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Existence and uniqueness of solution for $p(x)$-Laplacian problems

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ABSTRACT: This paper shows the existence and uniqueness of weak solution of a problem which involves the $p(x)$-Laplacian with some different boundary conditions. The proof of the result is made by Browder Theorem.

Key Words: $p(x)-$Laplacian, Weak solutions, variational methods

Contents

1. Introduction

At the turn of the millennium, a large number of papers is scattered to study of elliptic equations and variational problems with variable exponent, it is of considerable importance in the theory of partial differential equations. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids or image processing and restoration... For more inquiries on modeling physical phenomena involving $p(x)$-growth condition we refer to [2,3,4,5,6,8].

In this work, we consider the following problems involving $p(x)$-Laplacian

. Dirichlet problem

$$\begin{gathered} -\Delta_{p(x)} u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

. Neumann problem

$$\begin{gathered} -\Delta_{p(x)} u + |u|^{p(x)-2}u = f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

. No flux problem

$$\begin{gathered} -\Delta_{p(x)} u = f(x,u) \quad \text{in } \Omega, \\ u = constant \quad \text{on } \partial\Omega, \\ \int_{\partial\Omega} |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = 0. \end{gathered} \tag{1.3}$$

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. Steklov problem

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2} u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= g(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

. Robin problem

$$\begin{aligned} \Delta_{p(x)} u &= f(x,u) \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta |u|^{p(x)-2} u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^N, \Delta_{p(x)} u = div(|\nabla u|^{p(x)-2} \nabla u)$ is the $p(x)-$Laplacian operator, $p \in C(\Omega)$, with $1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) = p^+ < \infty$ and $\frac{\partial u}{\partial \nu}$ denotes the out normal derivative of u, $\lambda \in \mathbb{R}$ and $\beta : \partial\Omega \to \mathbb{R} \in L^\infty(\partial\Omega)$ is real function with $\beta^- = \inf_{x\in\partial\Omega} \beta(x) > 0$.

Assume that

(f_1) f and g are Carathéodory functions which are decreasing with respect to the second variable.

(f_2) There exist $b > 0, c > 0$ and $q \in C(\Omega), r \in C(\overline{\Omega})$ such that

$$|f(x,t| \leq b(1 + |t|^{q(x)}), \ a.e \ x \in \Omega, t \in \mathbb{R},$$

and

$$|g(x,t) \leq c(1 + |t|^{r(x)}), \ a.e.x \in \partial\Omega, t \in \mathbb{R}.$$

where

$$1 < q(x) \leq \sup_{\Omega} q(x) = q^+ < p^-,$$

and

$$1 < r(x) \leq \sup_{\overline{\Omega}} r(x) = r^+ < p^-$$

(f_3) $f(x,0) \neq 0, \ g(x,0) \neq 0$.

We report our main result,

Theorem 1.1. *Suppose that f satisfies the conditions* $(f_1), (f_2)$ *and* (f_3). *Then the problems* (1.1)– (1.5) *have a unique weak solution.*

When $p = 2$, theses problems are normal Schrödinger equations which has been extensively studied. There are several studies of the existence of solutions such problems on a bounded domain of $\mathbb{R}^N$. We mention the results obtained in [1], [8] and [11] for the case when p is constant. In recent years, more and more attention is paid to the quasilinear elliptic with a variable exponent. The main difficulty in the study of $p(x)-$Laplacian equations arises from its inhomogeneity.

Define the operators I, J , L and K : $X \to X^*$ by

$$\langle I(u), v \rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u. \nabla v dx,$$

$$\langle J(u), v\rangle = \int_\Omega |u|^{p(x)-2} uv dx,$$

$$\langle L(u), v\rangle = \int_\Omega f(x,u) v dx,$$

$$\langle K(u), v\rangle = \int_{\partial\Omega} g(x,u) v dx.$$

Definition 1.2. *Let $X = W_0^{1,p(x)}(\Omega), W^{1,p(x)}(\Omega)$ or $W_0^{1,p(x)}(\Omega) \oplus \mathbb{R}, u \in X$.*

(i) We say that u is a weak solution of (1.1)and (1.3) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla v dx = \int_\Omega f(x,u) v dx,$$

for all $v \in X$.

(ii) Let $u \in W^{1,p(x)}(\Omega)$. We say that u is a weak solution of (1.2) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla v dx + \int_\Omega |u|^{p(x)-2} uv dx = \int_\Omega f(x,u) v dx,$$

for all $v \in W^{1,p(x)}(\Omega)$.

