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$\mathcal{I}_g-$Submaximal Spaces

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abstract: In this paper, we define $\mathcal{I}_g$- submaximal spaces and study its charecterizations and properties.

Key Words: Ideal topological space, $\star$- dense sets, $\mathcal{I}_g$- open sets, $\mathcal{I}_g$- closed sets, $\mathcal{I}$- submaximal spaces, g- submaximal spaces, $\mathcal{I}_g$- locally $-\star-$ closed set.

Contents

1. Introduction and Preliminaries.

An *ideal* $\mathcal{I}$ on X [11] is a collection of subsets of X satisfying the following: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$, and (ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. A topological space (X, τ) together with an ideal $\mathcal{I}$ is called an *ideal topological space* and is denoted by $(X, \tau, \mathcal{I})$. For each subset A of X, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every open set U containing $x\}$ is called the *local function* of A [11] with respect to $\mathcal{I}$ and τ. We simply write $A^\star$ instead of $A^\star(\mathcal{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [8] without mentioning it. Moreover, $cl^\star(A) = A \cup A^\star$ [15] defines a Kuratowski closure operator for a topology $\tau^\star$, which is finer than τ. An ideal $\mathcal{I}$ is a *boundary ideal* [15] or a *codense ideal* [8] if $\tau \cap \mathcal{I} = \{\emptyset\}$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be pre$-\mathcal{I}-$open [3], if $A \subset intcl^\star(A)$ where int is the interior operator in (X, τ). A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$- *locally* -$\star$- *closed set* [13], if there exists an open set U and a $\star$- closed set F such that $A = U \cap F$. A subset A of a topological space (X, τ) is said to be g- closed [10], if $cl(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a g- closed set is called a g-open set [10]. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$- closed [4], if $A^\star \subset U$ whenever $A \subset U$ and U is open. The complement of an $\mathcal{I}_g$- closed set is called an $\mathcal{I}_g$- open set [4]. A space in which every $\mathcal{I}_g$- closed set is a $\star$- closed set is called a $T_{\mathcal{I}}$- *space* [4]. An *ideal topological space* $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}$-*submaximal* [1] space, if every $\star$-dense set is open. A *topological space* (X, τ) is said to be an g- submaximal [2] space if every dense set is g- open. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_g$-*locally*-$\star$-*closed set* [13], if there exists an $\mathcal{I}_g$-

2000 *Mathematics Subject Classification:* 54A05, 54A10 open set U and a $\star$- closed set F such that $A = U \cap F$. The following lemmas will be useful in the sequel.

Lemma 1.1. *[1] For a subset A of an ideal space $(X, \tau, \mathcal{I})$, the following properties are equivalent.*
(a) A is pre$-\mathcal{I}-$open.
(b) $A = G \cap B$ where G is open and B is $\star$- dense.

Lemma 1.2. *[14] Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. If $A \subset A^\star$, then $A^\star = cl(A^\star) = cl(A) = cl^\star(A)$.*

Lemma 1.3. *[4] Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then the finite union of $\mathcal{I}_g$- closed sets is an $\mathcal{I}_g$- closed set.*

2. $\mathcal{I}_g$- Submaximal Spaces

An *ideal topological space* $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}_g$- submaximal space if every $\star$- dense set is $\mathcal{I}_g$- open. The following Theorem 2.1 shows that every $\mathcal{I}$-submaximal space is an $\mathcal{I}_g$- submaximal space and Example 2.2 below shows that the converse is not true.

Theorem 2.1. *Every $\mathcal{I}$- submaximal space is an $\mathcal{I}_g$- submaximal space.*

Proof: Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}$- submaximal space and A be $\tau^\star$- dense in X. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}$- submaximal, A is open in X. Since every open set is $\mathcal{I}_g$- open, A is $\mathcal{I}_g$- open. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$-submaximal space. □

The following Example 2.2. shows that the converse of Theorem 2.1. is not true.

