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# On Timelike Parallel $p_i$-Equidistant Ruled Surfaces with a Timelike Base Curve in the Minkowski 3-Space $R_1^3$

Melek Masal and Nuri Kuruoğlu

ABSTRACT: In this paper, timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve are defined and the shape operators, shape tensor, the $q^{th}$ fundamental forms and the characteristic polynomials of the shape tensors of these surfaces are obtained. Then, some relations between them are found. Finally, an example for the timelike parallel $p_2$ equidistant ruled surfaces by a timelike base curve in the Minkowski 3-space $R_1^3$ is given.

Key Words: Lorentz space, Minkowski space, parallel $p_i$-equidistant, ruled surface, timelike, curvatures.

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## 1. Introduction

I. E. Valeontis defined parallel $p$-equidistant ruled surfaces in $E^3$ and given some results related with striction curves of ruled surfaces , [10]. Then he also studied on existence theorem related with homothety of parallel $p$-equidistant ruled surfaces. M. Masal, N. Kuruoğlu obtained some new characteristic properties related with dralls, integral invariants, shape operators, Gaussian curvatures, mean curvatures and the $q^{th}$ fundamental form of parallel $p$-equidistant ruled surfaces in $E^3$, [4,5]. And also, A. Turgut, H. H. Hacısalihoğlu defined timelike ruled surfaces and gave some theorems related to the distribution parameter in the three dimensional Minkowski space, [7,8,9].
In this paper, timelike parallel $p_i$-equidistant ruled surfaces with a timelike base curve are defined in the 3-dimensional Minkowski space and the shape operators, shape tensor (or second fundamental form tensor), the $q^{th}$ fundamental forms and characteristic polynomials of the shape tensors of these ruled surfaces are obtained. After all, an example for the timelike parallel $p_2$-equidistant ruled surfaces with a timelike base curve is given.

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## 2. Preliminaries

Let $R_1^3$ denote the three-dimensional Minkowski space, i.e. three dimensional vector space $R^3$ equipped with the flat metric $g = -dx_1^2 + dx_2^2 + dx_3^2$, where $(x_1, x_2, x_3)$ is rectangular coordinate system of $R_1^3$. Since $g$ is indefinite metric, recall that a vector $\vec{v}$ in $R_1^3$ can have one of three casual characters: It can be spacelike if $g(\vec{v},\vec{v}) > 0$ or $\vec{v} = \vec{0}$, timelike if $g(\vec{v},\vec{v}) < 0$ and null if $g(\vec{v},\vec{v}) = 0$ and $\vec{v} \neq \vec{0}$. The norm of a vector $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{|g(\vec{v},\vec{v})|}$. Therefore, $\vec{v}$ is a unit vector if $g(\vec{v},\vec{v}) = \pm 1$. Furthermore, vectors $\vec{v}$ and $\vec{w}$ are said to be orthogonal if $g(\vec{v},\vec{w}) = 0$, [6].

For any vectors $\vec{v} = (v_1, v_2, v_3), \vec{w} = (w_1, w_2, w_3) \in R_1^3$, the Lorentzian product $\vec{v} \wedge \vec{w}$ of $\vec{v}$ and $\vec{w}$ is defined as, [1]

$$\vec{v} \wedge \vec{w} = (v_2 w_3 - v_3 w_2, v_1 w_3 - v_3 w_1, v_2 w_1 - v_1 w_2).$$

A regular curve $\alpha : I \to R_1^3, I \subset R$ in $R_1^3$ is said to be spacelike, timelike and null curve if the velocity vector $\vec{\alpha}'(t)$ is a spacelike, timelike or null vector, respectively, [3]. Let $M$ be a semi-Riemannian surface in $R_1^3$, $D$ and $N$ represent Levi-Civita connection of $R_1^3$ and unit normal vector field of $M$, respectively. For all $X \in \chi(M)$ the transformation

$$S(X) = -D_X N \tag{2.1}$$

is called a shape operator of $M$, where $\chi(M)$ is the space of vector fields of $M$. Then the function is defined as

$$II(X,Y) = \varepsilon g(S(X),Y)N, \quad for \quad all \quad X, Y \in \chi(M) \tag{2.2}$$

is bilinear and symmetric. $II$ is called the shape tensor (or second fundamental form tensor) of $M$, where $\varepsilon = g(N,N)$, [6].

