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Vector Valued Multiple Sequence Spaces Defined by Orlicz Function

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abstract: In this article we define some vector valued multiple sequence spaces defined by Orlicz function. We study some of their properties like solidness, symmetry, completeness etc and some inclusion results.

Key Words: Orlicz function; completeness; semi-norm; regular convergence; solid space.

Contents

1. Introduction

Throughout this article the space of all, bounded, convergent in Pringsheim's sense, null in Pringsheim's sense, regularly convergent and regularly null multiple sequences defined over a semi-normed space (X, q), semi-normed by q will be denoted by ${}_k\omega\,(q)$, ${}_k\ell_\infty\,(q)$, ${}_kc\,(q)$, ${}_kc_0\,(q)$, ${}_kc^R\,(q)$, ${}_kc_0^R\,(q)$. For $X = C$, the field of complex numbers, these spaces represent the corresponding scalar sequence spaces. Throughout this article θ represents the zero element of X. The zero element of a single sequence space is denoted by $\overline{\theta} = (\theta, \theta, ...)$. The zero element of a multiple sequence is denoted by ${}_k\overline{\theta}$, a multiple infinite array of θ's.

An Orlicz function M is a mapping $M : [0, \infty) \to [0, \infty)$ such that it is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

The idea of Orlicz function was used by Lindenstrauss and Tzafriri [6] to construct the sequence space.

$$\ell^M = \left\{(x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right),\ \text{for some } \rho > 0\right\},$$

which is a Banach space normed by

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$||x_k|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{x_k}{\rho}\right) \leq 1 \right\}$

The space ℓ^M is an Orlicz sequence space with $M(x) = |x|^p$ for $1 \leq p < \infty$, which is closely related to the sequence space ℓ^p.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq K(Mu)$, where $u \geq 0$.

Recently Tripathy [9], Tripathy and Mahanta [11], Altin et. al. [1] and many others investigated the Orlicz sequence spaces from sequence point of view and related with the summability theory.

Remark 1.1. *Let $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$.*

2. Definition and Preliminaries.

Throughout this article a multiple sequence is denoted by $A = < a_{n_1 n_2 \ldots n_k} >$, a multiple infinite array of elements $a_{n_1 n_2 \ldots n_k} \in X$ for all $n_1, n_2, .., n_k \in N$.

Initial works on double sequences is found in Bromwich [3]. Hardy [5] introduced the notion of regular convergence for double sequences. Moricz [7] studied some properties of double sequences of real and complex numbers. Recently different types of double sequence have been introduced and investigated from different aspects by Basarir and Sonalcan [2], Colak and Turkmenoglu [4], Turkmenoglu [14], Moricz and Rhodes [8], Tripathy [10], Tripathy and Sarma ([12], [13])and many others.

Definition 2.1. *A multiple sequence space E is said to be solid if $< \alpha_{n_1 n_2 \ldots n_k} a_{n_1 n_2 \ldots n_k} > \in E$, whenever $< a_{n_1 n_2 \ldots n_k} > \in E$ for all multiple sequences $< \alpha_{n_1 n_2 \ldots n_k} >$ of scalars with $|\alpha_{n_1 n_2 \ldots n_k}| \leq 1$, for all $n_1, n_2, ..., n_k \in N$.*

Definition 2.2. *A multiple sequence space E is said to be symmetric if $< a_{n_1 n_2 \ldots n_k} > \in E$, implies $< a_{\pi(n_1, n_2, \ldots, n_k)} > \in E$, where $\pi(n_1, n_2, ..., n_k)$ are permutations of $N \times N ... \times N$.*

Definition 2.3. *A multiple sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Definition 2.4. *A multiple sequence space E is said to be convergence free if $< b_{n_1 n_2 \ldots n_k} > \in E$, whenever $< a_{n_1 n_2 \ldots n_k} > \in E$ and $b_{n_1 n_2 \ldots n_k} = \theta$ whenever $a_{n_1 n_2 \ldots n_k} = \theta$.*