(iii) Let $u \in W^{1,p(x)}(\Omega)$, a weak solution u for Steklov problem (1.4) provided

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u.\nabla v + |u|^{p(x)-2} uv) dx = \int_{\partial\Omega} g(x,u) v \ dx,$$

for all $v \in W^{1,p(x)}(\Omega)$.

(iv) Let $u \in W^{1,p(x)}(\Omega)$, u is a weak solution of the Robin problem (1.5) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u.\nabla v dx + \int_{\partial\Omega} \beta |u|^{p(x)-2} uv dx = \int_\Omega f(x,u) v \ dx,$$

for all $v \in W^{1,p(x)}(\Omega)$.

Theorem 1.3. *(cf. [9]) Let T be a reflexive real Banach space. Moreover, let $T : X \to X^*$ be an operator which is: bounded, demicontinuous, coercive, and monotone on the space X . Then, the equation $T(u) = f$ has at least one solution $u \in X$ for each $f \in X^*$. If moreover, T is strictly monotone operator, then for every $f \in X^*$ the equation $T(u) = f$ has precisely one solution $u \in X$.*

Define the operator $T : \ X \to X^*$ by
$T = I + aJ - bL - cK - d\int_{\partial\Omega} \beta(x)|u|^{p(x)-2} u dx$, with a, b, c et $d \geq 0$.

Definition 1.4. *Let X be a real Hilbert space. An operator $I : X \to X^*$ verifies*

$$\langle I(u) - I(v), u - v\rangle \geq 0, \tag{1.6}$$

for any $u, v \in X$ is called a monotone operator. An operator I is called strictly monotone if for $u \neq v$ the strict inequality holds in (1.6). An operator I is called strongly monotone if there exists $C > 0$ such that

$$\langle I(u) - I(v), u - v\rangle \geq C \parallel u - v \parallel^2,$$

for any $u, v \in X$.

2. Preliminary Notes

In order to deal with these problems, we need some theory of variable exponent Sobolev Space. For convenience, we only recall some basic facts which will be used later, we refer to [7] and references therein for more details.

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$,
$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \; measurable \; : \; \int_\Omega \mid u \mid^{p(x)} dx < \infty\}$ then $L^{p(x)}(\Omega)$ endowed with the norm

$$\mid u \mid_{p(x)} = \inf\{\lambda > 0 \; : \; \int_\Omega \mid \frac{u}{\lambda} \mid^{p(x)} dx \leq 1\}$$

becomes a Banach separable and reflexive space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\| u \| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

which is a separable reflexive Banach space.

Proposition 2.1. *Set,* $\rho(u) = \int_\Omega \mid \nabla u \mid^{p(x)} dx$, *if* $u \in W^{1,p(x)}(\Omega)$ *we have*
(1) $\| u \| \geq 1 \Rightarrow \| u \|^{p^-} \leq \rho(u) \leq \| u \|^{p^+}$.
(2) $\| u \| \leq 1 \Rightarrow \| u \|^{p^+} \leq \rho(u) \leq \| u \|^{p^-}$.
(3) $\lim\limits_{n\to\infty} \| u_n \| = 0 (resp \; +\infty) \; \Leftrightarrow \lim\limits_{\mathrm{n}\to+\infty} \rho(\mathrm{u_n}) = 0 \; (\mathrm{resp} \; +\infty)$.

Remark 2.2. *We have similar results (1) and (2) for* $\rho_1(u) = \int_\Omega \mid u \mid^{p(x)} dx$ *as in the above.*

Proposition 2.3. *For any* $u \in L^{p(x)}(\Omega), \; v \in L^{p'(x)}(\Omega)$, *we have*

$$\mid \int_\Omega uv \; dx \mid \leq 2 \mid u \mid_{p(x)} \mid v \mid_{p'(x)},$$

with

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Lemma 2.4. *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *for* $x \in \overline{\Omega}$, *then there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N; \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Lemma 2.5. *If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and*

$$|f(x,s)| \leq a(x) + b|s|^{\frac{p_1(x)}{p_2(x)}}, \quad a.e. x \in \overline{\Omega}, for\ all\ t \in \mathbb{R},$$

where $p_1(x), p_2(x) \in C(\overline{\Omega})$, $a(x) \in L^{p_2(x)}(\Omega)$, $p_1(x) > 1$, $p_2(x) > 1$, $a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ to $L^{p_2(x)}(\Omega)$ defined by $N_f(u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Define

$$p^{\partial}(x) = (p(x))^{\partial} := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Lemma 2.6. *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$, where $1 \leq q(x) < p^{\partial}(x)$, $\forall x \in \partial\Omega$.*

