theta p^+ < \beta^-$, dividing the above inequality by $\|(u_n,v_n)\|$ and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that the sequence $\{(u_n,v_n)\}$ is bounded in E.

Thus, there exists $(u,v) \in E$ such that passing to a subsequence, still denoted by $\{(u_n,v_n)\}$, it converges weakly to (u,v) in E. By (1.2) and the conditions (**A**) and (**B**), the embedding from E to the weighted spaces $L^{\alpha(x)}(\Omega_1,a(x))$ and $L^{\beta(x)}(\Omega_2,b(x))$ are compact. Then, using the Hölder inequalities, Propositions

2.2-2.5, we have

$$\begin{aligned}\Big|\int_{\Omega_1} a(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)dx\Big| &\le \int_{\Omega_1} a(x)|u_n|^{\alpha(x)-1}|u_n-u|dx\\ &\le c_1\Big|\Big(a(x)|u_n|^{\alpha(x)}\Big)^{\frac{\alpha(x)-1}{\alpha(x)}}\Big|_{\alpha'(x)}|u_n-u|_{a(x),\alpha(x),\Omega_1}\\ &\le c_2\Big|a(x)|u_n|^{\alpha(x)}\Big|_{L^1(\Omega_1)}^{\frac{\alpha^+-1}{\alpha^+}}|u_n-u|_{a(x),\alpha(x),\Omega_1}\\ &\le c_3|u_n|_{a(x),\alpha(x),\Omega_1}^{\frac{\alpha^+-1}{\alpha^+}}|u_n-u|_{a(x),\alpha(x),\Omega_1}\\ &\le c_4\|u_n\|^{\frac{\alpha^+-1}{\alpha^+}}|u_n-u|_{a(x),\alpha(x),\Omega_1},\end{aligned}$$

where $\frac{1}{\alpha(x)}+\frac{1}{\alpha'(x)}=1$ for a.e $x\in\Omega$. As $n\to\infty$, we deduce

$$\lim_{n\to\infty}\int_{\Omega_1} a(x)|u_n|^{\alpha(x)-2}u_n(u_n-u)dx=0. \tag{3.9}$$

Similarly, we get

$$\lim_{n\to\infty}\int_{\Omega_2} b(x)|v_n|^{\beta(x)-2}v_n(v_n-v)dx=0. \tag{3.10}$$

On the other hand, by (3.8), we have

$$\lim_{n\to\infty}\langle J^{'}(u_n,v_n),(u_n-u,v_n-v)\rangle=0. \tag{3.11}$$

From (3.9), (3.10) and (3.11), we get

$$\lim_{n\to\infty}\langle \Phi^{'}(u_n,v_n),(u_n-u,v_n-v)\rangle=0. \tag{3.12}$$

Hence,

$$\lim_{n\to\infty} M\Big(\int_{\Omega_1}\frac{|\nabla u_n|^{p(x)}}{p(x)}dx\Big)\int_{\Omega_1}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx=0,$$

and

$$\lim_{n\to\infty} M\Big(\int_{\Omega_2}\frac{|\nabla v_n|^{p(x)}}{p(x)}dx\Big)\int_{\Omega_2}|\nabla v_n|^{p(x)-2}\nabla v_n(\nabla v_n-\nabla v)dx=0.$$

From $(\mathbf{M}_1)$ and $(\mathbf{M}_2)$, it follows that

$$\lim_{n\to\infty}\int_{\Omega_1}|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx=0,$$

and

$$\lim_{n\to\infty}\int_{\Omega_2}|\nabla v_n|^{p(x)-2}\nabla v_n(\nabla v_n-\nabla v)dx=0.$$

Eventually, by [12, Theorem 3.11], we get that (u_n,v_n) converges strongly to (u_1,v_1) in E, so we conclude that functional J satisfies the Palais-Smale condition.