Remark 2.5. *A sequence space E is solid implies E is monotone.*

Let M be an Orlicz function. Now we introduce the following multiple sequence spaces:

$$ {}_k\ell_\infty(M,q) = \left\{ < a_{n_1 n_2 \dots n_k} > \in {}_k\omega(q) : \sup_{n_1 n_2 \dots n_k} M\left(q\left(\frac{a_{n_1 n_2 \dots n_k}}{\rho}\right)\right) < \infty \right. $$
$$ \left. \text{for some } \rho > 0 \right\} $$

$$ {}_k c(M,q) = \left\{ < a_{n_1 n_2 \dots n_k} > \in {}_k\omega(q) : M\left(q\left(\frac{a_{n_1 n_2 \dots n_k} - L}{\rho}\right)\right) \to 0 \right. $$
$$ \left. \text{as } n_1, n_2, .., n_k \to \infty \text{ for some } \rho > 0 \right\} $$

$A = < a_{n_1 n_3 \dots n_k} > \in {}_k c^R(M,q)$, i.e., regularly convergent if $< a_{n_1 n_2 \dots n_k} > \in {}_k c(M,q)$ and the following limits hold:

There exists $L_{n_2 n_3 \dots n_k}, L_{n_1 n_3 \dots n_k}, L_{n_1 n_2 n_4 \dots n_k}, \dots, L_{n_1 n_2 \dots n_{i-1} n_{i+1} \dots n_k}, \dots, L_{n_1 n_2 \dots n_{k-1}} \in X$ such that

$M\left(q\left(\frac{a_{n_1 n_2 \dots n_k} - L_{n_2 n_3 \dots n_k}}{\rho}\right)\right) \to 0$ as $n_1 \to \infty$ for some $\rho > 0$ and $n_2, n_3, \dots, n_k \in N$.

$M\left(q\left(\frac{a_{n_1 n_2 \dots n_k} - L_{n_1 n_3 \dots n_k}}{\rho}\right)\right) \to 0$ as $n_2 \to \infty$ for some $\rho > 0$ and $n_1, n_3, \dots, n_k \in N$.

$M\left(q\left(\frac{a_{n_1 n_2 \dots n_k} - L_{n_1 n_2 n_4 \dots n_k}}{\rho}\right)\right) \to 0$ as $n_3 \to \infty$ for some $\rho > 0$ and $n_1, n_2, n_4, \dots, n_k \in N$.

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$M\left(q\left(\frac{a_{n_1 n_2 \dots n_k} - L_{n_1 n_2 \dots n_{i-1} n_{i+1} \dots n_k}}{\rho}\right)\right) \to 0$ as $n_i \to \infty$ for some $\rho > 0$ and $n_1, n_2, \dots, n_{i-1}, n_{i+1}, .., n_k \in N$.

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$M\left(q(\frac{a_{n_1 n_2 \dots n_k} - L_{n_1 n_2 \dots n_{k-1}}}{\rho})\right) \to 0$ as $n_k \to \infty$ for some $\rho > 0$ and $n_1, n_2, \dots, n_{k-1} \in N$.

Without loss of generality, ρ can be chosen to be same for all the above cases.

The definition of ${}_kc_0\,(M,q)$ and ${}_kc_0^R\,(M,q)$ follows from the above definition on taking

$L = L_{n_2n_3...n_k} = L_{n_1n_3n_4...n_k} = ... = L_{n_1n_2...n_{i-1}n_{i+1}...n_k} = ... = L_{n_1n_2...n_{k-1}} = \theta$ for all $n_1, n_2, ..., n_k \in N$

Remark 2.6. *The space ${}_kc_0^R\,(M,q)$ has the following definition too.*

$${}_kc_0^R(M,q) = \left\{ < a_{n_1n_2...n_k} > \in {}_k\omega(q) : M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right) \to 0 \text{ as } \max\{n_1, n_2, ..., n_k\} \to \infty \text{ for some } \rho > 0 \right\}.$$

We also define

$${}_kc^B\,(M,q) = {}_kc\,(M,q) \cap {}_k\ell_\infty\,(M,q) \text{ and } {}_kc_0^B\,(M,q) = {}_kc_0\,(M,q) \cap {}_k\ell_\infty\,(M,q).$$

3. Main Results

Theorem 3.1. *The classes $Z\,(M,q)$ for $Z = {}_k\ell_\infty$, ${}_kc$, ${}_kc_0$, ${}_kc^B$, ${}_kc^R$, and ${}_kc_0^R$ of multiple sequences are linear spaces.*

Proof: We prove it for the case ${}_k\ell_\infty\,(M,q)$ and other cases can also be proved similarly.