Example 2.2. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. If $A = \{b, c, d\}$, then A is $\mathcal{I}_g$- open but not open.*

Theorem 2.3. *If $(X, \tau, \mathcal{I})$ is an T_I- ideal space and A is a $\star$-dense - in -itself, $\mathcal{I}_g$-closed subset of X, then A is closed.*

Proof: Let A be $\star$-dense-in -itself, $\mathcal{I}_g$- closed subset of X. Then by Theorem 2.1 [12], there exists no nonempty closed set in $cl^\star(A) - A$. Since $(X, \tau, \mathcal{I})$ is a T_I-space, A is $\star$- closed and so $cl^\star(A) - A = \emptyset$. Since A is $\star$- dense-in -itself, $cl^\star(A) = cl(A), cl(A) - A = \emptyset$ which implies that $cl(A) = A$. Thus A is closed. □

Theorem 2.4. *Every g- submaximal space is an $\mathcal{I}_g$- submaximal space.*

Proof: Let $A \subset X$ be $\star$- dense in a g- submaximal space $(X, \tau, \mathcal{I})$. Since every $\star$-dense set is a dense set, A is dense in X. Since$(X, \tau, \mathcal{I})$ is g- submaximal, A is g-open. By Remark 2 of [6], every g- open set is an $\mathcal{I}_g$- open set and so A is $\mathcal{I}_g$- open. Therefore every g- submaximal space is an $\mathcal{I}_g$- submaximal space. □

The following Example 2.5. shows that the converse of Theorem 2.4. is not true.

Example 2.5. *Let* $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$. *If* $A = \{a, c\}$, *then* A *is* $\mathcal{I}_g$- *open but not* g- *open.*

Theorem 2.6. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ *is an* $\mathcal{I}_g$- *submaximal space.*
(b) Every pre$-\mathcal{I}-$*open set is* $\mathcal{I}_g$- *open.*

Proof: $(a) \Rightarrow (b)$. Suppose that $(X, \tau, \mathcal{I})$ is $\mathcal{I}_g$- submaximal and $A \subset X$ be pre$-\mathcal{I}-$open. Then $A = U \cap D, U \in \tau, D$ is $\star$-dense. Since X is $\mathcal{I}_g$- submaximal, D is $\mathcal{I}_g$- open. Also U is $\mathcal{I}_g$- open. Since intersection of two $\mathcal{I}_g$- open sets is an $\mathcal{I}_g$- open set by Lemma 1.3, A is an $\mathcal{I}_g$- open set.
$(b) \Rightarrow (a)$. Let A be $\star$-dense in $(X, \tau, \mathcal{I})$. Then A is pre$-\mathcal{I}-$open which implies that A is $\mathcal{I}_g$- open, by hypothesis. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. □

Theorem 2.7. *Let* $(X, \tau, \mathcal{I})$ *be an ideal space. Then the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ *is an* $\mathcal{I}_g$- *submaximal space.*
(b) Every subset is $\mathcal{I}_g$*-locally-*$\star$*-closed set.*
(c) Every $\star$- *dense subset of* X *is an intersection of a* $\star$*-closed set and an* $\mathcal{I}_g$- *open subset of* X.

Proof: $(a) \Rightarrow (b)$. Let $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. Since every $\star$- dense set is open in $(X, \tau, \mathcal{I})$, by Corollary 4.7(b) of [13], every subset is $\mathcal{I}_g$- locally-$\star$-closed set.
$(b) \Rightarrow (c)$. Let A be $\star$- dense in $(X, \tau, \mathcal{I})$. By hypothesis, A is an $\mathcal{I}_g$ -locally-$\star$-closed set. Then by Theorem 4.3 of [13], there exists an $\mathcal{I}_g$- open set U such that $A = U \cap cl^\star(A)$. It follows that $A = U \cap X = U$ which implies that A is $\mathcal{I}_g$- open. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.
$(c) \Rightarrow (a)$. Let A be $\star$- dense in $(X, \tau, \mathcal{I})$. By hypothesis, $A = U \cap F$, where U is an $\mathcal{I}_g$- open and F is a $\star$- closed set. Since $A \subset F$, F is $\star$-dense and so $F = X$. Hence $A = U$ which is an $\mathcal{I}_g$- open set. Thus $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. □

Theorem 2.8. *For an ideal space* $(X, \tau, \mathcal{I})$*,the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ *is an* $\mathcal{I}_g$- *submaximal space.*
(b) For every subset $A \subset X$, *if* A *is not an* $\mathcal{I}_g$- *open set, then*
$A - intcl^\star(A) \neq \emptyset$.
(c) $\eta = \{U - A : U$ *is* $\mathcal{I}_g$*-open and* $int^\star(A) = \emptyset\}$ *where* η *is the family of all* $\mathcal{I}_g$*-open sets.*