Let $S(P)$ be a shape operator of $M$ at point $P$ then $K : M \to R$, $K(P) = \det S(P)$ function is called the Gaussian curvature function of $M$. In this case the value of $K(P)$ is defined to be the Gaussian curvature of $M$ at the point $P$. Similarly, the function $H : M \to R$, $H(P) = \frac{traceS(P)}{\dim M}$ is called the mean curvature of $M$ at the point $P$.

Let us suppose that $\alpha$ be a curve in $M$. If

$$S(T) = \lambda T \tag{2.3}$$

then the curve $\alpha$ is named curvature line (principal curve) in $M$, where $T$ is the tangential vector field of $\alpha$ and $\lambda$ is non-zero scalar. If the following equation holds

$$g(S(T),T) = 0 \tag{2.4}$$

then $\alpha$ is called an asymptotic curve. If $\alpha$ is a geodesic curve in $M$, then we have

$$\bar{D}_T T = 0, \tag{2.5}$$

where $\bar{D}$ is the Levi-Civita connection of $M$.
For $X_P, Y_P \in T_M(P)$,
**i)**If $II\,(X_P, Y_P) = 0$, then $X_P, Y_P$ are called the conjugate vectors.
**ii)** If $II\,(X_P, X_P) = 0$, then $X_P$ is called the asymptotic direction.
The $q^{th}$ fundamental form $I^q$, $1 \leq q \leq 3$, on $M$ such that

$$I^q\,(X,Y) = g(S^{q-1}(X), Y) \quad for\ all\ \ X, Y \in \chi\,(M) \tag{2.6}$$

is called the $q^{th}$ fundamental form of $M$.
If $P_S(\lambda)$ is the characteristic polynomial of the shape operator of $M$, then we have

$$P_S(\lambda) = det\,(\lambda I - S) \tag{2.7}$$

where $I$ is an unit matrix, $\lambda$ is a scalar.
If the induced metric on $M$ is Lorentz metric, then $M$ is called a timelike surface.

**Theorem 2.1.** *A surface in the 3-dimensional Minkowski space $R_1^3$ is a timelike surface if and only if a normal vector field of surface is a spacelike vector field, [2].*

The one parameter family of lines in $R_1^3$ is called the ruled surface and each of these lines of this family is named as the rulings of the ruled surface. Thus the parametrization of the ruled surface is given by

$$\varphi(t,v) = \alpha(t) + vX(t)$$

where $\alpha$ and $X$ are the base curve and unit vector in the direction of the rulings of the ruled surface, respectively. For the striction curve of ruled surface $\varphi(t,v)$ we can write

$$\overline{\alpha} = \alpha - \frac{g(\alpha', X')}{g(X', X')}X. \tag{2.8}$$

The drall ( distribution parameter) of the ruled surface $\varphi(t,v)$ is defined by

$$P_X = -\frac{\det(\alpha', X, X')}{g(X', X')} \quad , \quad g(X', X') \neq 0. \tag{2.9}$$

## 3. Timelike Parallel $p_i$-Equidistant Ruled Surfaces

Let $\alpha : I \to R_1^3$, $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ be a differentiable timelike curve parameterized by arc-length in the Minkowski 3-space, where $I$ is an open interval in $R$ containing the origin. The tangent vector field of $\alpha$ is denoted by $V_1$. Let $D$ be the Levi-Civita connection on $R_1^3$ and $D_{V_1}V_1$ be a spacelike vector. If $V_1$ moves along $\alpha$, then the timelike ruled surface which is given by the parametrization

$$\varphi\,(t,v) = \alpha\,(t) + vV_1(t) \tag{3.1}$$

can be obtained in the Minkowski 3-space. The timelike ruled surface by a timelike base curve is denoted by $M$. $\{V_1, V_2, V_3\}$ is an orthonormal frame field along $\alpha$ in

$R_1^3$, where $V_2$ and $V_3$ are spacelike vectors. If $k_1$ and $k_2$ are the natural curvature and torsion of $\alpha(t)$, respectively, then the Frenet formulas are given by

$$\frac{dV_1}{dt} = k_1 V_2, \quad \frac{dV_2}{dt} = k_1 V_1 - k_2 V_3, \quad \frac{dV_3}{dt} = k_2 V_2. \tag{3.2}$$

Using $V_1 = \alpha'$ and $V_2 = \frac{\alpha''}{\|\alpha''\|}$, we have

$$k_1 = \|\alpha''\| > 0.$$

For the timelike ruled surface $M$ with a timelike base curve given with the parametrization 3.1 we see

$$\varphi_t = V_1 + v k_1 V_2, \quad \varphi_v = V_1, \quad \varphi_t \wedge \varphi_v = v k_1 V_3.$$

It is obvious that $\varphi_t \wedge \varphi_v \in \chi^{\perp}(M)$.

