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μ - k - Connectedness in GTS

Shyamapada Modak and Takashi Noiri

ABSTRACT: Császár [4] introduced μ - semi - open sets, μ - preopen sets, μ - α - open sets and μ - β - open sets in a GTS (X, μ). By using the μ - σ - closure, μ - π - closure, μ - α - closure and μ - β - closure in (X, μ), we introduce and investigate the notions μ - k - separated sets and μ - k - connected sets in (X, μ).

Key Words: μ - k - separated, μ - k - connected, μ - k - component.

Contents

1. Introduction and preliminaries

Let X be a set and exp X denote the power set of X. We call a class $\mu \subset$exp X a generalized topology [2] (briefly GT) if $\emptyset \in \mu$ and any union of elements of μ belongs to μ. A set with GT is called a generalized topological space (briefly GTS). For a GTS (X, μ), the elements of μ are called μ - open sets and the complements of μ - open sets are called μ - closed sets. For $A \subset X$, we denote by $c_\mu(A)$ the intersection of all μ - closed sets containing A and $i_\mu(A)$ the union of all μ - open sets contained in A. Then we recall that $i_\mu(i_\mu(A)) = i_\mu(A)$, $c_\mu(c_\mu(A)) = c_\mu(A)$ and $i_\mu(A) = X - c_\mu(X - A)$. Also, we consider by [1] that $i_\mu(c_\mu(i_\mu(c_\mu(A)))) = i_\mu(c_\mu(A))$ and $c_\mu(i_\mu(c_\mu(i_\mu(A)))) = c_\mu(i_\mu(A))$. A set $A \subset X$ is said to be be μ - semi - open (resp. μ - preopen, μ - α - open, μ - β - open) [4] if $A \subset c_\mu(i_\mu(A))$ (resp. $A \subset i_\mu(c_\mu(A))$, $A \subset i_\mu(c_\mu(i_\mu(A)))$, $A \subset c_\mu(i_\mu(c_\mu(A))))$. We denote by $\sigma(\mu)$ (resp. $\pi(\mu)$, $\alpha(\mu)$, $\beta(\mu)$) the class of all μ - semi - open sets (resp. μ - preopen sets, μ - α - open sets, μ - β - open sets). The complement of a μ - α - open (resp. μ - σ - open, μ - π - open, μ - β - open) set is said to be μ - α - closed (resp. μ - σ - closed, μ - π - closed, μ - β - closed) [5]. $i_\alpha(A)$ (resp. $i_\sigma(A)$, $i_\pi(A)$, $i_\beta(A)$) denotes the union of μ - α - open (resp. μ - σ - open, μ - π - open, μ - β - open) sets included in A, and $c_\alpha(A)$ (resp. $c_\sigma(A)$, $c_\pi(A)$, $c_\beta(A)$) [5] denotes the intersection of μ - α - closed (resp. μ - σ - closed, μ - π - closed, μ - β - closed) sets including A.

Obviously $\mu \subset \alpha(\mu) \subset \sigma(\mu) \subset \beta(\mu)$ and $\alpha(\mu) \subset \pi(\mu) \subset \beta(\mu)$.

Given U, $V \subset X$, let us say U and V are γ - separated [3] if $c_\mu(U) \cap V = c_\mu(V) \cap U = \emptyset$.
Let us say that a set $S \subset X$ is γ - connected if $S = U \cup V$, U and V are γ - separated imply $U = \emptyset$ or $V = \emptyset$. The space X is said to be γ - connected if it is a γ - connected subset of itself (here space X means GTS (X, μ)).

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The purpose of this paper is to introduce and investigate the notions of μ - k - connected sets by using $c_\mu(A)$, $c_\alpha(A)$, $c_\sigma(A)$, $c_\pi(A)$ and $c_\beta(A)$ in GTS (X, μ).

2. μ - k - separated sets

Definition 2.1. *Let (X, μ) be a GTS. Two nonempty subsets U, V of X are said to be μ - k - separated if $c_\mu(U) \cap c_k(V) = \emptyset = c_k(U) \cap c_\mu(V)$, where $k = \alpha$, σ, π or β.*

If we assign the values $k = \sigma$, π, α, β, then we get different types μ - k - separated sets.

Observe that two μ - k - separated sets are disjoint. Moreover, if U and V are μ - k - separated, $U' \subset U$, $V' \subset V$, then U' and V' are μ - k - separated as well. Again every μ - k - separated sets is a γ - separated set.

From the above definition we obtain the following diagram:

DIAGRAM I

$$\begin{array}{ccccc}
 & & \mu\text{-}\sigma\text{-}separated & \Longleftarrow & \mu\text{-}\alpha\text{-}separated \\
 & & \Downarrow & & \Downarrow \\
\gamma\text{-}separated & \Longleftarrow & \mu\text{-}\beta\text{-}separated & \Longleftarrow & \mu\text{-}\pi\text{-}separated
\end{array}$$

Definition 2.2. *A subset A of a GTS X is said to be μ - k - connected if A is not the union of two μ - k - separated sets in X.*

From the above definition for a subset of a GTS the following diagram holds:

DIAGRAM II

$$\begin{array}{ccccc}
\gamma\text{-}connected & \Longrightarrow & \mu\text{-}\beta\text{-}connected & \Longrightarrow & \mu\text{-}\pi\text{-}connected \\
 & & \Downarrow & & \Downarrow \\
 & & \mu\text{-}\sigma\text{-}connected & \Longrightarrow & \mu\text{-}\alpha\text{-}connected
\end{array}$$

In the sequel, a GTS is briefly called a space.

