



Analytical solution of settling behavior of a particle in incompressible Newtonian fluid by using Parameterized Perturbation Method

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ABSTRACT: The problem of solid particle settling is a well known problem in the field of fluid mechanics. The parametrized Perturbation Method is applied to analytically solve the unsteady motion of a spherical particle falling in a Newtonian fluid using the drag of the form given by Oseen/Ferreira, for a range of Reynolds numbers. Particle equation of motion involved added mass term and ignored the Basset term. By using this new kind of perturbation method called parameterized perturbation method (PPM), analytical expressions for the instantaneous velocity, acceleration and position of the particle were derived. The presented results show the effectiveness of PPM and high rate of convergency of the method to achieve acceptable answers.

Key Words: Drag, Sphere, Acceleration motion, Non-linear equation, Parameterized Perturbation Method (PPM);

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1. Introduction

The settling of a particle in a fluid or gas happens in a wide series of natural and engineering phenomenon. There are numerous applications of settling velocity in many diverse fields of science and engineering, industrial process, alluvial channels [1,2] chemical and powder processing [3,4] and Sediment transport and deposition in pipe lines [5,6] are just a few examples. There are many researchers who have published some research in this filed Shukla [7] et al B. Željko et al [8]. A particle falling in a fluid under the influence of gravity will accelerate until the gravitational force is exactly balanced by the resisting forces including buoyancy and drag. The constant velocity reaches at a stage called the “terminal velocity” or “settling velocity”. The resistive drag force depends upon drag coefficient. During the past decades, a vast body of knowledge has been accumulated on the steady-state motion of spheres in incompressible Newtonian fluids and extensive sets of data have been collected resulting in several theoretical and empirical correlations

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for the drag coefficient, C_D in the terms of the Reynolds number, Re . These relationships for spherical particles were presented in treatises and review papers by Clift et al. [9], Khan and Richardson [10] and Chhabra [11], among others. A comparison between a number of these correlations for spheres by Hartman and Yates [12] showed relatively low deviations. In contrast to steady-state motion of particles much less has been reported about the acceleration motion of spherical particles in incompressible Newtonian fluids. The accelerated motion is relevant to many processes such as particle classification, centrifugal and gravity particle collection and/or separation, where it is often necessary to determine the trajectories of particles accelerating in a fluid [13]. Also for other particular situations, like viscosity measurement using the falling-ball method or rain-drop terminal velocity measurement it is necessary to know the time and distance required for particles to reach their terminal velocities. For very small Reynolds numbers (creeping flows) Stokes developed an analytical expression for the drag coefficient which is given as,

$$C_D = \frac{24}{Re} \quad (1.1)$$

a, b, c, d	Constants	PPM	Parameterized perturbation method
A	General differential operator	Re	Reynolds number
Acc	acceleration ($\cdot/s^2$)	t	time (s)
B	boundary operator	u	velocity (m/s)
C^D	Drag coefficient	Greek Symbols	
D	particle diameter (m)	Γ	boundary of domain
G	acceleration due to gravity (m/s^2)	μ	Dynamic viscosity (kg/ms)
L	linear part of equation	ρ	Fluid density (kg/m^3)
M	particle mass (kg)	ρ_s	particle density (kg/m^3)
N	non-linear part of equation	ϕ	Constant
P	embedding parameter	ψ	Constant
		Ω	Domain

The creeping flow equation neglects the effect of inertia and is acceptable for $Re < 0.4$. For larger Reynolds numbers, at first the boundary layer around the sphere particle has the steady laminar regime. At higher speeds, the boundary layer around the particle tends to separate resulting in vortex shedding and wake formation, and the fluid inertia becomes important. The vortex shedding does not initiate until Reynolds number increases to more than twenty [14]. For Reynolds greater than that of creeping flow there is a different regime of motion, also because of development of the boundary layer, the drag coefficient becomes higher than that predicted by Eq. (1.1). Another well-known analytical relationship is presented by Oseen [15]. The inertial effect has been developed approximately to correct to Stokes' drag given as:

$$C_D = \frac{24}{Re} + \frac{9}{2} \quad (1.2)$$

Reviewing the literature, shows that most of the pervious works on spherical particles were performed experimentally [16,17]. Ferreira et al. [16], in their analytical study, suggested another correlation given as:

$$C_D = \frac{24}{Re} + \frac{1}{2} \quad (1.3)$$

Eq. (1.3) was found to be in good agreement with the experimental data in a wide range of Reynolds numbers, $Re \leq 10^5$. Also Ferreira and Chhabra [13] presented an analytical solution for the transient motion of a sphere falling through a Newtonian fluid using the drag of the form given by Abraham [17] and Wadell [18]. In this work, we studied the motion of a sphere with a drag coefficient of form given by Eqs. (1.2) and (1.3) which has the following general expression:

$$C_D = \frac{\phi}{Re} + \psi \quad (1.4)$$

Lapple [19] shows the variations of C_D versus Re for Eqs (1.2) and (1.3) in a log-log diagram Fig. 1. This figure shows the drag given by Eq. (1.3) is much closer to the experimental data when compared with Eq. (1.2).

In this manuscript analytical expression for the accelerated motion of a falling spherical particle was derived. To analyze the problem the drag coefficient in form of Eq. (1.4) and the parameterized perturbation method (PPM) were used. Analysis of falling particle equation is a new application of parameterized perturbation method (PPM) which has been used for other engineering applications [20,21,22,23,24,25,26,27,28,29].

2. Problem description

Consider a small, rigid, spherical, particle with diameter D , mass m and density ρ_s falling in an infinite extent of an incompressible Newtonian fluid of density ρ and viscosity μ . Let u represent the velocity of the particle at any instant time, t , and g the acceleration due to gravity. Fig. 2 shows a schematic diagram of the falling sphere. The unsteady motion of the particle in a fluid can be described by the BBO equation. For a dense particle falling in light fluids and by assuming $\rho \ll \rho_s$, Basset History force is negligible. Thus, the equation of particle motion is given as

$$m \frac{du}{dt} = mg \left(1 - \frac{\rho}{\rho_s} \right) - \frac{1}{8} \pi D^2 C_D u^2 - \frac{1}{12} \pi D^3 \rho \frac{du}{dt} \quad (2.1)$$

Where C_D is the drag coefficient In the right hand side of the Eq. (2.1), the first term represents the buoyancy affect, the second term corresponds to drag resistance, and the last term is due to the added mass effect which is due to acceleration of fluid around the particle. The main difficulty in solution of Eq. (2.1) hidden in the non-linear terms due to non-linearity nature of the drag coefficient C_D Substituting Eq. (1.4) in Eq. (2.1) and by rearranging parameters, Eq. (2.1) could be rewritten as follow:

$$a \frac{du}{dt} + bu + cu^2 - d = 0, u(0) = 0 \quad (2.2)$$

Where

$$a = (m + \frac{1}{12}\pi D^3 \rho) \quad (2.3)$$

$$b = \frac{\varphi}{8}\pi D\mu \quad (2.4)$$

$$c = \frac{\psi}{8}\pi D^2 \rho \quad (2.5)$$

$$d = mg(1 - \frac{\rho}{\rho_s}) \quad (2.6)$$

Eq. (2.2) is a non-linear ordinary differential equation which could be solved by numerical techniques such Runge–Kutta method. We employed PPM and compared our results with numerical solution of 4th order Runge–Kutta method using the Maple package.

3. Basic idea of parameterized perturbation method

Perturbation method is very well-known method in solving nonlinear equation. In this method equations have a small parameter such as ε but in a wide series of equation in science and engineering there is not a small parameter so perturbation method can not be applied in these conditions to use a kind of this method called parameterized perturbation method (PPM) [20,21,22,23,24,25,26,27,28,29]. PPM makes a small parameter in equation then by considering zero value for negative powers of ε , singular points of differential equation are omitted. Finally perturbation method can be applied to solve equation and obtaining results. For making small parameter in equation we guess the answer such as $u(t)$ will be:

$$u(t) = \varepsilon v(t) + \beta \quad (3.1)$$

Substituting these sentences in equation and dividing it by ε gives an equation for $v(t)$ in which small parameter exists, for deleting sentences with negative powers of ε a suitable β must be chosen. Boundary condition and initial condition will change, considering relation between $u(t)$ and $v(t)$.

