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The Impact of Media on a New Product Innovation Diffusion: A Mathematical Model

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ABSTRACT: In this paper, we proposed a three compartment model consisting of non-adopter, adopter and frustrated classes of population to discuss the influence of media coverage in spreading and controlling of adopter of a particular product in a region. The model exhibits two equilibria:(i) a adopter-free and (ii) unique interior equilibrium. Stability analysis of the model shows that the adopter-free equilibrium is always locally asymptotically stable if the influence number of adopter (R_0), which depends on parameters of the system is less than unity. Otherwise if $R_0 > 1$, a unique interior equilibrium exists, it is locally asymptotically stable under some set of conditions. Further analytically and numerically it is observed that the region for backward bifurcation of adopter population increases with the decrease of the valid contact rate before media alert. Finally, numerically experimentation are presented to establish the effect of different media alert rate on adopter and non adopter population.

Key Words: Innovation diffusion model, Impact of media, Influence number, Asymptotically stable, Backward bifurcation.

Contents

1. Introduction

A manager seeking to introduce a new product into the potential market has a limited number of variables under his control. The marketing manager must understand how these decision variables impact the diffusion process if he hopes to use them effectively [1,25,26]. A review on the theory of adoption of new products by a social system has been presented in [2]. These ideas have been expressed mathematically in diffusion models which emerged early in epidemiology and population models [3]-[12]. The diffusion of an innovation has traditionally been defined as the process by which an innovation is communicated through certain channels over time among the members of a particular geographical region [2,26].

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The diffusion process has frequently been modeled via a two-stage single differential equation approach, representing the epidemics manner in which the penetration and adoption of the innovation are influenced simultaneously by external and internal sources [13,14,15,16]. The price and advertising variables have been typically incorporated in these models to determine the basic parameters of the differential equation [17,18,19].

The theory of innovation diffusion when viewed as a theory of communication, centers around communication channels, which transmitted information to social system and also with in social system. The communication channels considered in the theory are two: mass media and interpersonal communication. The first one mass media facilitate gain if of information of innovation by the individuals, it is more effective in imparting knowledge, whereas second interpersonal communication plays a decisive role of persuasion level in the society, where face-to-face exchange of view is a continues process [24]. In developing this model the basic behavior theory which stipulates that the innovation is at first adopted by innovators them self which encourages its adoption by the society via interpersonal communication. The diffusion process in marketing has been described by classical bass 1969 [13] model as this differential equation.

$$\frac{dN}{dt} = p[m - N(t)] + \frac{q}{m} N(t)[m - N(t)], \tag{1.1}$$

where $N(t)$ is the cumulative number of adopter at time t, m is the total population of potential adopters, p is the coefficient of innovation and q is the coefficient of imitation.The first term in equation (1) denotes the adoption by innovators and the second term denotes the adoption by imitators.

Therefore, in this paper, we propose a non-linear mathematical model to study the media alert effect on innovation diffusion by using the stability theory of differential equations. In section 2, we have developed and analyzed a model to incorporate the media impact considering three classes of population namely, non-adopter, adopter and frustrated. The calculation of adopter free and endemic equilibrium, the basic influence number and proof of the local stability of adopter free equilibrium and the local stability of the endemic equilibrium are presented in section 3. Again in section 4 and 5, we have discussed existence of backward bifurcation and numerical simulation of the system respectively.

2. Mathematical Model

We proposed a non-linear dynamical mathematical model considering three types of population classes, first is non-adopter class second is adopter class and third is frustrated class with population densities $N(t)$, $A(t)$ and $R(t)$ respectively at time t. Let r is the recruitment rate of population which will join non-adopter class, ν is the rate of frustration from adopter population which will join frustrated class, δ is the coefficient of discontinuance rate of adopters, d_1 is the natural death rate of population for all classes, β_1 is the contact rate before media alert and $\beta(A) = \beta_1 + \frac{\beta_2 A}{m+A}$ is the contact rate after media alert. We choose this function to model the media alert with the assumption that $\frac{\beta_2 A}{m+A}$ to reflect the transmission rate when adopter individuals appear and are reported. When $A \to \infty$, the increased value of the transmission rate approaches its maximum β_2, and the increased value of the transmission rate equals half of the maximum β_2 when the reported adopter arrives at m (i.e., half saturation period).