Theorem 2.3. *Let X be a space. If A is a μ - k - connected subset of X and H, G are μ - k - separated subsets of X with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.*

Proof: Let A be a μ - k - connected set. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $c_k(A \cap G) \cap c_\mu(A \cap H) \subset c_k(G) \cap c_\mu(H) = \emptyset$ and $c_\mu(A \cap G) \cap c_k(A \cap H) \subset c_\mu(G) \cap c_k(H) = \emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then A is not μ - k - connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. $\square$

Theorem 2.4. *If A and B are μ - k - connected sets of a space X such that A and B are not μ - k - separated, then $A \cup B$ is μ - k - connected.*

Proof: Let A and B be μ - k - connected sets in X. Suppose $A \cup B$ is not μ - k - connected. Then, there exist two nonempty μ - k - separated sets G and H such that $A \cup B = G \cup H$. Since A and B are μ - k - connected, by Theorem 2.3, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now if $A \subset G$ and $B \subset H$, then $c_\mu(A) \cap c_k(B) \subset c_\mu(G) \cap c_k(H) = \emptyset$ and $c_k(A) \cap c_\mu(B) \subset c_k(G) \cap c_\mu(H) = \emptyset$. Thus, A and B are μ - k - separated, which is a contradiction. In case $B \subset G$ and $A \subset H$ a contradiction is similarly shown. Hence, $A \cup B$ is μ - k - connected. □

Theorem 2.5. *If $\{M_i : i \in I\}$ is a nonempty family of μ - k - connected sets of a space X, with $\bigcap_{i \in I} M_i \neq \emptyset$, then $\bigcup_{i \in I} M_i$ is μ - k - connected.*

Proof: Suppose $\bigcup_{i \in I} M_i$ is not μ - k - connected. Then we have $\bigcup_{i \in I} M_i = H \cup G$, where H and G are nonempty μ - k - separated sets in X. Since $\bigcap_{i \in I} M_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \bigcup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then M_i and H intersect for each $i \in I$. By Theorem 2.3, $M_i \subset H$ or $M_i \subset G$. Since H and G are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\bigcup_{i \in I} M_i \subset H$. This implies that G is empty. This is a contradiction. Suppose that $x \in G$. By the similar way, we have that H is empty. This is a contradiction. Thus, $\bigcup_{i \in I} M_i$ is μ - k - connected. □

Theorem 2.6. *Let X be a space, $\{A_\alpha : \alpha \in \triangle\}$ be a family of μ - k - connected sets and A be a μ - k - connected set. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \triangle$, then $A \cup (\cup_{\alpha \in \triangle} A_\alpha)$ is μ - k - connected.*

Proof: Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \triangle$, by Theorem 2.5, $A \cup A_\alpha$ is μ - k - connected for each $\alpha \in \triangle$. Moreover, $A \cup (\cup A_\alpha) = \cup (A \cup A_\alpha)$ and $\cap (A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 2.5, $A \cup (\cup A_\alpha)$ is μ - k - connected. □

Theorem 2.7. *If A is a μ - k - connected subset of a space X and $A \subset B \subset c_k(A)$, then B is also a μ - k - connected subset of X.*

Proof: Suppose B is not a μ - k - connected subset of X then there exist μ - k - separated sets H and G such that $B = H \cup G$. This implies that H and G are nonempty and $c_k(G) \cap c_\mu(H) = \emptyset = c_\mu(G) \cap c_k(H)$. By Theorem 2.3, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $c_k(A) \subset c_k(H)$ and $c_\mu(G) \cap c_k(A) \subset c_\mu(G) \cap c_k(H) = \emptyset$. This implies that $G \subset B \subset c_k(A)$ and $G = c_k(A) \cap G \subset c_k(A) \cap c_\mu(G) = \emptyset$. Thus G is an empty set. Since G is nonempty, this is a contradiction. Hence, B is μ - k - connected. □

Corollary 2.8. *If A is a μ - k - connected subset of X, then $c_k(A)$ is also a μ - k - connected subset of X.*

Definition 2.9. *Let X be a space and $x \in X$. Then union of all μ - k - connected subsets of X containing x is called the μ - k - component of X containing x.*

Theorem 2.10. *1. The set of all distinct μ - k - components of a space X forms a partition of X.*
2. Each μ - k - component of a space X is a k - closed set.

Proof: The proof of (2) follows from Corollary 2.8. □

Now we recall the following definition from [2] and [5]:

Definition 2.11. *Let (X, μ) and (X, μ') be GTS's. Then a function $f : X \to X'$ is said to be (μ, μ') - continuous if $f^{-1}(V)$ is μ - open set in X for every μ' - open set of X'.*

Theorem 2.12. *The (μ, μ') - continuous image of a γ - connected space is a μ - k - connected space.*

Proof: The proof is obvious from the Theorem 2.2 of [3] and the DIAGRAM II. □

Definition 2.13. *([6]) Let (X, μ) be a GTS and $G \subset X$.*
(1) G is called $\mu -$ dense if $c\mu(G) = X$.
(2) (X, μ) is called hyperconnected if G is $\mu -$dense for every $\mu -$open subset $G = \emptyset$ of (X, μ).

Remark 2.14. *([6]) For a GTS (X, μ), the following holds: (X, μ) is hyperconnected $\to$ (X, μ) is connected. This implication is not reversible as shown in [6].*

Theorem 2.15. *([6]) Let (X, μ) be a GTS. The following properties are equivalent:*
(1) (X, μ) is hyperconnected,
(2) $G \cap H = \emptyset$ for every nonempty $\mu -$ open subsets G and H of (X, μ).

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Shyamapada Modak
Department of Mathematics, University of Gour Banga, P.O. Mokdumpur,
Malda - 732103, India,
E-mail address: spmodak2000@yahoo.co.in

and

Takashi Noiri
2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumomoto-ken, 869-5142 JAPAN
E-mail address: t.noiri@nifty.com