\( \mu - k \)-Connectedness in GTS

Shyamapada Modak and Takashi Noiri

ABSTRACT: Császár [4] introduced \( \mu \)-semi-open sets, \( \mu \)-preopen sets, \( \mu \)-\( \alpha \)-open sets and \( \mu \)-\( \beta \)-open sets in a GTS \( (X, \mu) \). By using the \( \mu \)-\( \sigma \)-closure, \( \mu \)-\( \pi \)-closure, \( \mu \)-\( \alpha \)-closure and \( \mu \)-\( \beta \)-closure in \( (X, \mu) \), we introduce and investigate the notions \( \mu - k \)-separated sets and \( \mu - k \)-connected sets in \( (X, \mu) \).

Key Words: \( \mu - k \)-separated, \( \mu - k \)-connected, \( \mu - k \)-component.

Contents

1 Introduction and preliminaries 159

2 \( \mu - k \)-separated sets 160

1. Introduction and preliminaries

Let \( X \) be a set and \( \exp X \) denote the power set of \( X \). We call a class \( \mu \subset \exp X \) a generalized topology \([2]\) (briefly GTS) if \( \emptyset \in \mu \) and any union of elements of \( \mu \) belongs to \( \mu \). A set with GT is called a generalized topological space (briefly GTS).

For a GTS \( (X, \mu) \), the elements of \( \mu \) are called \( \mu \)-open sets and the complements of \( \mu \)-open sets are called \( \mu \)-closed sets. For \( A \subset X \), we denote by \( c_{\mu}(A) \) the intersection of all \( \mu \)-closed sets containing \( A \) and \( i_{\mu}(A) \) the union of all \( \mu \)-open sets contained in \( A \). Then we recall that \( i_{\mu}(i_{\mu}(A)) = i_{\mu}(A) \), \( c_{\mu}(c_{\mu}(A)) = c_{\mu}(A) \) and \( i_{\mu}(i_{\mu}(A)) = c_{\mu}(A) \). A set \( A \subset X \) is said to be \( \mu \)-semi-open (resp. \( \mu \)-preopen, \( \mu \)-\( \alpha \)-open, \( \mu \)-\( \beta \)-open) if \( A \subset c_{\mu}(i_{\mu}(A)) \) (resp. \( A \subset i_{\mu}(c_{\mu}(A)), A \subset i_{\mu}(i_{\mu}(A)) \)). We denote by \( \sigma(\mu) \) (resp. \( \pi(\mu), \alpha(\mu), \beta(\mu) \)) the class of all \( \mu \)-semi-open sets (resp. \( \mu \)-preopen sets, \( \mu \)-\( \alpha \)-open sets, \( \mu \)-\( \beta \)-open sets). The complement of a \( \mu \)-\( \alpha \)-open (resp. \( \mu \)-\( \sigma \)-open, \( \mu \)-\( \pi \)-open, \( \mu \)-\( \beta \)-open) set is said to be \( \mu \)-\( \alpha \)-closed (resp. \( \mu \)-\( \sigma \)-closed, \( \mu \)-\( \pi \)-closed, \( \mu \)-\( \beta \)-closed) \([5]\). \( i_{\mu}(A) \) (resp. \( i_{\sigma}(A), i_{\pi}(A), i_{\beta}(A) \)) denotes the union of \( \mu \)-\( \alpha \)-open (resp. \( \mu \)-\( \sigma \)-open, \( \mu \)-\( \pi \)-open, \( \mu \)-\( \beta \)-open) sets included in \( A \) and \( c_{\alpha}(A) \) (resp. \( c_{\sigma}(A), c_{\pi}(A), c_{\beta}(A) \)) \([5]\) denotes the intersection of \( \mu \)-\( \alpha \)-closed (resp. \( \mu \)-\( \sigma \)-closed, \( \mu \)-\( \pi \)-closed, \( \mu \)-\( \beta \)-closed) sets including \( A \).

Obviously \( \mu \subset \alpha(\mu) \subset \sigma(\mu) \subset \beta(\mu) \) and \( \alpha(\mu) \subset \pi(\mu) \subset \beta(\mu) \).

Given \( U, V \subset X \), let us say \( U \) and \( V \) are \( \gamma \)-separated \([3]\) if \( c_{\mu}(U) \cap V = c_{\mu}(V) \cap U = \emptyset \).

Let us say that a set \( S \subset X \) is \( \gamma \)-connected if \( S = U \cup V, U \) and \( V \) are \( \gamma \)-separated imply \( U = \emptyset \) or \( V = \emptyset \). The space \( X \) is said to be \( \gamma \)-connected if it is a \( \gamma \)-connected subset of itself (here space \( X \) means GTS \( (X, \mu) \)).

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The purpose of this paper is to introduce and investigate the notions of $\mu - k$ - connected sets by using $c_\mu(A)$, $c_\sigma(A)$, $c_\pi(A)$ and $c_\beta(A)$ in GTS $(X, \mu)$.