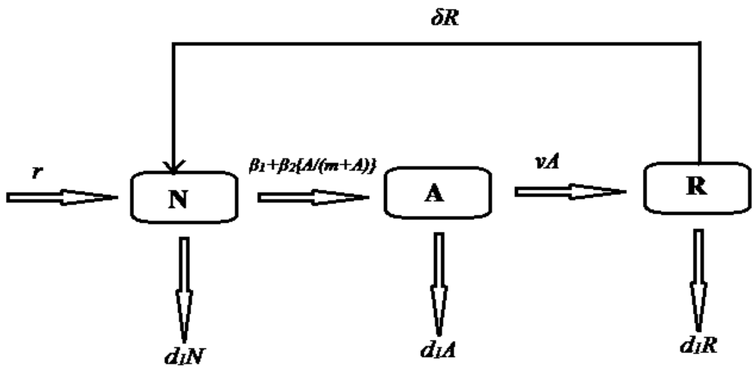

Figure 1: Schematic model flow diagram

In real life, it is true, that almost everybody will take measures to affect themselves from adopter as soon as adopted individuals are reported by media coverage, which will raise the transmission rate more or less. Generally speaking, the more individuals become adopter. The schematic flow diagram of our proposed system is show in figure 1. Hence our proposed three compartment model is govern by following system of equations:

$$\frac{dN}{dt} = r - d_1 N - (\beta_1 + \frac{\beta_2 A}{m + A})NA + \delta R, \tag{2.1}$$

$$\frac{dA}{dt} = (\beta_1 + \frac{\beta_2 A}{m + A})NA - (d_1 + \nu)A, \tag{2.2}$$

$$\frac{dR}{dt} = \nu A - (d_1 + \delta)R. \tag{2.3}$$

where all the paraments are positive. In the next section, we will examine the steady state behavior of the system.

3. Steady State, Basic Influence Number and Stability

The system (2.1)-(2.3) has one adopter free equilibrium: $E_0 = (\frac{r}{d_1}, 0, 0)$ and the interior equilibrium(s): $E^* = (N^*, A^*, R^*)$ satisfies $N^* > 0$, $A^* > 0$, $R^* > 0$

and $N^* = \frac{d_1+\nu}{\beta_1+\beta_2\frac{A}{A+m}}$, $R^* = \frac{\nu A}{(d_1+\delta)}$ and A^* is the positive solution of the following equation:

$$XA^{*2} + YA^* + Z = 0, \tag{3.1}$$

where $X = \frac{(\beta_1+\beta_2)d_1(d_1+\delta+\nu)}{d_1+\delta}$, $Y = -(r\beta_1(1-\frac{1}{\mathcal{R}_0}) + r\beta_2 - (d_1+\nu)\beta_1 m + \frac{\delta\nu\beta_1 m}{d_1+\delta})$ and $Z = (d_1(d_1+\nu)m\ (1\text{-}\mathcal{R}_0))$.

The local stability of E_0 can be obtained through a straightforward calculation of the eigenvalues. It follows that for the proposed compartmental model, local stability of adopter free equilibrium is governed by the basic influence number of model.The basic influence number R_0, is defined as the expected number of secondary adoption caused by an adopter individual upon entering a totally non-adopter population, as similar to the disease spreading models [22]. Using the notation in [22,23], we have two vectors F and V to represent the new infection term and remaining transfer terms, respectively:

$$\mathcal{F} = \begin{pmatrix} (\beta_1+\beta_2\frac{A}{A+m})AN \\ 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} (d_1+\nu)A \\ (d_1+\delta)R - \nu A \end{pmatrix}. \tag{3.2}$$