Let $< a_{n_1n_2...n_k} >, < b_{n_1n_2...n_k} > \in {}_k\ell_\infty\,(M,q)$.

Then we have

$$\sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) < \infty, \text{ for some } \rho_1 > 0 \tag{3.1}$$

$$\sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_2}\right)\right) < \infty \text{ for some } \rho_2 > 0. \tag{3.2}$$

Let α, β be scalars and $\rho = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$.

Then

$$\sup_{n_1,n_2,...,n_k} M\left(q\left(\frac{\alpha a_{n_1n_2...n_k} + \beta b_{n_1n_2...n_k}}{\rho}\right)\right)$$
$$\leq \frac{1}{2} \sup_{n_1,n_2,...,n_k} M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) + \frac{1}{2} \sup_{n_1,n_2,...,n_k} M\left(q\left(\frac{b_{n_1n_2...n_k}}{\rho_2}\right)\right) < \infty.$$

Hence $< \alpha a_{n_1n_2\ldots n_k} + \beta b_{n_1n_2\ldots n_k} > \in \ {}_k\ell_\infty(M,q)$.

Thus ${}_k\ell_\infty(M,q)$ is a linear space. $\square$

Theorem 3.2. *The spaces $Z(M,q)$ for $Z = \ {}_k\ell_\infty,\ {}_kc^B,\ {}_kc_0^B,\ {}_kc^R,\ {}_kc_0^R$ are semi-normed spaces, semi-normed by*

$$f(< a_{n_1n_2\ldots n_k} >) = \inf\left\{\rho > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\frac{a_{n_1n_2\ldots n_k}}{\rho}\right)\right) \le 1\right\}. \quad (3.3)$$

Proof: Clearly $f({}_k\overline{\theta}) = 0$ and $f(- < a_{n_1n_2\ldots n_k} >) = f(< a_{n_1n_2\ldots n_k} >)$ for all $< a_{n_1n_2\ldots n_k} > \in \ {}_k\ell_\infty(M,q)$. Let $\lambda \in C$, then we have

$$f(< a_{n_1n_2\ldots n_k} >) = \inf\left\{\rho > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}}{\rho}\right)\right) \le 1\right\}$$

$$\begin{aligned} f(\lambda < a_{n_1n_2\ldots n_k} >) &= \inf\left\{\rho > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{\lambda a_{n_1n_2\ldots n_k}}{\rho}\right)\right) \le 1\right\} \\ &= |\lambda| \inf\left\{r > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}}{r}\right)\right) \le 1\right\} \text{ where } r = \tfrac{\rho}{|\lambda|} \\ &= |\lambda| f(< a_{n_1n_2\ldots n_k} >). \end{aligned}$$

Next let $< a_{n_1n_2\ldots n_k} >, < b_{n_1n_2\ldots n_k} > \in \ {}_k\ell_\infty(M,q)$. Then we have

$$f(< a_{n_1n_2\ldots n_k} >) = \inf\left\{\rho_1 > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}}{\rho_1}\right)\right) \le 1\right\},$$

$$f(< b_{n_1n_2\ldots n_k} >) = \inf\left\{\rho_2 > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{b_{n_1n_2\ldots n_k}}{\rho_2}\right)\right) \le 1\right\}.$$

Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}}{\rho_1}\right)\right) \le 1$$

and

$$\sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{b_{n_1n_2\ldots n_k}}{\rho_2}\right)\right) \le 1.$$

Let $\rho = \rho_1 + \rho_2$, then we have

$$\sup_{n_1n_2\ldots n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}+b_{n_1n_2\ldots n_k}}{\rho}\right)\right)$$

$$\le \tfrac{\rho_1}{\rho_1+\rho_2} \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{a_{n_1n_2\ldots n_k}}{\rho_1}\right)\right) + \tfrac{\rho_2}{\rho_1+\rho_2} \sup_{n_1,n_2,\ldots,n_k} M\left(q\left(\tfrac{b_{n_1n_2\ldots n_k}}{\rho_2}\right)\right).$$

