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Infinitely many solutions for a nonlinear Navier boundary systems involving $(p(x), q(x))$-biharmonic

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ABSTRACT: In this article, we study the following $(p(x), q(x))$-biharmonic type system

$$\begin{aligned} \Delta(|\Delta u|^{p(x)-2}\Delta u) &= \lambda F_u(x,u,v) \quad \text{in } \Omega,\\ \Delta(|\Delta v|^{q(x)-2}\Delta v) &= \lambda F_v(x,u,v) \quad \text{in } \Omega,\\ u = v = \Delta u &= \Delta v = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

We prove the existence of infinitely many solutions of the problem by applying a general variational principle due to B. Ricceri and the theory of the variable exponent Sobolev spaces.

Key Words: Navier value problem; infinitely many solutions; variable exponent Sobolev space; Ricceri's variational principle

Contents

1. Introduction

The study of differential equations and variational problems with variable exponents has attracted intense research interests in recent years. Such problems arise from the study of electrorheological fluids, image processing, and the theory of nonlinear elasticity (see [15,20]). In this paper, we consider the existence of solutions for the following system

$$\begin{aligned} \Delta(|\Delta u|^{p(x)-2}\Delta u) &= \lambda F_u(x,u,v) \quad \text{in } \Omega,\\ \Delta(|\Delta v|^{q(x)-2}\Delta v) &= \lambda F_v(x,u,v) \quad \text{in } \Omega,\\ u = v = \Delta u &= \Delta v = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$, $\lambda \in (0,\infty)$ and $p,\ q \in C(\overline{\Omega})$ with $\frac{N}{2} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$, $\frac{N}{2} < q^- := \inf_{x\in\overline{\Omega}} q(x) \leq q^+ := \sup_{x\in\overline{\Omega}} q(x) < +\infty$. $F : \overline{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is a

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function such that $F(.,s,t)$ is continuous in $\overline{\Omega}$, for all $(s,t) \in \mathbb{R}^2$ and $F(x,.,.)$ is C^1 in $\mathbb{R}^2$ for every $x \in \Omega$, and $\sup_{\{|s|\leq\theta,|t|\leq\theta\}}\Big(|F_u(.,s,t)| + |F_v(.,s,t)|\Big) \in L^1(\Omega)$ for all $\theta > 0$, with F_u, F_v denote the partial derivatives of F, with respect to u, v respectively.

There are many works devoted to the existence of solutions for variable exponent problems, both on bounded domain and unbounded domain, we refer to [1,4,7,19] as examples. For existence results on elliptic systems, we refer to [8,16,18].

The investigation of existence and multiplicity of solutions for problems involving biharmonic, p-biharmonic and $p(x)$-biharmonic operators has drawn the attention of many authors, see [2,3,5,6,11] and references therein. Candito and Livrea [5] considered the nonlinear elliptic Navier boundary-value problem

$$\begin{aligned} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) \quad &\text{in } \Omega, \\ u = \Delta u = 0 \quad &\text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

There the authors established the existence of infinitely many solutions.

In the present paper, we look for the existence of infinitely many solutions of system (1.1). More precisely, we will prove the existence of well precise intervals of parameters such that problem (1.1) admits either an unbounded sequence of solutions provided that $F(x,u,v)$ has a suitable behaviour at infinity or a sequence of nontrivial solutions converging to zero if a similar behaviour occurs at zero.

In the case when $p(x) \equiv p$ and $q(x) \equiv q$ are two constants, we know that the problem (1.1) has infinitely many solutions from [12]. Here we point out that the $p(x)$-biharmonic operator possesses more complicated nonlinearities than p-biharmonic, for example, it is inhomogeneous and usually it does not have the so-called first eigenvalue, since the infimum of its principle eigenvalue is zero.

This article is organized as follows. In Section 2, we introduce the generalized Lebesgue-Sobolev spaces and some important related results. In section 3, we give the main results of this paper. In section 4, we use the general variational principle by B. Ricceri to prove the main results.

2. Preliminaries

To study $p(x)$-Laplacian problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{k,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian used later.