The Adopter compartment is A, hence a straightforward calculation of jacobian matrices gives

$$F = J(\mathcal{F}(E_0)) = \begin{pmatrix} \frac{r\beta_1}{d_1} & 0 \\ 0 & 0 \end{pmatrix}, \quad V = J(\mathcal{V}(E_0)) = \begin{pmatrix} (d_1+\nu) & 0 \\ -\nu & (d_1+\delta) \end{pmatrix}, \tag{3.3}$$

where F is non-negative and V is a non-singular M-matrix, therefore FV^{-1} is non-negative, and

$$V^{-1} = \begin{pmatrix} \frac{1}{d_1+\nu} & \frac{\nu}{(d_1+\nu)(d_1+\delta)} \\ 0 & \frac{1}{d_1+\delta} \end{pmatrix}, FV^{-1} = \begin{pmatrix} \frac{\beta_1 r}{d_1(d_1+\nu)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4}$$

Hence the influence number is given $\rho(FV^{-1})$ and

$$\mathcal{R}_0 = \frac{\beta_1 r}{d_1(d_1+\nu)}. \tag{3.5}$$

Here the basic influence number ($\mathcal{R}_0$) define as on average the number of non-adopter population become adopter under the influence of an adopter over the course of its adopter period. Again the interior equilibrium $E^* = (N^*, A^*, R^*)$ to exist, the solution of (3.1) must be real and positive. Since $X > 0$ and we can easily summarize the following conditions for the existence:

Lemma 3.1. *The system (2.1)-(2.3) has*
(i) a unique interior equilibrium if $Z < 0 \Leftrightarrow R > 1$;
(ii)a unique interior equilibrium if $Y < 0$ and $Z = 0$ or $Y^2 - 4XZ = 0$;
(iii)two interior equilibrium if $Z > 0, Y < 0$ and $Y^2 - 4XZ > 0$;
(iv)no interior equilibrium otherwise.

Now, we will show that all the solutions of the system (2.1)-(2.3) are bounded in a region $B \subset R^3_+$. We consider the following function:

$$\omega(t) = N(\tau) + A(\tau) + R(\tau),$$

and substituting the values from (2.1)-(2.3), we get

$$\frac{d\omega}{d\tau} = r - (N + A + R)d_1, \tag{3.6}$$

$$\frac{d\omega}{d\tau} = r - \omega d_1, \tag{3.7}$$

which implies as $\tau \to \infty$, $\omega \to \frac{r}{d_1}$. Hence consider the set:

$$B = \{(N, A, R) \in R^3_+ : 0 \leq N(\tau) + A(\tau) + R(\tau) \leq \frac{r}{d}, N(\tau), A(\tau), R(\tau) \geq 0\},$$

we can state the following lemma:

Lemma 3.2. *The system (2)-(4) is bounded in the region $B \subset R^3_+$.*

Now we will state and prove the local stability of all steady states.

Theorem 3.3. *For the system (2.1)-(2.3),*
(i)if $\mathcal{R}_0 < 1$, has a unique adopter free equilibrium $E_0 = (\frac{r}{d_1}, 0, 0)$ which is always locally asymptotically stable.
(ii) if $\mathcal{R}_0 > 1$, has a unique interior equilibrium $E^*(N^*, A^*, R^*)$ and it is locally asymptotically stable if $q_1 = \frac{\beta_2 m A^*(d_1+\nu)}{(\beta_1 m+(\beta_1+\beta_2)A^*)(m+A^*))} < p_1 = (\beta_1 + \frac{\beta_2 A^*}{m+A^*})A^*$.