**Definition 3.1.** *The planes which are corresponding to the subspaces $Sp\{V_1, V_2\}$, $Sp\{V_2, V_3\}$ and $Sp\{V_3, V_1\}$ are called asymptotic plane, polar plane and central plane, respectively.*
*Let us suppose that $\alpha^* = \alpha^*(t^*)$ is another differentiable timelike curve with arc-length and $\{V_1^*, V_2^*, V_3^*\}$ is Frenet frame of this curve in three dimensional Minkowski space $R_1^3$. Hence, we define timelike ruled surface $M^*$ parametrically as follows*

$$\varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*), (t^*, v^*) \in I \times R.$$

**Definition 3.2.** *Let $M$ and $M^*$ be two timelike ruled surfaces by a timelike base curve in $R_1^3$ with the generators $V_1$ of $M$ and $V_1^*$ of $M^*$. Let $p_1, p_2$ and $p_3$ be the distances between the polar planes, central planes and asymptotic planes, respectively. If*
***i)*** *the generator vectors of $M$ and $M^*$ are parallel,*
***ii)*** *the distances $p_i$, $1 \leq i \leq 3$, are constants,*
*then the pair of ruled surfaces $M$ and $M^*$ are called the timelike parallel $p_i$ equidistant ruled surfaces with a timelike base curve in $R_1^3$. If $p_i = 0$, then the pair of $M$ and $M^*$ are called the timelike parallel $p_i$ equivalent ruled surfaces by a timelike base curve in $R_1^3$.*
*From the Definition 3.2, the timelike parallel $p_i$ equidistant ruled surfaces with a timelike base curve with the following parametric representations can be obtained that*

$$M : \varphi(t, v) = \alpha(t) + v V_1(t), \quad (t, v) \in I \times R.$$

$$M^* : \varphi^*(t^*, v^*) = \alpha^*(t^*) + v^* V_1(t^*), \ (t^*, v^*) \in I \times R \tag{3.3}$$

where $t$ and $t^*$ are arc parameters of curves $\alpha$ and $\alpha^*$, respectively.
Throughout this paper $M$ and $M^*$ will be used for the timelike parallel $p_i$ equidistant ruled surfaces with a timelike base curve.
Now we consider the Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ of ruled surfaces

$M$ and $M^*$. From the Definition 3.1 it is obvious that $V_1^*(t^*) = V_1(t)$. Furthermore, from $\frac{dV_i}{dt} = \frac{dV_i^*}{dt^*}\frac{dt^*}{dt}$, $1 \le i \le 3$, and the equation 3.2, we find $V_2^*(t^*) = V_2(t)$ and $V_3^*(t^*) = V_3(t)$, for $\frac{dt^*}{dt} > 0$.
Hence the following theorem will be given without the proof:

**Theorem 3.3.** ***i)*** *The Frenet frames $\{V_1, V_2, V_3\}$ and $\{V_1^*, V_2^*, V_3^*\}$ are equivalent at the corresponding points in $M$ and $M^*$, respectively.(For $\frac{dt^*}{dt} > 0$.)*
***ii)*** *If $k_1$ and $k_1^*$ are the natural curvatures and $k_2$, $k_2^*$ are the torsions of base curves of $M$ and $M^*$, respectively, then we have*

$$k_i^* = k_i \frac{dt}{dt^*} \quad 1 \le i \le 2.$$

Now, we'll study the matrices of the shape operators of timelike parallel $p_i$ - equidistant ruled surfaces with a timelike base curve. These surfaces can be given by

$$M : \varphi(t, v) = \alpha(t) + vV_1(t), \quad (t, v) \in I \times R$$

and

$$M^* : \varphi^*(t^*, v) = \alpha^*(t^*) + vV_1(t^*), \quad (t^*, v) \in I \times R.$$

Thus, we have

$$\varphi_t = V_1 + vk_1V_2, \quad \varphi_v = V_1$$

where $g(\varphi(t), \varphi(v)) \neq 0$. By the Gram-Schmidt method, we obtain the orthogonal base $\{X, Y\}$ of $\chi(M)$ with

$$X = \varphi_v = V_1, \quad Y = vk_1V_2 \tag{3.4}$$

where $X$ is a timelike vector and $Y$ is a spacelike vector. If $N$ is a normal vector field of $M$, then we have

$$N = X \wedge Y = -vk_1V_3.$$

If $N_0$ is a unit normal vector field of $M$, then we find

$$N_0 = \begin{cases} -V_3, & for \quad v > 0 \\ V_3, & for \quad v < 0. \end{cases} \tag{3.5}$$