2. $\mu - k$ - separated sets

**Definition 2.1.** Let $(X, \mu)$ be a GTS. Two nonempty subsets $U, V$ of $X$ are said to be $\mu - k$ - separated if $c_\mu(U) \cap c_k(V) = \emptyset = c_k(U) \cap c_\mu(V)$, where $k = \alpha, \sigma, \pi$ or $\beta$.

If we assign the values $k = \sigma, \pi, \alpha, \beta$, then we get different types $\mu - k$ - separated sets.

Observe that two $\mu - k$ - separated sets are disjoint. Moreover, if $U$ and $V$ are $\mu - k$ - separated, $U' \subset U$, $V' \subset V$, then $U'$ and $V'$ are $\mu - k$ - separated as well.

Again every $\mu - k$ - separated sets is a $\gamma$ - separated set.

From the above definition we obtain the following diagram:

**DIAGRAM I**

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\begin{array}{ccc}
\mu-\sigma\text{-separated} & \rightarrow & \mu-\alpha\text{-separated} \\
\downarrow & & \downarrow \\
\gamma\text{-separated} & \rightarrow & \mu-\beta\text{-separated} \\
& & \downarrow \\
& & \mu-\pi\text{-separated}
\end{array}
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**Definition 2.2.** A subset $A$ of a GTS $X$ is said to be $\mu - k$ - connected if $A$ is not the union of two $\mu - k$ - separated sets in $X$.

From the above definition for a subset of a GTS the following diagram holds:

**DIAGRAM II**

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\begin{array}{ccc}
\gamma\text{-connected} & \rightarrow & \mu-\beta\text{-connected} \\
& & \downarrow \\
& & \mu-\pi\text{-connected} \\
\downarrow & & \downarrow \\
\mu-\sigma\text{-connected} & \rightarrow & \mu-\alpha\text{-connected}
\end{array}
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In the sequel, a GTS is briefly called a space.

**Theorem 2.3.** Let $X$ be a space. If $A$ is a $\mu - k$ - connected subset of $X$ and $H, G$ are $\mu - k$ - separated subsets of $X$ with $A \subset H \cup G$, then either $A \subset H$ or $A \subset G$.

**Proof:** Let $A$ be a $\mu - k$ - connected set. Let $A \subset H \cup G$. Since $A = (A \cap H) \cup (A \cap G)$, then $c_k(A \cap G) \cap c_\mu(A \cap H) \subset c_k(G) \cap c_\mu(H) = \emptyset$ and $c_\mu(A \cap G) \cap c_k(A \cap H) = \emptyset$, $c_\mu(A \cap G) \cap c_k(H) = \emptyset$. Suppose $A \cap H$ and $A \cap G$ are nonempty. Then $A$ is not $\mu - k$ - connected. This is a contradiction. Thus, either $A \cap H = \emptyset$ or $A \cap G = \emptyset$. This implies that $A \subset H$ or $A \subset G$. □
Theorem 2.4. If $A$ and $B$ are $\mu$ - $k$ - connected sets of a space $X$ such that $A$ and $B$ are not $\mu$ - $k$ - separated, then $A \cup B$ is $\mu$ - $k$ - connected.

Proof: Let $A$ and $B$ be $\mu$ - $k$ - connected sets in $X$. Suppose $A \cup B$ is not $\mu$ - $k$ - connected. Then, there exist two nonempty $\mu$ - $k$ - separated sets $G$ and $H$ such that $A \cup B = G \cup H$. Since $A$ and $B$ are $\mu$ - $k$ - connected, by Theorem 2.3, either $A \subset G$ and $B \subset H$ or $B \subset G$ and $A \subset H$. Now if $A \subset G$ and $B \subset H$, then $\cap_{\mu}(A) \cap \cap_{\mu}(B) \subset \cap_{\mu}(G) \cap \cap_{\mu}(H) = \emptyset$ and $\cap_{\mu}(A) \cap \cap_{\mu}(B) \subset \cap_{\mu}(G) \cap \cap_{\mu}(H) = \emptyset$. Thus, $A$ and $B$ are $\mu$ - $k$ - separated, which is a contradiction. In case $B \subset G$ and $A \subset H$ a contradiction is similarly shown. Hence, $A \cup B$ is $\mu$ - $k$ - connected. □

Theorem 2.5. If $\{M_i : i \in I\}$ is a nonempty family of $\mu$ - $k$ - connected sets of a space $X$, with $\cap_{i \in I} M_i \neq \emptyset$, then $\cup_{i \in I} M_i$ is $\mu$ - $k$ - connected.

Proof: Suppose $\cup_{i \in I} M_i$ is not $\mu$ - $k$ - connected. Then we have $\cup_{i \in I} M_i = H \cup G$, where $H$ and $G$ are nonempty $\mu$ - $k$ - separated sets in $X$. Since $\cap_{i \in I} M_i \neq \emptyset$, we have a point $x \in \cap_{i \in I} M_i$. Since $x \in \cup_{i \in I} M_i$, either $x \in H$ or $x \in G$. Suppose that $x \in H$. Since $x \in M_i$ for each $i \in I$, then $M_i$ and $H$ intersect for each $i \in I$. By Theorem 2.3, $M_i \subset H$ or $M_i \subset G$. Since $H$ and $G$ are disjoint, $M_i \subset H$ for all $i \in I$ and hence $\cup_{i \in I} M_i \subset H$. This implies that $G$ is empty. This is a contradiction.