Proof: The jacobian matrix $J = [j_{lk}]$ for the system of equations (2.1)-(2.3) evaluated at equilibrium $E_0 = (\frac{r}{d_1}, 0, 0)$ is

$$J(\frac{r}{d_1}, 0, 0) = \begin{pmatrix} -d_1 & -\frac{\beta_1 r}{d_1} & \delta \\ 0 & (d_1 + \nu)(\mathcal{R}_0 - 1) & 0 \\ 0 & \nu & -(d_1 + \delta) \end{pmatrix}. \tag{3.8}$$

The characteristic equation about E_0 is given by

$$(d_1 + \lambda)[((d_1 + \nu)(\mathcal{R}_0 - 1) - \lambda)][(d_1 + \delta) + \lambda] = 0. \tag{3.9}$$

The eigenvalues of the characteristic equation of $J(\frac{r}{d_1}, 0, 0)$ are $\lambda_1 = -d_1$, $\lambda_2 = (d_1 + \nu)(\mathcal{R}_0 - 1)$ and $\lambda_3 = -(d_1 + \delta)$. Thus all the eigenvalue of equation (12) are negative real when $\mathcal{R}_0 < 1$. It is observed from the above eigenvalues that the equilibrium point E_0 is always locally asymptotically stable of (2.1)-(2.3) for $\mathcal{R}_0 < 1$. Thus, the adopter-free equilibrium is locally asymptotically stable.
The jacobian matrix $J = [j_{lk}]$ for the system of equations (2.1)-(2.3) evaluated at equilibrium $E^* = (N^*, A^*, R^*)$ is

$$J(N^*, A^*, R^*) = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{pmatrix}, \tag{3.10}$$

where $\mathcal{A}_{11} = -d_1 - (\beta_1 + \frac{\beta_2 A^*}{m+A^*})A^*$, $\mathcal{A}_{12} = -(d_1 + \nu) - \frac{\beta_2 m A^*(d_1+\nu)}{(m+A^*)[\beta_1 m+(\beta_1+\beta_2)A^*]}$, $\mathcal{A}_{13} = \delta$, $\mathcal{A}_{21} = (\beta_1 + \frac{\beta_2 A^*}{m+A^*})A^*$, $\mathcal{A}_{22} = \frac{\beta_2 m A^*(d_1+\nu)}{(m+A^*)[\beta_1 m+(\beta_1+\beta_2)A^*]}$, $\mathcal{A}_{23} = 0$, $\mathcal{A}_{31} = 0$, $\mathcal{A}_{32} = \nu$, $\mathcal{A}_{33} = -(d_1 + \delta)$. The characteristic equation about E^* is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{3.11}$$

After taking $p_1 = (\beta_1 + \frac{\beta_2 A^*}{m+A^*})A^*$ and $q_1 = \frac{\beta_2 m A^*(d_1+\nu)}{(\beta_1 m+(\beta_1+\beta_2)A^*)(m+A^*))}$, we have $a_1 = (2d_1+p_1+\delta)-q_1$, $a_2 = d_1^2+d_1(2p_1+\delta)+p_1(\delta+\nu)-q_1(2d_1+\delta)$, $a_3 = d_1(p_1-q_1)(d_1+\delta)$ and $a_1a_2 - a_3 = (d_1 + p_1 - q_1)(2d_1 + \delta)(d_1 + p_1 - q_1 + \delta) + p_1(2d_1 + p_1 - q_1 + \delta)\nu$.

Now from Routh-Hurwitz criteria that all the eigenvalue of (3.11) have negative real part for $\mathcal{R}_0 > 1$, iff $q_1 < 2d_1 + p_1 + \delta$, $q_1 < p_1$, $q_1 < p_1 + \frac{1}{2d_1+\delta}[d_1^2 + d_1\delta + p_1\nu]$, i.e, $q_1 < min\{2d_1 + p_1 + \delta, p_1 + \frac{1}{2d_1+\delta}[d_1^2 + d_1\delta + p_1\nu], p_1\} = p_1$. Hence, for $\mathcal{R}_0 > 1$, the interior steady state E^* is locally asymptotically stable if $q_1 < p_1$. □

As similar as in [27], we will establish that $\mathcal{R}_0 = 1$ is a bifurcation point, in fact, across $\mathcal{R}_0 = 1$ the adopter free equilibrium changes its stability properties. In the following we consider system (2.1)-(2.3) and investigate the nature of the bifurcation involving the adopter-free equilibrium E_0 for $\mathcal{R}_0 = 1$. More precisely, we look for conditions on the parameter values that cause a forward or a backward bifurcation to occur. In order to do that, we will make use of the result summarized below, which has been obtained in [20] and is based on the use of general center manifold theory [21].