Application

By considering Eq. (2.2) for using PPM, in [24,25], the expanding parameter is introduced by a linear transformation:

$$u(t) = \varepsilon v(t) + \beta \quad (3.2)$$

Where ε is the introduced as perturbation parameter, β is a constant. With these guesses, general equation and initial condition from Eq.6 will be:

$$a \cdot \frac{dv(t)}{dt} + b \cdot v(t) + c \cdot \varepsilon v^2(t) + 2c \cdot v(t) + \frac{(c \cdot \beta^2 + b\beta - d)}{\varepsilon} = 0 \quad (3.3)$$

$$v(0) = -\frac{\beta}{\varepsilon} \quad (3.4)$$

For omitting the sentence of equation with negative power of epsilon terms, the answer of following equation will be chosen for β . This action will simplify the procedure of solution:

$$c.\beta^2+b\beta-d=0 \quad (3.5)$$

The final form of Eq. (3.4) will be changed in the following form:

$$a.\frac{dv(t)}{dt}+b.v(t)+c.\varepsilon.v^2(t)+2c.v(t)=0 \quad (3.6)$$

By using perturbation method for $v(t)$ assuming that solution of the Eq. (3.4) can be written in the form:

$$v=v_0+\varepsilon v_1+\varepsilon^2 v_2+\varepsilon^3 v_3+\dots \quad (3.7)$$

Unlike the traditional perturbation methods, we keep

$$v_0(0)=v(0) \quad (3.8)$$

And

$$\sum_{i=1} v_i(0)=0 \quad (3.9)$$

By considering coefficient of different fluids from Table. 1 and using the coefficient of general equation Eq. (2.2) from Table. 2 and continuing procedure of solution, velocity and acceleration function will be obtained.

By considering following values for Water by

$$a=1.0863*10^{-5}, b=2.8273*10^{-5}, c=1.5848*10^{-2}, d=2.3665*10^{-5}$$

The velocity profile will be as follows:

$$\begin{aligned} u(t) = & -0.06656852263e^{(-104.9486898t)} + 0.05391265951e^{(-209.8973795t)} \\ & -0.3243531202e^{(-314.8460693t)} + 0.01199744606e^{(-419.7947591t)} \\ & -0.001982766901e^{(-524.7434489t)} + 0.03507649599 \end{aligned} \quad (3.10)$$

Results of Numerical Runge-Kutta method and PPM for Water at Eq. (1.2) and (1.3) were compared in table. 3 and 4.

The mentioned method was applied for analyzing settling manner of a spherical particle in a Newtonian fluid. A single spherical Aluminum particle of 3 mm diameter was assumed to fall in an infinity body of Water, Glycerin or Ethylene glycol. Required physical properties of selected materials are given in Table 1. Inserting the above properties into Eqs.(2.3–2.6), and using Eqs. (1.2) and (1.3), six different combinations are gained which are classified in Table 2. By substituting the above coefficients in Eq. (2.2), six different non-linear equations are achieved. Parameterized Perturbation Method was used to solve applied equations. The results of

this method were compared with those from the numerical solution. Figs. 4 and 5 depict the variation of falling velocity of the particle versus time in different fluids and drag coefficients obtained from Eq. (1.2) and (1.3). Figs. 6 and 7 depict the variation of falling acceleration of the particle versus time in different fluids by using drag coefficient obtained from Eqs. (1.2) and (1.3). Presented results demonstrate an excellent agreement between PPM and the numerical method.

By increasing the fluid viscosity, terminal velocity (shown in Figs. 4 and 5) and acceleration duration (shown in Figs. 6 and 7) decline. Moreover the difference between the results of Eqs. (1.2) and (1.3) were decreased by the reduction of fluid viscosity that this object can be seen in Figs. 10 and 11 for velocity values and in Fig. 12 and 13 for acceleration values. Fig. 8 and 9 demonstrate the position of the falling particle in different fluids for each instant during the falling procedure, Comparing Figs. 8 and 9 reveals the difference between the positions predicted by Eqs. (1.2) and (1.3), which is stronger for the higher viscose media due to a larger calculated drag coefficient by Eq. (1.2) in low Reynolds number.