Proof: $(a) \Rightarrow (b)$. Suppose that $A - intcl^\star(A) = \emptyset$. Then $A \subset intcl^\star(A)$ which implies A is pre$-\mathcal{I}-$open. Since X is $\mathcal{I}_g$- submaximal, A is $\mathcal{I}_g$- open which is a contradiction. Hence $A - intcl^\star(A) \neq \emptyset$.
$(b) \Rightarrow (a)$. Let A be pre$-\mathcal{I}-$open. Suppose that A is not an $\mathcal{I}_g$- open set. Then by hypothesis, $A - intcl^\star(A) \neq \emptyset$ which implies $A \nsubseteq intcl^\star(A)$ which is a contradiction. Hence A is $\mathcal{I}_g$- open which implies that $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.
$(a) \Rightarrow (c)$. Suppose that $\sigma = \{U - A : U$ is $\mathcal{I}_g$-open and $int^\star(A) = \emptyset\}$. Let $G \in \eta$.

Since $G = G - \emptyset$, and $int^\star(\emptyset) = \emptyset$, then $\eta \subset \sigma$. Let $G \in \sigma$. Then $G = U - A$, where U is $\mathcal{I}_g$- open and $int^\star(A) = \emptyset$. Then $G = U \cap (X - A)$. Since $int^\star(A) = \emptyset, X - int^\star(A) = cl^\star(X - A) = X$. Since X is $\mathcal{I}_g$- submaximal, $X - A$ is $\mathcal{I}_g$- open. By Lemma 1.3, G is $\mathcal{I}_g$- open. Hence $\sigma \subset \eta$.
$(c) \Rightarrow (a)$. Let A be a pre$-\mathcal{I}-$open set. By Lemma 1.1, $A = G \cap B$, where G is open and B is $\star$- dense. Hence $cl^\star(B) = X$ and so $int^\star(X - B) = \emptyset$. This implies $A = G - (X - B)$ and $int^\star(X - B) = \emptyset$. Since every open set is $\mathcal{I}_g$- open, G is $\mathcal{I}_g$- open. Hence by Lemma 1.3, A is $\mathcal{I}_g$- open. □

Theorem 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.
(b) $cl^\star(A) - A$ is $\mathcal{I}_g$- closed for every $A \subset X$.

Proof: $(a) \Rightarrow (b)$. Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}_g$- submaximal space and $A \subset X$. Consider $(X - (cl^\star(A) - A)) = (X - cl^\star(A)) \cup A$. Then $cl^\star(X - (cl^\star(A) - A)) = cl^\star((X - cl^\star(A)) \cup A) \supset (X - (cl^\star(A)) \cup cl^\star(A) = X$. Thus $cl^\star(X - (cl^\star(A) - A)) = X$. Hence $X - (cl^\star(A) - A)$ is $\mathcal{I}_g$- open which implies that $cl^\star(A) - A$ is $\mathcal{I}_g$- closed for every $A \subset X$.
$(b) \Rightarrow (a)$. Suppose that (b) holds. Let A be $\star$-dense in $(X, \tau, \mathcal{I})$. Since $cl^\star(A) - A$ is $\mathcal{I}_g$- closed for every $A \subset X$, $X - A$ is $\mathcal{I}_g$-closed which implies that A is an $\mathcal{I}_g$-open set for every $A \subset X$. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. □

Theorem 2.10. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.
(b) $A \cap (A^\star - A)^\star$ is $\mathcal{I}_g$- open for every $A \subset X$.