Consider the following general system of ordinary differential equations with a parameter ϕ :

$$\frac{dx}{dt} = f(x, \phi), \;\; f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R} \;\; and \quad f \in \mathbf{C}^2(\mathbf{R}^n \times \mathbf{R}). \tag{3.12}$$

Without loss of generality, it is assumed that $x = 0$ is an equilibrium for system (3.12) for all values of the parameter ϕ, (that is f(0,ϕ)= 0, for all ϕ).

Theorem 3.4. *[20] Assume*
$(A_1) : A = D_x f(0, 0)$ *is the is the liberalization matrix of the system (3.12) around the equilibrium $x = 0$ with ϕ evaluated at 0. Zero is a simple eigenvalue of A and other eigenvalues of A have negative real parts;*
(A2): Matrix A has a nonnegative right eigenvector w and a left eigenvector v (each corresponding to the zero eigenvalue).
Let f_k be the k^{th} component of f, and

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k(0, 0)}{\partial x_i \partial x_j},$$

$$b = \sum_{k,i,j=1}^{n} v_k w_i \frac{\partial^2 f_k(0,0)}{\partial x_i \partial \phi}.$$

Then, the local dynamics of the system (3.12) around $x = 0$ is totally determined by a and b. Moreover, the requirement of nonnegative components of w is not necessary.

1. *$a > 0$, $b > 0$. When $\phi < 0$, with $|\phi| \ll 1$, $x = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \phi \ll 1$, 0 is unstable and there exists a negative, locally asymptotically stable equilibrium;*
2. *$a < 0$, $b < 0$. When $\phi < 0$, with $|\phi| \ll 1$, $x = 0$ is unstable; when $0 < \phi \ll 1$, 0 is locally asymptotically stable equilibrium, and there exists a positive unstable equilibrium;*
3. *$a > 0$, $b < 0$. When $\phi < 0$, with $|\phi| \ll 1$, $x = 0$ is unstable and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi \ll 1$, 0 is stable and a positive unstable equilibrium appears;*
4. *$a < 0$, $b > 0$. When ϕ changes from negative to positive, $x = 0$ changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable.*

It clearly appears that, at $\phi = 0$ a transcritical bifurcation takes place: more precisely, when $a < 0$ and $b > 0$, such a bifurcation is forward; when $a > 0$ and $b > 0$, the bifurcation at $\phi = 0$ is backward. Now let $\phi = \beta_1$ be the bifurcation parameter, such that $\mathcal{R}_0 < 1$ for $\phi < 0$ and $\mathcal{R}_0 > 1$ for $\phi > 0$, such that x_0 is a adopter-free equilibrium for all values of ϕ. Consider the system $\frac{dx}{dt} = f(x, \phi)$, where f is continuously differentiable at least twice in both x and ϕ. The adopter-free equilibrium is the $(x_0; \phi)$ and the local stability of the adopter-free equilibrium changes at the point $(x_0; \phi)$ [23]. Now we want to show that there are nontrivial equilibrium near the bifurcation point $(x_0; \phi)$. Let $N = x_1$, $A = x_2$, $R = x_3$, the system (2.1)-(2.3) reduces to

$$\frac{dx_1(t)}{dt} = r - d_1 x_1(t) - (\beta_1 + \frac{\beta_2 x_2(t)}{m + x_2(t)}) x_1(t) x_2(t) + \delta x_3(t) := f_1, \tag{3.13}$$

$$\frac{dx_2(t)}{dt} = (\beta_1 + \frac{\beta_2 x_2(t)}{m + x_2(t)}) x_1(t) x_2(t) - (d_1 + \nu) x_2(t) := f_2, \tag{3.14}$$