4. Conclusions

In this study behavior of settling a spherical particle in a Newtonian fluid analytically is expanded. By considering the different drag coefficient, governing equation of particle's behavior was achieved in non-linear differential form. In this equation there wasn't a small parameter, this matter makes a problem for using perturbation method. A new kind of perturbation method called parameterized perturbation method (PPM) was used to implement small parameter artificially and solve the non-linear governing equation. The instantaneous velocity, acceleration and position of particle were obtained by using PPM. Results were compared with numerical Runge-Kutta method which a very good approximation was observed. Analysis of the results has presented relation between viscosity of fluid on the behavior of settling and velocity and acceleration regime of particle. This presented work reveals high flexibility and accuracy of Parameterized parameter method (PPM) for solving a wide series of engineering and natural problems without a small parameter. PPM can be used in future for other non-linear equations.

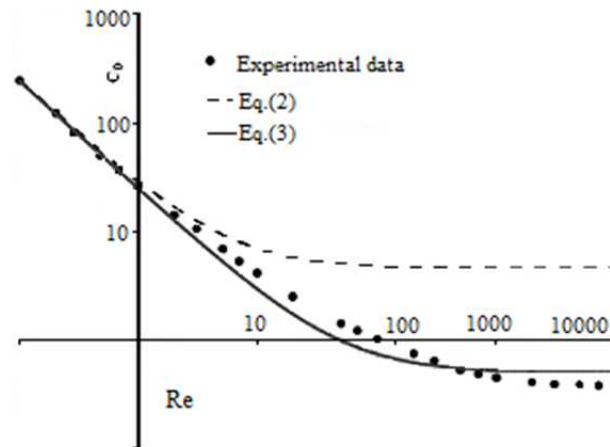


Figure 1: Drag coefficient as a function of Reynolds number.

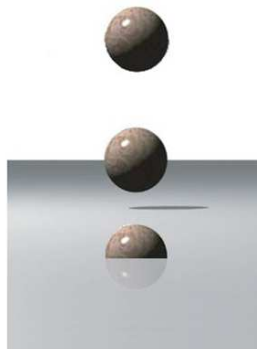


Figure 2: Schematic picture of free falling spherical particle in a Newtonian fluid

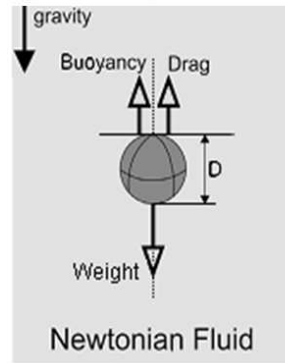


Figure 3: free body diagram of forces

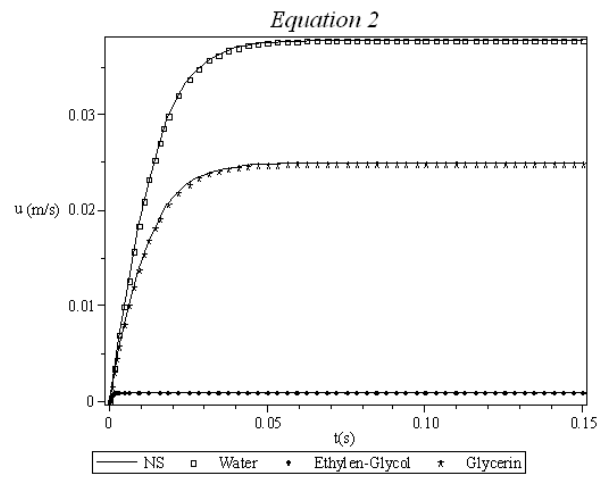


Figure 4: velocity variation for different fluids, Drag Eq.(1.2)

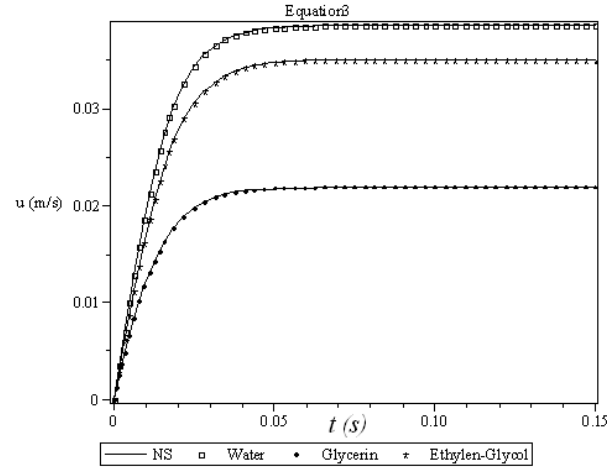


Figure 5: elocity variation for different fluids, Drag Eq.(1.3)