If $S$ is the matrix of the shape operator of $M$, then we have

$$S(X) = aX + bY, \quad S(Y) = cX + dY$$

or

$$S = \begin{bmatrix} \frac{g(S(X),X)}{g(X,X)} & \frac{g(S(X),Y)}{g(Y,Y)} \\ \\ \frac{g(S(Y),X)}{g(X,X)} & \frac{g(S(Y),Y)}{g(Y,Y)} \end{bmatrix}.$$

For $v < 0$, we obtain

$$S(X) = -D_XN_0 = 0, \quad S(Y) = -D_YN_0 = -k_2V_2$$

and we see

$$S=\begin{bmatrix} 0 & 0 \\ 0 & \frac{-k_2}{vk_1} \end{bmatrix}. \tag{3.6}$$

Similarly, if $S^*$ is the matrix of the shape operator of $M^*$, then $S^*$ is given by

$$S^*=\begin{bmatrix} 0 & 0 \\ 0 & \frac{-k_2^*}{vk_1^*} \end{bmatrix}.$$

For $v>0$ we obtain

$$S(X)=-D_X N_0=0, \quad S(Y)=-D_Y N_0=k_2 V_2$$

and we have

$$S=\begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2}{vk_1} \end{bmatrix}. \tag{3.7}$$

Similarly, $S^*$ is seen as

$$S^*=\begin{bmatrix} 0 & 0 \\ 0 & \frac{k_2^*}{vk_1^*} \end{bmatrix}.$$

From the theorem 3.3.ii, we can give

$$S^*=S. \tag{3.8}$$

**Theorem 3.4.** *The geodesic curves in $M$ are the geodesic curves in $M^*$, too.*

**Proof:** If $V,W \in \chi(M)$ and $V,W \in \chi(M^*)$, then from the definition of Gaussian equation we can write

$$D_V W=\overline{D}_V W+\varepsilon g(S(V),W)N_0 \quad \text{and} \quad D_V W=\overline{D}_V^* W+\varepsilon g(S^*(V),W)N_0^*$$

where $\bar{D}$ and $\bar{D}^*$ are Levi-Civita connections of $M$ and $M^*$, respectively. From the equations 3.5, 3.8 and Theorem 3.3.i, we have,

$$\overline{D}_V^* W=\overline{D}_V W$$

If $\alpha$ is a geodesic curve of $M$ and $T$ is a tangent vector field of $\alpha$, we get

$$\overline{D}_T^* T=\overline{D}_T T$$

then $\alpha$ is a geodesic curve of $M^*$ □

**Theorem 3.5.** *Let $II$ and $II^*$ be the shape tensors of $M$ and $M^*$, respectively, then we have*

$$II^*(V,W)=II(V,W),$$

*where $V,W\in\chi(M)$ and $V,W \in \chi(M^*)$.*

**Proof:** Using the definition of shape tensor, we can write,

$$II\,(V,W) = g(S(V),W)N_0, \quad \text{for all } V,W \in \chi(M)$$

and

$$II^*\,(V,W) = g(S^*(V),W)N_0^*, \quad \text{for all} \quad V,W \in \chi(M^*).$$

From the Theorem 3.3.i and the equation 3.8, we find

$$II^*\,(V,W) = II\,(V,W)\,.$$

□

**Result 3.6.** *Let $M$ and $M^*$ be the timelike parallel $p_i$ -equidistant ruled surfaces.*
***i)****The conjugate vectors in $M$ are the conjugate vectors in $M^*$, too.*
***ii)****Asymptotic directions in $M$ are the asymptotic directions in $M^*$, too.*

**Theorem 3.7.** *Let $M$ and $M^*$ be timelike ruled surfaces. If $T_{\bar{M}}(P)$ is the set of vectors $V_P = (0,y)|_P$, then all points in $\bar{M}$ are umbilic points, where $\bar{M} : \varphi\,(t,v) = \alpha\,(t) + vV_1(t), \quad for \quad v = \pm\frac{k_2}{k_1}$.*