3. Proof of the main result

(A) I, J, K and L are bounded, in fact, let $\| u \| \leq M$,

Since I and J are the Fréchet derivative of the functional $\int_\Omega \frac{1}{p(x)}(| \nabla u |^{p(x)} + | u |^{p(x)})dx$ and $\int_\Omega \frac{1}{p(x)} | u |^{p(x)} dx$ respectively, therefore I and J are bounded.

We have the same deduction for $\int_{\partial\Omega} \beta(x)|u|^{p(x)-2}u dx$.

Moreover, from proposition 2.3 and lemma 2.4, there exists $C_1 > 0$ such that

$$\begin{aligned} \| L(u) \|_{X^*} &= \sup_{\|v\|=1} | \langle L(u), v \rangle | \\ &\leq \sup_{\|v\|=1} 2 | f |_{p'(x)} | v |_{p(x)} \\ &\leq C_1 | f |_{p'(x)} . \end{aligned}$$

Similarly, in view of lemma 2.6 there exists $C_2 > 0$, such that

$$\| K(u) \|_{X^*} \leq C_2 | g |_{l^{p'(x)}(\partial\Omega)} .$$

so K is a bounded operator.

(B) I , J , K and L are continuous operators,

We have I and J are continuous operators because that are the Frêchet derivative of the functional $\int_\Omega \frac{1}{p(x)}(| \nabla u |^{p(x)} + | u |^{p(x)})dx$ and $\int_\Omega \frac{1}{p(x)} | u |^{p(x)} dx$ respectively, and then I with J are continuous.

On the other hand, Let $(u_n)_n \subset X$ be a sequence such that $u_n \rightharpoonup u$. Since there is a compact embedding of X into $L^{q(x)}(\Omega)$, there is a subsequence, denoted also by $(u_n)_n$, such that $u_n \to u$ in $L^{q(x)}(\Omega)$. According to the Krasnoselki's theorem, the Nemytskii operator

$$\begin{aligned} N_f : \ & L^{q(x)}(\Omega) \to L^{\frac{q(x)}{q(x)-1}}(\Omega) \\ & u \mapsto f(., u) \end{aligned}$$

is continuous. Hence, $N_f(u_n) \to N_f(u)$ in $L^{\frac{q(x)}{q(x)-1}}(\Omega)$. Using Hölder's inequality and the continuous embedding of X into $L^{q(x)}(\Omega)$, we obtain

$$\begin{aligned} \mid \langle L(u_n) - L(u), v\rangle \mid &= \mid \int_\Omega (f(x,u_n) - f(x,u))v(x)dx \mid \\ &\leq 2 \parallel N_f(u_n) - N_f(u) \parallel_{\frac{q(x)}{q(x)-1}} \mid v(x) \mid_{q(x)} \\ &\leq C \mid N_f(u_n) - N_f(u) \mid_{\frac{q(x)}{q(x)-1}} \parallel v \parallel . \end{aligned}$$

Thus, $L(u_n) \to L(u)$.

Further, it is known that the Nemytskii operator $N_g: \ u \mapsto g(x,u)$ is a continuous bounded operator from $L^{r(x)}(\partial\Omega)$ into $L^{\frac{r(x)}{r(x)-1}}(\partial\Omega)$, and analogously, K is completely continuous.

(C) T is strongly monotone,

We set

$$U_p = \{x \in \Omega : p(x) \geq 2\},$$
$$V_p = \{x \in \Omega : 1 < p(x) < 2\}.$$