Suppose that $x \in G$. By the similar way, we have that $H$ is empty. This is a contradiction. Thus, $\cup_{i \in I} M_i$ is $\mu$ - $k$ - connected. □

Theorem 2.6. Let $X$ be a space, $\{A_\alpha : \alpha \in \Delta\}$ be a family of $\mu$ - $k$ - connected sets and $A$ be a $\mu$ - $k$ - connected set. If $A \cap A_\alpha \neq \emptyset$ for every $\alpha \in \Delta$, then $A \cup (\cup_{\alpha \in \Delta} A_\alpha)$ is $\mu$ - $k$ - connected.

Proof: Since $A \cap A_\alpha \neq \emptyset$ for each $\alpha \in \Delta$, by Theorem 2.5, $A \cup A_\alpha$ is $\mu$ - $k$ - connected for each $\alpha \in \Delta$. Moreover, $A \cup (\cup A_\alpha) = (A \cup A_\alpha)$ and $\cap (A \cup A_\alpha) \supset A \neq \emptyset$. Thus by Theorem 2.5, $A \cup (\cup A_\alpha)$ is $\mu$ - $k$ - connected. □

Theorem 2.7. If $A$ is a $\mu$ - $k$ - connected subset of a space $X$ and $A \subset B \subset c_k(A)$, then $B$ is also a $\mu$ - $k$ - connected subset of $X$.

Proof: Suppose $B$ is not a $\mu$ - $k$ - connected subset of $X$ then there exist $\mu$ - $k$ - separated sets $H$ and $G$ such that $B = H \cup G$. This implies that $H$ and $G$ are nonempty and $c_k(G) \cap c_k(H) = \emptyset = c_\mu(G) \cap c_\mu(H)$. By Theorem 2.3, we have that either $A \subset H$ or $A \subset G$. Suppose that $A \subset H$. Then $c_k(A) \subset c_k(H)$ and $c_\mu(G) \cap c_k(A) \subset c_\mu(G) \cap c_k(H) = \emptyset$. This implies that $G \subset B \subset c_k(A)$ and $G = c_k(A) \cap G \subset c_k(A) \cap c_\mu(G) = \emptyset$. Thus $G$ is an empty set. Since $G$ is nonempty, this is a contradiction. Hence, $B$ is $\mu$ - $k$ - connected. □

Corollary 2.8. If $A$ is a $\mu$ - $k$ - connected subset of $X$, then $c_k(A)$ is also a $\mu$ - $k$ - connected subset of $X$. 


Definition 2.9. Let $X$ be a space and $x \in X$. Then union of all $\mu$-k-connected subsets of $X$ containing $x$ is called the $\mu$-k-component of $X$ containing $x$.

Theorem 2.10. 1. The set of all distinct $\mu$-k-components of a space $X$ forms a partition of $X$.
2. Each $\mu$-k-component of a space $X$ is a $k$-closed set.

Proof: The proof of (2) follows from Corollary 2.8. □

Now we recall the following definition from [2] and [5]:

Definition 2.11. Let $(X, \mu)$ and $(X, \mu')$ be GTS’s. Then a function $f : X \to X'$ is said to be $(\mu, \mu')$-continuous if $f^{-1}(V)$ is $\mu$-open set in $X$ for every $\mu'$-open set of $X'$.

Theorem 2.12. The $(\mu, \mu')$-continuous image of a $\gamma$-connected space is a $\mu$-k-connected space.

Proof: The proof is obvious from the Theorem 2.2 of [3] and the DIAGRAM II. □

Definition 2.13. ([6]) Let $(X, \mu)$ be a GTS and $G \subset X$.
1) $G$ is called $\mu$-dense if $c_\mu(G) = X$.
2) $(X, \mu)$ is called hyperconnected if $G$ is $\mu$-dense for every $\mu$-open subset $G = \emptyset$ of $(X, \mu)$.

Remark 2.14. ([6]) For a GTS $(X, \mu)$, the following holds: $(X, \mu)$ is hyperconnected $\rightarrow$ $(X, \mu)$ is connected. This implication is not reversible as shown in [6].

Theorem 2.15. ([6]) Let $(X, \mu)$ be a GTS. The following properties are equivalent:
1) $(X, \mu)$ is hyperconnected,
2) $G \cap H = \emptyset$ for every nonempty $\mu$-open subsets $G$ and $H$ of $(X, \mu)$.

References
μ - k - Connectedness in GTS

Shyamapada Modak
Department of Mathematics, University of Gour Banga, P.O. Mokdumpur,
Malda - 732103, India,
E-mail address: spmodak2000@yahoo.co.in

and

Takashi Noiri
2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142 JAPAN
E-mail address: t.noiri@nifty.com