Since ρ_1 and ρ_2 are non-negative, so we have

$$f\left(< a_{n_1n_2...n_k} > + < b_{n_1n_2...n_k} >\right)$$
$$= \inf\left\{\rho = \rho_1 + \rho_2 > 0 : \sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{a_{n_1n_2...n_k} + b_{n_1n_2...n_k}}{\rho}\right)\right) \leq 1\right\}$$
$$\leq \inf\left\{\rho_1 > 0 : \sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) \leq 1\right\}$$
$$+ \inf\left\{\rho_2 > 0 : \sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{b_{n_1n_2...n_k}}{\rho_2}\right)\right) \leq 1\right\}$$
$$= f\left(< a_{n_1n_2...n_k} > + < b_{n_1n_2...n_k} >\right)$$

Hence f is a semi-norm on $Z(M,q)$ for $Z = {}_k\ell_\infty, {}_kc^B, {}_kc_0^B, {}_kc^R, {}_kc_0^R$.

□

Theorem 3.3. *The spaces* ${}_k\ell_\infty(M,q)$ *and* ${}_kc_0^R(M,q)$ *are symmetric where as the spaces* $Z(M,q)$ *for* $Z = {}_kc, {}_kc_0, {}_kc^B, {}_kc_0^B, {}_kc^R, {}_kc_0^R$ *are not symmetric.*

Proof: The space ${}_k\ell_\infty(M,q)$ is symmetric is obvious. We prove it for ${}_kc_0^R(M,q)$.

Let $< a_{n_1n_2...n_k} > \in {}_kc_0^R(M,q)$. Then for a given $\varepsilon > 0$ there exists positive integers $k_1, k_2, k_3, ..., k_k, k_{k+1}$ such that

$M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_1 > k_1$ for all $n_2, n_3, ..., n_k \in N$

$M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_2 > k_2$ for all $n_1, n_3, ..., n_k \in N$

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$M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_k > k_k$ for all $n_1, n_2, n_3, ..., n_{k-1} \in N$

$M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_1 > k_{k+1}, n_2 > k_{k+1}, ..., n_k > k_{k+1}$

and without loss of generality ρ can be chosen to be same for the $(k+1)$ cases.

Let $k_0 = \max\{k_1, k_2, k_3, ..., k_k, k_{k+1}\}$.

Let $< b_{n_1n_2...n_k} >$ be a rearrangement of $< a_{n_1n_2...n_k} >$. Then we have

$a_{i_1i_2...i_k} = b_{n_{i_1}n_{i_2}n_{i_3}...n_{i_k}}$ for all $i_1, i_2, ..., i_k \in N$.

Let

$k_{k+2} = \max\{n_{1_1}, n_{2_1}, ..., n_{k_1}, n_{(k+1)_1}, n_{(k_0)_1}, n_{1_2}, ..., n_{k_2}, n_{(k+1)_2}, n_{(k_0)_2}, ..., n_{1_k}, n_{2_k}, ..., n_{n_k}, n_{(k+1)_k}, n_{(k_0)_k}\}$.

Then we have

$M\left(q\left(\frac{b_{n_1n_2...n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_1 > k_{k+2}, n_2 > k_{k+2}, ..., n_k > k_{k+2}$.

Thus $< b_{n_1n_2...n_k} > \in {}_kc_0^R(M,q)$. Hence ${}_kc_0^R(M,q)$ is a symmetric space. □

To show that ${}_kc^R(M,q)$ is not symmetric, we consider the following example.