Proof: $(a) \Rightarrow (b)$. Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}_g$- submaximal space. Then by Theorem 2.9, every subset is $\mathcal{I}_g$- locally -$\star$- closed and so by Theorem 4.8 of [13], $A \cap (A^\star - A)^\star$ is $\mathcal{I}_g$- open for every $A \subset X$.
$(b) \Rightarrow (a)$. Let A be a $\star-$dense set in an ideal space $(X, \tau, \mathcal{I})$. Then by hypothesis, $A \cap (A^\star - A)^\star$ is $\mathcal{I}_g$- open for every $A \subset X$. Since A is $\star$- dense, $A^\star - A = cl^\star(A) - A = X - A$. This implies that $A \cap (A^\star - A)^\star = (A^\star - A)^\star \cap A = (X - A)^\star \cap A = (X - A)^\star - (X - A)$. Since $(X - A)^\star - (X - A)$ is $\mathcal{I}_g$- open, by Theorem 2.12 of [13], $X - A$ is $\mathcal{I}_g$- closed which implies that A is $\mathcal{I}_g$- open in $(X, \tau, \mathcal{I})$. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. □

A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\star$-*codense* [5], if $X - A$ is $\star$-dense. The following Theorem 2.12 follows from the definition of $\mathcal{I}_g$- submaximal spaces.

Theorem 2.11. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*
(a) $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.
(b) Every $\star$- codense subset A of X is $\mathcal{I}_g$- closed.

3. $\mathcal{I}_g$- Submaximal Subspaces

If $(X, \tau, \mathcal{I})$ is an ideal topological space and $A \subset X$, then $(A, \tau_A, \mathcal{I}_A)$, where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap \mathcal{J} : \mathcal{J} \in \mathcal{I}\}$ is an ideal topological space.

Lemma 3.1. *[9] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $B \subset A \subset X$. Then $B^\star(\tau_A, \mathcal{I}_A) = B^\star(\tau, \mathcal{I}) \cap A$.*

Lemma 3.2. *[7] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $B \subset A \subset X$. Then $cl^\star_A(B) = cl^\star(B) \cap A$.*

Theorem 3.3. *If $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space, then every open subspace $(A, \tau_A, \mathcal{I}_A)$ is an $\mathcal{I}_g$- submaximal space.*

Proof: Let B be $\star$- dense in $(A, \tau_A, \mathcal{I}_A)$. Let $U = B \cup (X - A)$. Then $cl^\star(U) = cl^\star(B) \cup cl^\star(X - A) \supset cl^\star_A(B) \cup cl^\star(X - A) = A \cup cl^\star(X - A)$. We have $cl^\star(U) \supset A \cup (X - A) \cup (X - A)^\star = X$. Therefore U is $\star$- dense in X. Since X is $\mathcal{I}_g$- submaximal, U is $\mathcal{I}_g$-open. Now $B = U \cap A$ is $\mathcal{I}_g$- open, since A is open. Therefore $(A, \tau_A, \mathcal{I}_A)$ is an $\mathcal{I}_g$- submaximal space. □

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ is codense. If every subset is $\mathcal{I}$- locally closed, then $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space.*

Proof: Let A be $\tau^\star$- dense. Since $\mathcal{I}$ is codense, A is $\mathcal{I}$- dense. Also A is $\mathcal{I}$- locally closed. Then by Theorem 4.8 of [14], A is open. Thus A is $\mathcal{I}_g$- open. Hence $(X, \tau, \mathcal{I})$ is an $\mathcal{I}_g$- submaximal space. □

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an $\mathcal{I}_g$- submaximal space and $\mathcal{I} \subset \mathcal{J}$ where $\mathcal{I}$ and $\mathcal{J}$ are ideals on X. Then $(X, \tau, \mathcal{J})$ is $\mathcal{J}_g$- submaximal.*

Proof: Let A be $\tau^\star(\mathcal{J})$- dense in $(X, \tau, \mathcal{J})$. Then $A \cup A^\star(\mathcal{J}) = X$. Since $\mathcal{I} \subset \mathcal{J}$, $A^\star(\mathcal{J}) \subset A^\star(\mathcal{I})$. Hence $X = A \cup A^\star(\mathcal{J}) \subset A \cup A^\star(\mathcal{I})$ which implies that $A \cup A^\star(\mathcal{I}) = X$. Thus A is $\tau^\star(\mathcal{I})$-dense. Since $(X, \tau, \mathcal{I})$ is $\mathcal{I}_g$- submaximal, A is $\mathcal{I}_g$- open. Since $\mathcal{I} \subset \mathcal{J}$, A is $\mathcal{J}_g$- open. Hence $(X, \tau, \mathcal{J})$ is $\mathcal{J}_g$- submaximal. □

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