By the elementary inequalities,(cf. [10]) we have $\forall x, y \in \mathbb{R}^N$

$$\mid x - y \mid^\gamma \leq 2^\gamma (\mid x \mid^{\gamma-2} x - \mid y \mid^{\gamma-2} y).(x-y) \ if \ \gamma \geq 2,$$

$$\mid x - y \mid^2 \leq \frac{1}{\gamma - 1} (\mid x \mid + \mid y \mid)^{2-\gamma} (\mid x \mid^{\gamma-2} x - \mid y \mid^{\gamma-2} y).(x-y) \ if \ 1 < \gamma < 2,$$

where $x.y$ denotes the usual inner product in Ω. It follows that

$$\begin{aligned} \langle (I+J)(u) &- (I+J)(v), u - v\rangle \qquad (3.1) \\ &= \int_\Omega [\{\mid \nabla u \mid^{p(x)-2} \nabla u - \mid \nabla v \mid^{p(x)-2} \nabla v)\} \nabla(u-v) \\ &\quad + \{(\mid u \mid^{p(x)-2} u - \mid v \mid^{p(x)-2} v)\}(u-v)]dx \\ &\geq \frac{1}{2^{p^+}} \int_{U_p} [\mid \nabla(u-v) \mid^{p(x)} + \mid u - v \mid^{p(x)}]dx + \\ &\quad (p^- - 1) \int_{V_p} [\mid \nabla(u-v) \mid^{p(x)} + \mid u - v \mid^{p(x)}]dx. \end{aligned}$$

From proposition 2.1, taking $c_0 = \min\{\frac{1}{2^{p^+}}, p^- - 1\}$, then we have

$$\langle (I+J)(u) - (I+J)(v), u - v\rangle \geq c_0 \begin{cases} \parallel u - v \parallel^{p^+} & if \ 0 \leq \parallel u - v \parallel \leq 1, \\ \parallel u - v \parallel^{p^-} & if \ \parallel u - v \parallel > 1 \end{cases} \qquad (3.2)$$

hence $I + J$ is strongly monotone(cf. [12]).

Since f is decreasing with respect to the second variable, then
$\langle L(u) - L(v), u - v\rangle = \int_\Omega (f(x,u) - f(x,v))(u-v)dx \geq 0$.

Also,
$\langle K(u)-K(v),u-v\rangle=\int_{\partial\Omega}(g(x,u)-g(x,v))(u-v)dx\geq 0.$
Consequently, T is strongly monotone.
(D) T is is coercive,
we have for $u\in X$ with $\| u \| > 1$,
wether $a=b=1, c=d=0$,

$$\begin{aligned}\frac{1}{\| u \|}\langle Tu,u\rangle &= \frac{1}{\| u \|}\int_{\Omega}(\mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)})dx - \int_{\Omega} f(x,u)udx\\ &\geq \frac{1}{\| u \|}[\| u \|^{p^-} -2 \mid f \mid_{p'(x)} \mid u \mid_{p(x)}\\ &\geq \frac{1}{\| u \|}[\| u \|^{p^-} -C_1 \| u \|].\end{aligned}$$

If $a=c=1, b=d=0$, we have

$$\begin{aligned}\frac{1}{\| u \|}\langle Tu,u\rangle &= \frac{1}{\| u \|}\int_{\Omega}(\mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)})dx - \int_{\partial\Omega} g(x,u)udx\\ &\geq \frac{1}{\| u \|}[\| u \|^{p^-} -2 \mid g \mid_{L^{r'(x)}(\partial\Omega)} \mid u \mid_{L^{r(x)}(\partial\Omega)}\\ &\geq \frac{1}{\| u \|}(\| u \|^{p^-} -C_2 \| u \|).\end{aligned}$$

For $a=c=0, b=d=1$,

$$\begin{aligned}\frac{1}{\| u \|_\beta}\langle Tu,u\rangle &= \frac{1}{\| u \|_\beta}\int_{\Omega}(\mid \nabla u \mid^{p(x)} + \mid \beta(x) u \mid^{p(x)})dx - \int_{\Omega} f(x,u)udx\\ &\geq \frac{1}{\| u \|_\beta}[\| u \|^{p^-} -2 \mid f \mid_{q'(x)} \mid u \mid_{q(x)}\\ &\geq \frac{1}{\| u \|_\beta}[\| u \|_\beta^{p^-} -C_1 \| u \|_\beta],\end{aligned}$$

with $\| u \|_\beta = |\nabla u|_{p(x)} + |u|_{L^{p(x)}(\partial\Omega)}$ is equivalent to $\| u \|$. It means that the coercivity of T holds. The previous steps guarantee the existence of solution of the problems (1.1)-(1.5).

For the uniqueness of weak solution for problems studied, suppose that u and v be a weak solutions such that $u\neq v$. By the strong monotonicity of T, it follows that

$$0=\langle Tu-Tv,u-v\rangle\geq C_p \| u-v \|^p \geq 0.$$

Then $u=v$ and the proof now is completed.

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