Example 3.4. *Let $X = C$ and define $< a_{n_1n_2...n_k} >$ by*

$a_{n_1n_2...n_k} = 1$, *for all $n_1 = 1$ and for all $n_2, n_3, ..., n_k \in N$.*

$= 0$, *otherwise.*

Then $< a_{n_1n_2...n_k} > \in {}_kc^R(M,q)$

Now consider the rearranged sequence $< b_{n_1n_2...n_k} >$ of $< a_{n_1n_2...n_k} >$ defined by

$b_{n_1n_2...n_k} = 1$ *for all $n_1 = n_2 = ... = n_k$*

$= 0$, *otherwise.*

Then $< b_{n_1n_2...n_k} > \notin {}_kc^R(M,q)$. Hence ${}_kc^R(M,q)$ is not symmetric.

Examples similar to the above can be constructed to establish that the other spaces are also not symmetric.

Theorem 3.5. *The spaces ${}_kc_0^R(M,q), {}_kc_0^B(M,q), {}_kc_0(M,q)$ and ${}_k\ell_\infty(M,q)$ are solid, but the spaces ${}_kc(M,q)$, ${}_kc^B(M,q)$ and ${}_kc^R(M,q)$ are not solid.*

Proof: The spaces $Z(M,q)$ for $Z = {}_k\ell_\infty$, ${}_kc_0$, ${}_kc_0^B$ and ${}_kc_0^R$, are solid follows from the following inequality.

$M\left(q\left(\frac{\alpha_{n_1n_2...n_k}a_{n_1n_2...n_k}}{\rho}\right)\right) \le M\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right)$ for all $n_1, n_2, ..., n_k \in N$ and scalars $< \alpha_{n_1n_2...n_k} >$ with $|\alpha_{n_1n_2...n_k}| \le 1$, for all $n_1, n_2, ..., n_k \in N$. □

To show that ${}_kc(M,q)$, ${}_kc^B(M,q)$ and ${}_kc^R(M,q)$ are not solid we consider the following example.

Example 3.6. *Let $X = C$, $M(x) = x$, $q(x) = |x|$. Define the sequence $< a_{n_1n_2...n_k} >$ by $a_{n_1n_2...n_k} = 1$ for all $n_1, n_2, ..., n_k \in N$. Consider the sequence*

$< \alpha_{n_1n_2...n_k} >$ of scalars defined by $\alpha_{n_1n_2...n_k} = (-1)^{n_1+n_2+...+n_k}$ for all n_1, n_2, ..., $n_k \in N$.

Then $< a_{n_1n_2...n_k} > \in Z(M,q)$, but $< \alpha_{n_1n_2...n_k} a_{n_1n_2...n_k} > \notin Z(M,q)$ for $Z = {}_kc$, ${}_kc^B$ and ${}_kc^R$. Hence $Z(M,q)$ is not solid for $Z = {}_kc$, ${}_kc^B$, ${}_kc^R$.

Theorem 3.7. *The spaces $Z(M,q)$ are monotone for $Z = {}_kc_0^R$, ${}_k\ell_\infty$, ${}_kc_0$, ${}_kc_0^B$, but are not monotone for $Z = {}_kc$, ${}_kc^B$, ${}_kc^R$.*

Proof: The first part follows from the Remark 2.5 and Theorem 3.5. For the second part, we consider the following example. □

Example 3.8. *Let $X = C$, $M(x) = x$, $q(x) = |x|$. Consider the sequence $< a_{n_1n_2...n_k} >$ defined by $a_{n_1n_2...n_k} = 1$ for all $n_1, n_2, ..., n_k \in N$. We consider its pre-images on the step space E defined by $< b_{n_1n_2...n_k} > \in E$, implies $b_{n_1n_2...n_k} = 0$, for $n_2, n_3, , ..., n_k$ even and for all $n_1 \in N$.*

Then the pre-image of $< a_{n_1n_2...n_k} > \notin Z(M,q)$, for $Z = {}_kc$, ${}_kc^B$, ${}_kc^R$.

Theorem 3.9. *Let X be a complex semi-normed space, then the spaces $Z(M,q)$ for $Z = {}_kc_0^R$, ${}_k\ell_\infty$, ${}_kc^R$, ${}_kc_0^B$ and ${}_kc^B$ are complete semi-normed spaces semi-normed f defined by (3.3).*

Proof: We prove it for the space ${}_k\ell_\infty(M,q)$ and other cases can be established following similar technique.