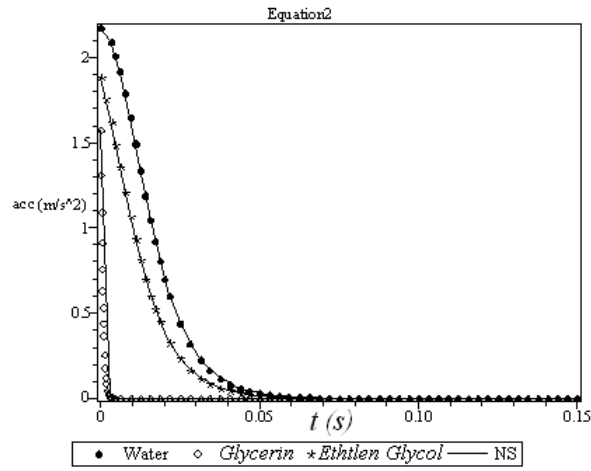


Figure 6: Acceleration variation for different fluids, Drag Eq.(1.2)

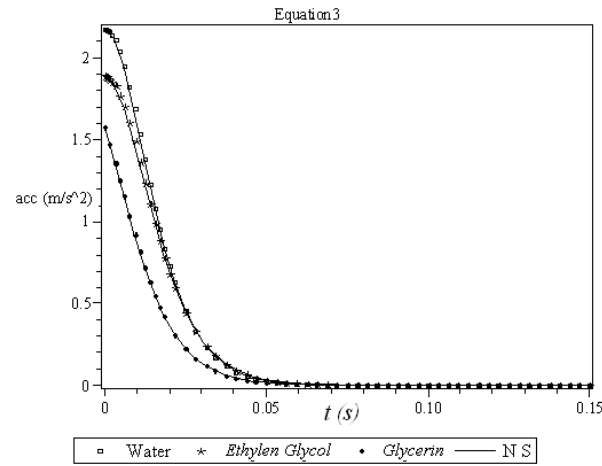


Figure 7: Acceleration variation for different fluids, Drag Eq.(1.3)

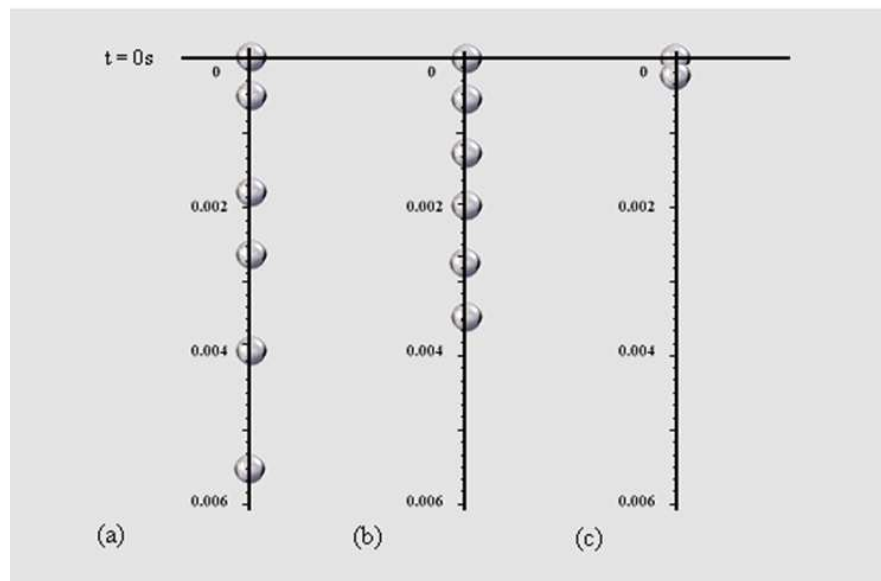


Figure 8: Positions of falling particle for different fluids at Time step = 0.03 s using Eq. (1.2)