$$\frac{dx_3(t)}{dt} = \nu x_2(t) - (d_1 + \delta) x_3(t) := f_3. \tag{3.15}$$

We will apply the result discussed above and explore the possibility of backward bifurcation in the system at $\mathcal{R}_0 = 1$. We consider the adopter free equilibrium $E_0 = (\frac{r}{d_1}, 0, 0)$ and observe that the condition $\mathcal{R}_0 = 1$ is equivalent to $\beta_1 = \beta_1^* = \frac{d_1(d_1+\nu)}{r}$. The eigenvalues of the matrix

$$J(E_0, \beta_1^*) = \begin{pmatrix} -d_1 & -\frac{\beta_1 r}{d_1} & \delta \\ 0 & \frac{\beta_1 r}{d_1} - (d_1 + \nu) & 0 \\ 0 & \nu & -(d_1 + \delta) \end{pmatrix}, \tag{3.16}$$

are given by $\lambda_1 = -d_1, \lambda_2 = -(d_1 + \delta), \lambda_3 = 0$. Thus $\lambda_3 = 0$ is simple zero eigenvalue of the matrix $J(E_0, \beta_1^*)$ and the other eigenvalues are real and negative. Therefore, we can use the center manifold theory. Hence, when $\beta_1 = \beta_1^*$ (or equivalently when $\mathcal{R}_0 = 1$), the Adopter free equilibrium.

Now we denote by $\mathbf{W} = (w_1, w_2, w_3)^{\mathrm{T}}$, a right eigenvector associated with the zero eigenvalue $\lambda_3 = 0$. Then

$$\begin{pmatrix} -d_1 & -\frac{\beta_1 r}{d_1} & \delta \\ 0 & 0 & 0 \\ 0 & \nu & -(d_1+\delta) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0,$$

which gives, $-d_1 w_1 - \frac{\beta_1^* r}{d1} w_2 + \delta w_3 = 0$ and $\nu w_2 - (d_1 + \delta) w_3 = 0$. Assuming $w_3 = \nu$, from above equations we obtain $w_1 = \frac{\delta\nu}{d_1} - \frac{\beta_1 r(d_1+\delta)}{d_1^2}$ and $w_2 = (d_1 + \delta)$. Therefore, the right eigenvector is

$$\mathbf{W} = (\frac{\delta\nu}{d_1} - \frac{\beta_1 r(d_1+\delta)}{d_1^2}, d_1 + \delta, \nu). \tag{3.17}$$

Furthermore, the left eigenvector $\mathbf{V}$ obtained from solving $\mathbf{V.J} = \mathbf{0}$ and $\mathbf{V.W} = \mathbf{1}$ is given by

$$\mathbf{V} = (0, \frac{1}{\delta + d_1}, 0). \tag{3.18}$$

Evaluating the partial derivatives at the adopter-free equilibrium, we obtain $\frac{\partial^2 f_1}{\partial x_1 \partial x_2} = \frac{\partial^2 f_1}{\partial x_2 \partial x_1} = -\beta_1, \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_1} = \beta_1, \frac{\partial^2 f_1}{\partial x_2^2} = \frac{-2\beta_2 r}{m d_1}, \frac{\partial^2 f_2}{\partial x_2^2} = \frac{2\beta_2 r}{m d_1}$, $\frac{\partial^2 f_1}{\partial x_2 \partial \phi} = \frac{-r}{d_1}$, and $\frac{\partial^2 f_2}{\partial x_2 \partial \phi} = \frac{r}{d_1}$. and all the other second-order partial derivatives are equal to zero. Thus, we can compute the coefficient $\mathbf{a}$ and $\mathbf{b}$, i.e,

$$\mathbf{a} = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k(E_0, \beta_1^*)}{\partial x_i \partial x_j},$$

$$\mathbf{b} = \sum_{k,i,j=1}^{n} v_k w_i \frac{\partial^2 f_k(E_0, \beta_1^*)}{\partial x_i \partial \phi}.$$