Let $A_i = < a^i_{n_1n_2...n_k} >$ be a Cauchy sequence in ${}_k\ell_\infty(M,q)$. Let $\varepsilon > 0$ be fixed and $r > 0$, we choose x_0 such that $M\left(\frac{rx_0}{2}\right) \geq 1$ and $rx_0 \geq 1$. Then there exists a positive integer m_0 such that

$f\left(a^i_{n_1n_2...n_k} - a^j_{n_2...n_k}\right) < \frac{\varepsilon}{rx_0}$ for all i, $j \geq m_0$.

Using definition of semi-norm we have

$$\inf\left\{r > 0 : \sup_{n_1, n_2, ..., n_k} M\left(q\left(\frac{a^i_{n_1n_2...n_k} - a^j_{n_1n_2...n_k}}{r}\right)\right) \leq 1\right\} < \frac{\varepsilon}{rx_0}, \quad (3.4)$$

for all $i, j \geq m_0$

and

$$\sup_{n_1n_2...n_k} M\left(q\left(\frac{a^i_{n_1n_2...n_k} - a^j_{n_1n_2...n_k}}{f\left(a^i_{n_1n_2...n_k} - a^j_{n_1n_2...n_k}\right)}\right)\right) \leq 1, \text{ for all } i,\ j \geq m_0.$$

It follows that

$M\left(q\left(\frac{a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}}{f\left(a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}\right)}\right)\right)\leq 1$, for all $i,j\geq m_0$.

For $r>0$ with $M\left(\frac{rx_0}{2}\right)\geq 1$ we have

$M\left(q\left(\frac{a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}}{f\left(a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}\right)}\right)\right)\leq M\left(\frac{rx_0}{2}\right)$, for all $i,j\geq m_0$.

Since M is continuous, so we have

$q\left(\frac{a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}}{f\left(a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}\right)}\right)\leq\left(\frac{rx_0}{2},\right)$, for all $i,j\geq m_0$.

$\Rightarrow q\left(a^i_{n_1n_2\ldots n_k}-a^j_{n_2,\ldots,n_k}\right)\leq\left(\frac{rx_0}{2}\right)\left(\frac{\varepsilon}{rx_0}\right)=\frac{\varepsilon}{2}$, for all $i,j\geq m_0$.

$\Rightarrow < a^i_{n_1n_2\ldots n_k} >$ is a Cauchy sequence in X.

Since X is complete, there exists $a_{n_1n_2\ldots n_k}\in X$ such that

$\lim\limits_{i\to\infty} a^i_{n_1n_2\ldots n_k}=a_{n_1n_2\ldots n_k}$ for all $n_1,n_2,....,n_k\in N$.

So we have from (3.4) for all $i,j\geq m_0$,

$\inf\left\{r>0:\sup\limits_{n_1,n_2,..,n_k} M\left(q\left(\frac{a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}}{r}\right)\right)\leq 1\right\}<\varepsilon$,

$\lim\limits_{j\to\infty}\inf\left\{r>0:\sup\limits_{n_1,n_2,..,n_k} M\left(q\left(\frac{a^i_{n_1n_2\ldots n_k}-a^j_{n_1n_2\ldots n_k}}{r}\right)\right)\leq 1\right\}<\varepsilon$, for all $i\geq m_0$.

On taking the infimum of such r's we have

$\inf\{r>0:\sup\limits_{n_1,n_2,\ldots,n_k} M\left(q\left(\frac{a^i_{n_1n_2\ldots n_k}-a_{n_1n_2\ldots n_k}}{r}\right)\right)\leq 1\}<\varepsilon$, for all $i\geq m_0$.

$\Rightarrow< a^i_{n_1n_2\ldots n_k}-a_{n_1n_2\ldots n_k} >\in\ {}_k\ell_\infty(M,q)$.

Since ${}_k\ell_\infty(M,q)$ is a linear space, we have for all $i\geq m_0$

$< a_{n_1n_2\ldots n_k} >=< a^i_{n_1n_2\ldots n_k} > - < a^i_{n_1n_2\ldots n_k}-a_{n_1n_2\ldots n_k} >\in_k \ell_\infty(M,q)$.