(a) Water, (b) Ethylene-glycol and (c) Glycerin

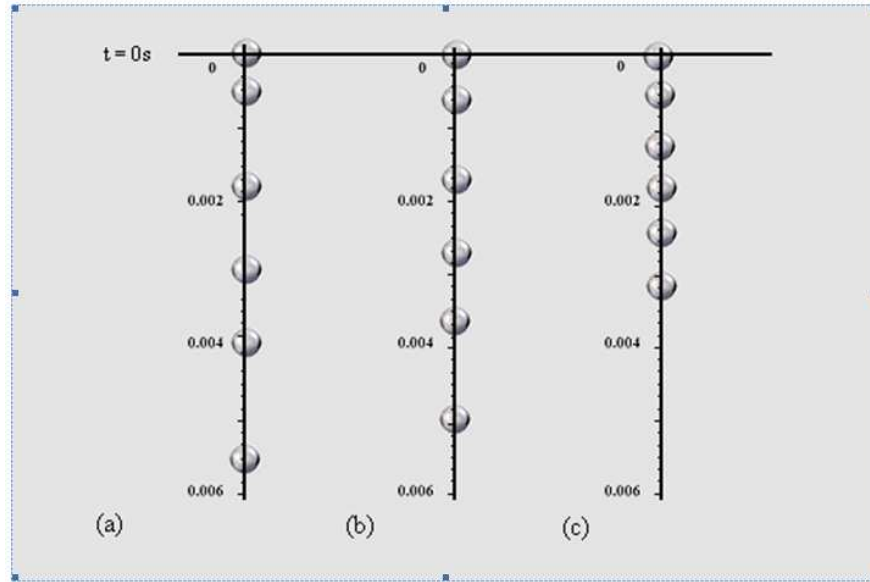


Figure 9: Positions of falling particle for different fluids at Time step = 0.03 s using Eq. (1.3)
 (a) Water, (b) Ethylene-glycol and (c) Glycerin