Define the generalized Lebesgue space by

$$L^{p(x)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\big\},$$

where $p \in C_+(\overline{\Omega})$ and

$$C_+(\overline{\Omega}) := \big\{p \in C(\overline{\Omega}) : p(x) > 1 \quad \forall x \in \overline{\Omega}\big\}.$$

Denote

$$p^+ = \max_{x\in\overline{\Omega}} p(x), \quad p^- = \min_{x\in\overline{\Omega}} p(x).$$

One introduces in $L^{p(x)}(\Omega)$ the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1 \right\}.$$

The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is a Banach space.

Proposition 2.1 ([10]). *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is separable, uniformly convex and its conjugate space is $L^{q(x)}(\Omega)$ where $q(x)$ is the conjugate function of $p(x)$, i.e*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1, \quad \forall x \in \Omega.$$

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

$$|\int_\Omega u(x)v(x)dx| \leq (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

The Sobolev space with variable exponents $W^{k,p(x)}(\Omega)$ is defined as

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \right\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u$ (the derivation in distributional sense) with $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{k,p(x)}(\Omega)$, equipped with the norm

$$|u|_{k,p(x)} := \sum_{|\alpha|\leq k} |D^\alpha u|_{p(x)},$$

also becomes a Banach, separable and reflexive space. For more details, we refer the reader to [9,10,13].

We denote by $W_0^{k,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

In this paper, we shall look for weak solutions of problem (1.1) in the space X defined by

$$X := (W^{2,p(x)}(\Omega) \bigcap W_0^{1,p(x)}(\Omega)) \times (W^{2,q(x)}(\Omega) \bigcap W_0^{1,q(x)}(\Omega)),$$

which is separable and reflexive Banach spaces with the norm

$$\|(u,v)\| = \|u\|_{p(x)} + \|v\|_{q(x)},$$

where $\|.\|_{p(x)}$ (resp. $\|.\|_{q(x)}$) is the norm of $W^{2,p(x)}(\Omega) \bigcap W_0^{1,p(x)}(\Omega)$ (resp. $W^{2,q(x)}(\Omega) \bigcap W_0^{1,q(x)}(\Omega)$),

$$\|u\|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega (|\frac{\triangle u}{\sigma}|^{p(x)} + |\frac{\nabla u}{\sigma}|^{p(x)} + |\frac{u}{\sigma}|^{p(x)}) dx \leq 1\},$$

and

$$\|u\|_{q(x)} = \inf\{\sigma > 0 : \int_\Omega (|\frac{\triangle u}{\sigma}|^{q(x)} + |\frac{\nabla u}{\sigma}|^{q(x)} + |\frac{u}{\sigma}|^{q(x)})dx \leq 1\}.$$

According to [17], the norm $|.|_{2,p(x)}$ is equivalent to the norm $|\triangle.|_{p(x)}$ in the space $W^{2,p(x)}(\Omega)\bigcap W_0^{1,p(x)}(\Omega)$. Consequently, the norms $|.|_{2,p(x)}$, $|\triangle.|_{p(x)}$ and $\|.\|_{p(x)}$ are equivalent.

The following proposition will plays an important role in our arguments.

Proposition 2.2. *The embedding $X \hookrightarrow C(\overline{\Omega})\times C(\overline{\Omega})$ is compact whenever $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. So there is a constant $K > 0$ such that*

$$K := \max\Big\{ \sup_{u\in W^{2,p(x)}(\Omega)\cap W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|u(x)|}{\|u\|_{p(x)}}, \sup_{v\in W^{2,q(x)}(\Omega)\cap W_0^{1,q(x)}(\Omega)\setminus\{0\}} \frac{\max_{x\in\overline{\Omega}}|v(x)|}{\|v\|_{q(x)}} \Big\} < \infty. \quad (2.1)$$

Proof: It is well known that $X \hookrightarrow W^{2,p(x)}(\Omega) \times W^{2,q(x)}(\Omega)$ and $W^{2,p(x)}(\Omega) \times W^{2,q(x)}(\Omega) \hookrightarrow W^{2,p^-}(\Omega) \times W^{2,q^-}(\Omega)$ are all continuous embedding. And the embedding $W^{2,p^-}(\Omega) \times W^{2,q^-}(\Omega) \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact when $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. So we get, the embedding $X \hookrightarrow C(\overline{\Omega}) \times C(\overline{\Omega})$ is compact when $p^- > \frac{N}{2}$ and $q^- > \frac{N}{2}$. $\square$

Using the similar proof method with [9], we have the following result.