Thus ${}_k\ell_\infty(M,q)$ is a complete semi-normed space. □

We state the following result without proof.

Theorem 3.10. *The spaces $Z(M,q)$ for $Z=\ {}_k\ell_\infty,\ {}_kc^B,\ {}_kc^B_0,\ {}_kc^R$ and ${}_kc^R_0$ are K-spaces.*

Since $Z(M,q) \subset {}_k\ell_\infty(M,q)$ for $Z = {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$, so the following result is a consequence of Theorem 3.9.

Theorem 3.11. *The spaces* $Z(M,q)$ *for* $Z = {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$ *are nowhere dense subsets of* ${}_k\ell_\infty(M,q)$.

Theorem 3.12. *Let* M_1 *and* M_2 *be Orlicz functions. Then we have*

(i) $Z(M_1,q) \subseteq Z(M_2 \circ M_1, q)$, *for* $Z = {}_k\ell_\infty,\ {}_kc,\ {}_kc_0,\ {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$.

(ii) $Z(M_1,q) \cap Z(M_2,q) \subseteq Z(M_1+M_2,q)$, *for* $Z = {}_k\ell_\infty,\ {}_kc,\ {}_kc_0,\ {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$.

(iii) $Z(M_1,q_1) \cap Z(M_1,q) \subseteq Z(M_1,q_1+q_2)$, *for* $Z = {}_k\ell_\infty,\ {}_kc,\ {}_kc_0,\ {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$, q_1, q_2 *are two semi-norms on* X.

(iv) If q_1 *is stronger than* q_2, *then* $Z(M_1,q_1) \subseteq Z(M_1,q_2)$, *for* $Z = {}_k\ell_\infty,\ {}_kc,\ {}_kc_0,\ {}_kc^B,\ {}_kc_0^B,\ {}_kc^R$ *and* ${}_kc_0^R$.

Proof: **(**i**)** We prove this for the space ${}_kc^R(M_1,q)$ and the other cases can be established in a similar way.

Let $< a_{n_1n_2\ldots n_k} > \in {}_kc^R(M_1,q)$. Then there exists $\rho > 0$ such that for a given $\varepsilon > 0$ with $0 < \frac{\varepsilon}{M_2(1)} < 1$. We have $n_{1_0}, n_{2_0}, n_{3_0}, \ldots, n_{k_0} \in N$ such that

$$M_1\left(q\left(\frac{a_{n_1n_2\ldots n_k} - L}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)} \text{ for all } n_1 > n_{1_0}, n_2 > n_{2_0}, \ldots., n_k > n_{k_0}. \tag{3.5}$$

$$M_1\left(q\left(\frac{a_{n_1n_2\ldots n_k} - L_{n_1}}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)} \text{ for all } n_1 > n_{1_0} \text{ and for all } n_2, n_3, \ldots, n_k \in N. \tag{3.6}$$

$$M_1\left(q\left(\frac{a_{n_1n_2\ldots n_k} - L_{n_2}}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)} \text{ for all } n_2 > n_{2_0} \text{ and for all } n_1, n_3, \ldots, n_k \in N. \tag{3.7}$$

....

....

$$M_1\left(q\left(\frac{a_{n_1n_2\ldots n_k} - L_{n_k}}{\rho}\right)\right) < \frac{\varepsilon}{M_2(1)}, \text{ for all } n_k > n_{k_0} \text{ and for all } n_1, n_2, \ldots, n_{k-1} \in N, \tag{3.8}$$

Thus from Remark 1.1 and from (3.5),(3.6), (3.7) and (3.8) we have

$(M_2 \circ M_1)\left(q\left(\frac{a_{n_1n_2...n_k}-L}{\rho}\right)\right) < \varepsilon$ for all $n_1 > n_{1_0}, n_2 > n_{2_0}, ..., n_k > n_{k_0}$.