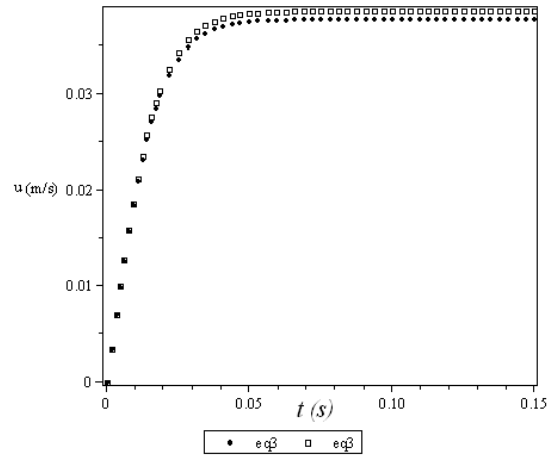


Figure 10: Velocity profile from Eq.2 and Eq.3 for Water

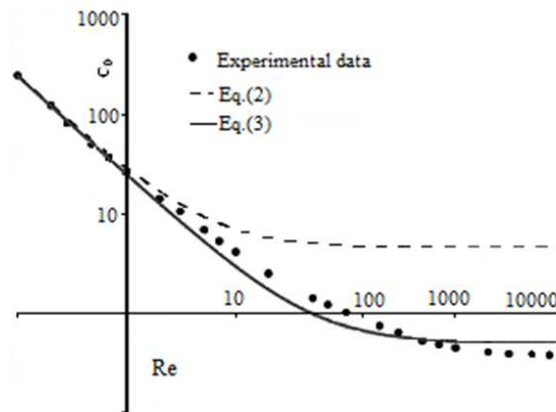


Figure 11: Velocity profile from Eq.2 and Eq.3 for Glycerin

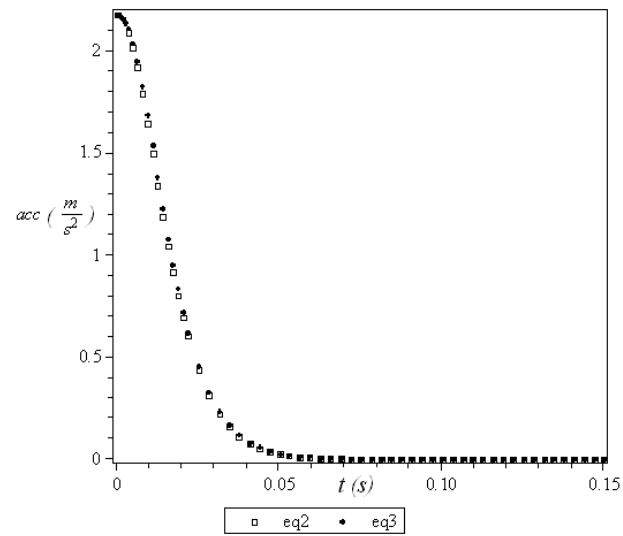


Figure 12: Acceleration profile from Eq.2 and Eq.3 for Water

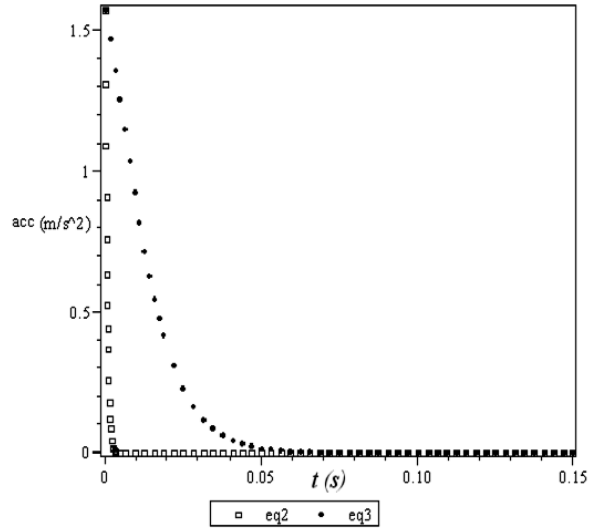


Figure 13: Acceleration profile from Eq.2 and Eq.3 for Glycerin

Table 1: Physical properties of materials.

Materials	Density(kg/m^3)	Viscosity($kg/m\ s$)
Water	996.51	0.001
Glycerin	1259	0.799
Ethylene-glycol	1111.40	0.0157
Aluminum	2702.00	—

Table 2: Selected coefficient of Eq. (1.2).

Solid	Fluid	Drag equation	A	B	c	d
Aluminum	Water	Eq.(1.2)	1.0863 e -5	2.8273 e-5	1.5848 e-2	2.3665 e-5
		Eq.(1.3)	1.0863 e -5	5.8903 e-8	1.5848 e-2	2.3665 e-5
	Glycerin	Eq.(1.2)	1.2719 e-5	2.2590 e-2	2.0022 e-2	2.0023 e-5
		Eq.(1.3)	1.2719 e-5	4.7063 e-4	2.0022 e-2	2.0023 e-5
	Ethylene-glycol	Eq.(1.2)	1.1675 e-5	4.4389 e-4	1.7675 e-2	2.2071 e-5
		Eq.(1.3)	1.1675 e-5	9.2477 e-6	1.7675 e-2	2.2071 e-5

Table 3: Comparing PPM and numerical results for Water at Eq. (1.2).

	u(t)		
T	PPM	Numerical	Error
0	0	0	0
0.015	0.023245	0.026197	0.002952
0.030	0.032267	0.035335	0.003067
0.045	0.034477	0.037302	0.002826
0.060	0.034951	0.037676	0.002724
0.075	0.035051	0.037745	0.002695
0.090	0.035071	0.037758	0.002687
0.105	0.035075	0.037760	0.002685
0.120	0.035076	0.037761	0.002684
0.135	0.035076	0.037761	0.002684
0.150	0.035076	0.037761	0.002684

Table 4: Comparing PPM and numerical results for Water at Eq. (1.3)

	u(t)		
T	PPM	Numerical	Error
0	0	0	0
0.015	0.02720542	0.02661540	0.00059002
0.030	0.03634794	0.03610248	0.00024546
0.045	0.03822480	0.03816010	0.00006470
0.060	0.03856669	0.03855171	0.00001498
0.075	0.03862758	0.03862431	0.00000327
0.090	0.03863839	0.03863770	0.00000068
0.105	0.03864030	0.03864017	0.00000013
0.120	0.03864064	0.03864062	0.00000002
0.135	0.03864070	0.03864070	0.00000000
0.150	0.03864071	0.03864072	0.00000000

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