It follows that $\mathbf{a} = 2v_2 w_1 w_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2} + v_2 w_2^2 \frac{\partial^2 f_2}{\partial x_2^2}$ and $\mathbf{b} = v_2 w_2 \frac{\partial^2 f_2}{\partial x_2 \partial \phi}$; in view (19) and (20), we get

$$\mathbf{a} = \frac{2(d_1+\delta)\beta_2 r}{m d_1} - \frac{2d_1(d_1+\nu)(d_1+\delta+\nu)}{r} \tag{3.19}$$

or $\mathbf{a} = \frac{2d_1(d_1+\nu)(d_1+\delta+\nu)}{r}(\frac{(d_1+\delta)\beta_2 r^2}{m d_1^2 (d_1+\nu)(d_1+\delta+\nu)} - 1)$ and $\mathbf{b} = \frac{r}{d_1}$. It is observe that the coefficient $\mathbf{b}$ is always positive so that, according to Theorem 1, it is the sign of the coefficient $\mathbf{a}$ which decides the local dynamics around the adopter-free equilibrium for $\beta_1 = \beta_1^*$. Define $\mathcal{R}_1^* = \frac{(d_1+\delta)\beta_2 r^2}{m d_1^2 (d_1+\nu)(d_1+\delta+\nu)}$. Note that if $\mathcal{R}_1^* < 1$, then $\mathbf{a} < 0$ and $\mathbf{a} > 0$ if $\mathcal{R}_1^* > 1$. Hence, we have the following theorem, which is similar to the result established in [27]:

Theorem 3.5. *If $\mathcal{R}_1^* > 1$, system (2.1)-(2.3) exhibits a backward bifurcation when $\mathcal{R}_0 = 1$. If $\mathcal{R}_1^* < 1$, system (2.1)-(2.3) exhibits a forward bifurcation when $\mathcal{R}_0 = 1$.*

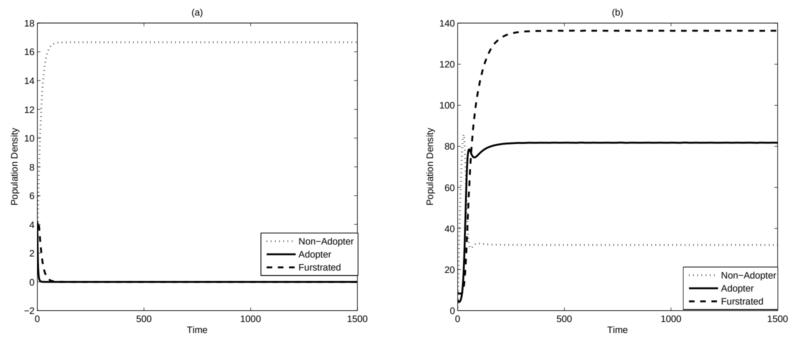

Figure 2: Population distributions with the existence of (a) Adopter free and (b) Interior equilibrium.

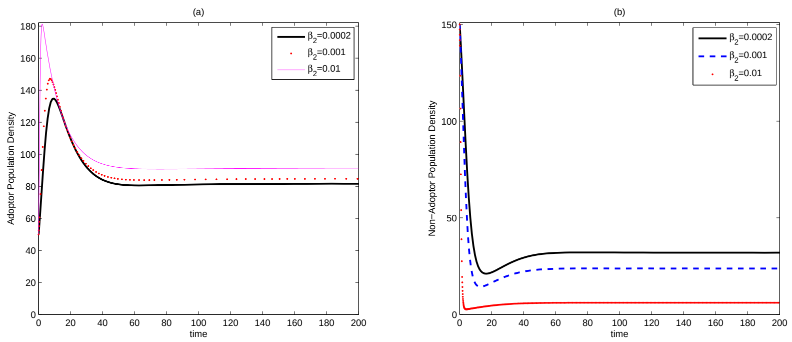

Figure 3: Population distributions with different media rates: (a) Adopter population and (d) Non-adopter population