Proof of Theorem 1.2. By Lemmas 3.1, 3.2 and 3.4, all assumptions of the mountain pass theorem in [2] are satisfied. Then, we deduce (u_1, v_1) as a nontrivial critical point of the functional J with $J(u_1, v_1) = c$ and thus a nontrivial weak solution of problem $(\mathcal{P})$.

Now, we prove that there exists a second nontrivial weak solution $(u_2, v_2) \in E$ such that $(u_2, v_2) \neq (u_1, v_1)$. Indeed, by Lemma 3.3, we have

$$\inf_{\partial B_\rho(0)} J \geq r > 0 \quad \text{and} \quad \inf_{\overline{B_\rho(0)}} J < 0.$$

Let us choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{\partial B_\rho(0)} J - \inf_{\overline{B_\rho(0)}} J. \tag{3.13}$$

Therefore, by applying the Ekeland's variational principle to the functional $J : \overline{B_\rho(0)} \to \mathbb{R}$, there exists $(u_\epsilon, v_\epsilon) \in \overline{B_\rho(0)}$ such that

$$J(u_\epsilon, v_\epsilon) < \inf_{\overline{B_\rho(0)}} J + \epsilon,$$

$$J(u_\epsilon, v_\epsilon) < J(u, v) + \epsilon \|(u - u_\epsilon, v - v_\epsilon)\|, \quad u \neq u_\epsilon, v \neq v_\epsilon.$$

Thus, by (3.13), it follows that $J(u_\epsilon, v_\epsilon) < \inf_{\partial B_\rho(0)} J$ and so, $(u_\epsilon, v_\epsilon) \in B_\rho(0)$.

Now, let us define $I : \overline{B_\rho(0)} \to \mathbb{R}$ by $I(u, v) = J(u, v) + \epsilon \|(u - u_\epsilon, v - v_\epsilon)\|$. It is easy to see that (u_ϵ, v_ϵ) is a minimum point of I, and thus

$$\frac{I(u_\epsilon + t.\zeta, v_\epsilon + t.\xi) - I(u_\epsilon, v_\epsilon)}{t} \geq 0,$$

for $t > 0$ small enough and any $(\zeta, \xi) \in B_\rho(0)$. The above expression shows that

$$\frac{J(u_\epsilon + t.\zeta, v_\epsilon + t.\xi) - J(u_\epsilon, v_\epsilon)}{t} + \epsilon \|(\zeta, \xi)\| \geq 0.$$

Letting $t \to 0^+$, we deduce that

$$\langle J^{'}(u_\epsilon, v_\epsilon), (u, v) \rangle \geq -\epsilon \|(u, v)\|.$$

It should be noticed that $-(u, v)$ also belongs to $B_\rho(0)$, so replacing (u, v) by $-(u, v)$, we get

$$\langle J^{'}(u_\epsilon, v_\epsilon), (u, v) \rangle \leq \epsilon \|(u, v)\|,$$

which helps us to deduce that $\|J^{'}(u_\epsilon, v_\epsilon)\|_{E^*} \leq \epsilon$.

Therefore, there exists a sequence $\{(u_n, v_n)\} \subset B_\rho(0)$ such that

$$J(u_n, v_n) \to \underline{c} := \inf_{\overline{B_\rho(0)}} J < 0, \quad \text{and } J^{'}(u_n, v_n) \to 0 \quad \text{in } E^* \quad \text{as } n \to \infty. \tag{3.14}$$

From Lemma 3.4, the sequence $\{(u_n, v_n)\}$ converges strongly to (u_2, v_2) as $n \to \infty$. Moreover, since $J \in C^1(E, \mathbb{R})$, by (3.14) it follows that $J(u_2, v_2) = \underline{c}$ and $J^{'}(u_2, v_2) = 0$. Thus, (u_2, v_2) is a non trivial weak solution of problem $(\mathcal{P})$.

Finally, we point out the fact that $(u_1, v_1) \neq (u_2, v_2)$ since $J(u_1, v_1) = \overline{c} > 0 > \underline{c} = J(u_2, v_2)$. The proof of Theorem 1.2 is complete.

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