**Proof:** Considering the definition of shape tensor,

$$II(V_P,W_P) = g(S(V_P),W_P)N_P, \quad for \quad V_P,W_P \in T_M(P).$$

If $P$ is an umbilic point in $M$, there exists $N_P \in T_M^{\perp}(P)$ such that

$$II(V_P,W_P) = g(V_P,W_P)N_P$$

Then we have

$$g(S(V_P),W_P)N_P = g(V_P,W_P)N_P$$

From inner product operation is bilinear and non-degenerate, we get

$$S(V_P) = V_P. \tag{3.9}$$

Taking into consideration to equations 3.6, 3.7 and 3.9, the followings can be find for $V_P = (x,y)|_P$,

$$\begin{cases} x = 0, v = -\frac{k_2}{k_1} \;\; for \;\; v < 0, \\ x = 0, v = \frac{k_2}{k_1} \;\; for \;\; v > 0. \end{cases}$$

Let $\bar{M} : \varphi\,(t,v) = \alpha\,(t) + vV_1\,(t)$, for $v = \pm\frac{k_2}{k_1}$ and $T_{\bar{M}}(P)$ be the set of vectors $V_P = (0,y)|_P$. Thus, for all $V_P,W_P \in T_{\bar{M}}(P)$ there exists $N_P \in T_{\bar{M}}^{\perp}(P)$ such that

$$II(V_P,W_P) = g(S(V_P),W_P)N_P.$$

Since, all points of $\bar{M}$ are umbilic points.
The following result can be given from the theorem 3.5. □

**Result 3.8.** ***i)*** *Umbilic points of $M$ are umbilic points of $M^*$, too.*
***ii)****Flat points of $M$ are flat points of $M^*$, too.*

**Theorem 3.9.** *If $I^q$ and $I^{*q}$ are the $q^{th}$ fundamental forms of $M$ and $M^*$, respectively, then the following relation can be satisfy*

$$I^{*q}(X,Y) = I^q(X,Y), \quad 1 \leq q \leq 3.$$

*where $X, Y \in \chi(M)$ and $X, Y \in \chi(M^*)$.*

**Proof:** Using the equations 2.6 and 3.8 the following relation is obvious

$$I^{*q}(X,Y) = I^q(X,Y), \quad 1 \leq q \leq 3.$$

□

**Theorem 3.10.** *If $P_S(\lambda)$ and $P_{S^*}(\lambda)$ are the characteristic polynomials of the shape operators of $M$ and $M^*$, respectively, then we obtain $P_{S^*}(\lambda) = P_S(\lambda)$.*

**Proof:** Considering the equations 2.7 and 3.8, the relation between the characteristic polynomials can be reached as follows

$$P_{S^*}(\lambda) = P_S(\lambda).$$

□

**Example 3.11.** *$M$ and $M^*$ are timelike parallel $p_2$-equidistant ruled surfaces in three-dimensional Minkowski space $R^3_1$ defined by the following parametric equations;*

$$M : \varphi(t,v) = (\sinh t + v\cosh t, 1, \cosh t + v \sinh t)$$

*and*

$$M^* : \varphi^*(t^*,v^*) = (2\sinh t^* + v^*\cosh t^*, 1, 2\cosh t^* + v^* \sinh t^*)$$

*where the curves $\alpha(t) = (\sinh t, 1, \cosh t)$ and $\alpha^*(t^*) = (2\sinh t^*, 1, 2\cosh t^*)$ are timelike base curves of $M$ and $M^*$, respectively, (Figure 1).*

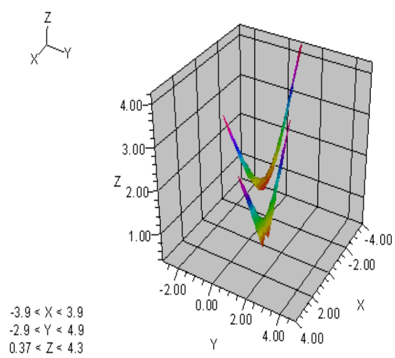


*Figure 1. Timelike Parallel $p_2$-Equidistant Ruled Surfaces*

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*Melek Masal*
*Faculty of Education,*
*Department of Elementary Education,*
*Sakarya University,*
*Turkey*
*E-mail address:* mmasal@sakarya.edu.tr

*and*

*Nuri Kuruoğlu*
*Faculty of Arts and Sciences,*
*Department of Mathematics and Computer Sciences,*
*Bahcesehir University,*
*Turkey*
*E-mail address:* kuruoglu@bahcesehir.edu.tr