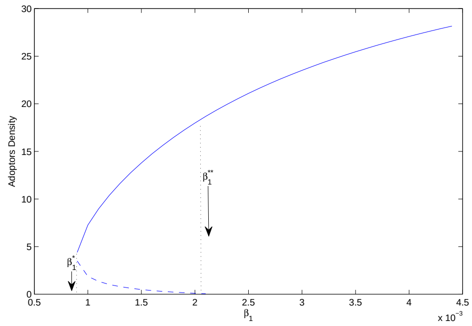

Figure 4: Bifurcation diagram in the plane (β_1, A). In this diagram, solid line indicate stability and the dashed lines indicate instability.

4. Numerical Simulations

In this section, we perform the numerical simulation of system (2.1)-(2.3) to verify the results obtained in previous sections. We choose following set of parametric values for the numerical experimentation:

1. For the following parametric values: $r = 1, \beta_1 = 0.02, \beta_2 = 0.02, \delta = 0.01, \nu = 0.4, d_1 = 0.06$ and $m = 5$, the condition of Theorem 1 is satisfied, i.e., $R_0 = 0.725 < 1$. The system (2.1)-(2.3) has an adopter-free equilibrium $E_0 = (16.67, 0, 0)$ is locally asymptotically stable (see Fig. 2(a)).

2. For the parameter values: $r = 5, \beta_1 = 0.002, \beta_2 = 0.0002, \delta = 0.01, \nu = 0.05, d_1 = 0.02$ and $m = 5$, we can obtain $R_0 = 7.1429 > 1$. In this case the system(2.1)-(2.3) has an interior equilibrium $E_1 = (31.9858, 81.75, 136.25)$ is locally asymptotically stable(see Fig. 2(b)).

3. For $r = 0.5, \beta_2 = 0.0018, \delta = 0.7, \nu = 0.1, d_1 = 0.01$ and $m = 0.725$. The phenomenon of backward bifurcation at $\beta_1 = \beta_1^*$ as shown in Fig. 4. There exists two threshold values of β_1, namely β_1^* and β_1^{**}, we observe that the adopter equilibrium E_0 is the only equilibrium for system $(2.1) - (2.3)$ for $\beta_1 < \beta_1^*$ and a interior equilibrium occurs for $\beta_1^* < \beta_1 < \beta_1^{**}$. Again, the interior stable interior equilibrium is locally asymptotically stable for $\beta_1 > \beta_1^{**}$ and adopter free equilibrium E_0 became unstable i.e. an unique interior equilibrium exists is LAS.

4. For the parameter values $r = 5, \beta_1 = 0.002, \delta = 0.01, \nu = 0.05, d_1 = 0.02, m = 5$, shown the media effect of adopter and non-adopter in Fig. 3(a)-(b) when media rate β_2 is high adopter reached maximum with in a short period of time, but the media is low then slowly adopter reached its maximum after a certain period of time adopter population finally settle down to its equilibrium

label, on the other hand as expected the opposite situation arise in the case of non-adopter population.

5. Conclusion

In the present paper, we have analyzed a innovation diffusion model consisting of three nonintersecting classes of population, namely, non-adopter, adopter and frustrated. Here, we presented a new measure, i.e., basic influence number of an individual adopter, which means that on an average the number of non-adopter population become adopter under the influence of an adopter over the course of its adoption period. We have studied the stability of adopter free equilibrium as well as interior equilibrium and it is shown that the adopter free equilibrium is locally asymmetrically stable if the basic influence number $\mathcal{R}_0 < 1$. Again, when $\mathcal{R}_0 > 1$ the interior equilibrium state exists and it is locally asymptotically stable under some parameter conditions. The existence of backward bifurcation of adopter population has been studied with respect to the valid contact rate before media alert (β_1) and a numerical threshold is determined for a particular set of parametric values.

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