$(M_2 \circ M_1)\left(q\left(\frac{a_{n_1n_2...n_k}-L_{n_1}}{\rho}\right)\right) < \varepsilon$ for all $n_1 > n_{1_0}$ and for all $n_2, n_3, ..., n_k \in N$

$(M_2 \circ M_1)\left(q\left(\frac{a_{n_1n_2...n_k}-L_{n_2}}{\rho}\right)\right) < \varepsilon$ for all $n_2 > n_{2_0}$ and for all $n_1, n_3, ..., n_k \in N$.

....

$(M_2 \circ M_1)\left(q\left(\frac{a_{n_1n_2...n_k}-L_{n_k}}{\rho}\right)\right) < \varepsilon$ for all $n_k > n_{k_0}$ and for all $n_1, n_2, ..., n_{k-1} \in N$.

Hence $< a_{n_1n_2...n_k} > \in\ _kc^R(M_2 \circ M_1, q)$.

Thus $_kc^R(M_1, q) \subseteq\ _kc^R(M_2 \circ M_1, q)$.

(ii) We prove the result for the case $_k\ell_\infty$. Other cases will follow similarly.

Let $< a_{n_1n_2...n_k} > \in\ _k\ell_\infty(M_1, q) \cap\ _k\ell_\infty(M_2, q)$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$, such that

$$\sup_{n_1,n_2,...,n_k} M_1\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) < \infty$$

and

$$\sup_{n_1,n_2,...,n_k} M_2\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_2}\right)\right) < \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then

$$\sup_{n_1,n_2,...,n_k} (M_1 + M_2)\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho}\right)\right)$$
$$\leq \sup_{n_1,n_2,...,n_k} M_1\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) + \sup_{n_1,n_2,...,n_k} M_2\left(q\left(\frac{a_{n_1n_2...n_k}}{\rho_2}\right)\right) < \infty.$$

Hence $< a_{n_1n_2...n_k} > \in\ _k\ell_\infty(M_1 + M_2, q)$.

(iii) Let $< a_{n_1n_2...n_k} > \in\ _k\ell_\infty(M_1, q_1) \cap\ _k\ell_\infty(M_1, q_2)$. Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_{n_1,n_2,...,n_k} M_1\left(q_1\left(\frac{a_{n_1n_2...n_k}}{\rho_1}\right)\right) < \infty$$

and

$$\sup_{n_1,n_2,\ldots,n_k} M_1\left(q_1\left(\frac{a_{n_1n_2\ldots n_k}}{\rho_2}\right)\right) < \infty.$$

Let $\rho = \max\{\rho_1, \rho_2\}$. Then

$$\sup_{n_1,n_2,\ldots,n_k} M_1\left((q_1+q_2)\left(\frac{a_{n_1n_2\ldots n_k}}{\rho}\right)\right)$$
$$\leq \sup_{n_1,n_2,\ldots,n_k} M_1\left(q_1\left(\frac{a_{n_1n_2\ldots n_k}}{\rho_1}\right)\right) + \sup_{n_1,n_2,\ldots,n_k} M_1\left(q_2\frac{a_{n_1n_2\ldots n_k}}{\rho_2}\right) < \infty.$$

Hence $< a_{n_1n_2\ldots n_k} > \in {}_k\ell_\infty(M_1, q_1+q_2)$. □

The following result is a consequence of Theorem 3.12(i).

Proposition 3.13. *Let M be an Orlicz function, then $Z(q) \subset Z(M,q)$, for $Z = {}_k\ell_\infty$, ${}_kc$, ${}_kc_0$, ${}_kc^B$, ${}_kc_0^B$, ${}_kc^R$, ${}_kc_0^R$.*

4. Particular cases

If we take X to be normed linear space, instead of a semi-normed space, then all the results of Section 3 will follow immediately. In that case spaces $Z(M, ||.||)$,where $Z = {}_k\ell_\infty$, ${}_kc^B$, ${}_kc_0^B$, ${}_kc^R$, ${}_kc_0^R$ will be normed linear spaces, normed by

$$f(< a_{n_1n_2.....n_k} >) = \inf\left\{\rho > 0 : \sup_{n_1,n_2,\ldots,n_k} M\left(\|\frac{a_{n_1n_2.....n_k}}{\rho}\|\right) \leq 1\right\}.$$

These spaces would be Banach spaces under f when X is a Banach space.

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