Proposition 2.3. *Let $I(u) = \int_\Omega |\triangle u|^{p(x)}\, dx$, for $u \in W^{2,p(x)}(\Omega)\bigcap W_0^{1,p(x)}(\Omega)$ we have*

1. *For $u \neq 0$, $\|u\|_{p(x)} = \beta \Leftrightarrow I(\frac{u}{\beta}) = 1$;*
2. *$\|u\|_{p(x)} < 1(= 1, > 1) \Leftrightarrow I(u) < 1(= 1, > 1)$;*
3. *$\|u\|_{p(x)} \leq 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq I(u) \leq \|u\|_{p(x)}^{p^-}$;*
4. *$\|u\|_{p(x)} \geq 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq I(u) \leq \|u\|_{p(x)}^{p^+}$;*
5. *$\lim_{k\to+\infty} \|u_k\|_{p(x)} = 0 \Leftrightarrow \lim_{k\to+\infty} I(u_k) = 0$;*
6. *$\lim_{k\to+\infty} \|u_k\|_{p(x)} = +\infty \Leftrightarrow \lim_{k\to+\infty} I(u_k) = +\infty$.*

Let us recall for the reader's convenience a smooth version of a previous result of Ricceri [14].

Proposition 2.4. *[14] Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$ then, for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds: either

(b1) I_λ possesses a global minimum,or

(b2) there is a sequence (u_n) of critical points (local minima) of I_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

(c) If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds: either

(c1) there is a global minimum of Φ which is a local minimum of I_λ,or

(c2) there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to global minimum of Φ.

For each$(u, v) \in X$, we define

$$\Phi(u, v) = \int_\Omega \frac{1}{p(x)} |\triangle u|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\triangle v|^{q(x)} dx.$$

Then, the operator $L := \Phi' : X \to X^*$, where X^* is the dual space of X, defined by

$$L(u, v)(\varphi, \psi) = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta\varphi dx + \int_\Omega |\Delta v|^{q(x)-2} \Delta v \Delta\psi dx \quad \forall(\varphi, \psi) \in X, \tag{2.2}$$

satisfies the assertions of the following proposition.

Proposition 2.5. *(see [4]).*

1. *L is continuous, bounded and strictly monotone.*
2. *L is of (S_+) type.*

3. Main results

Fix $x_0 \in \Omega$ and pick $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subseteq \Omega$. Set

$$\begin{aligned} L_{p^+} &:= \frac{\Gamma(1+N/2)}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}\pi^{N/2}}\Big(\frac{R_2^2-R_1^2}{2N}\Big)^{p^+}\frac{1}{R_2^N-R_1^N}, \\ L_{q^+} &:= \frac{\Gamma(1+N/2)}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}\pi^{N/2}}\Big(\frac{R_2^2-R_1^2}{2N}\Big)^{q^+}\frac{1}{R_2^N-R_1^N} \end{aligned} \tag{3.1}$$

where Γ denotes the Gamma function and K is given by (2.1).

Definition 3.1. We say that $(u, v) \in X$ is a weak solution of problem (1.1) if

$$\begin{aligned} &\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta\varphi\, dx + \int_\Omega |\Delta v|^{q(x)-2}\Delta v\Delta\psi\, dx \\ &- \lambda\int_\Omega F_u(x,u,v)\varphi\, dx - \lambda\int_\Omega F_v(x,u,v)\psi\, dx = 0, \end{aligned}$$

for all $(\varphi, \psi) \in X$.

Define the functional $J_\lambda : X \to \mathbb{R}$, by

$$J_\lambda(u,v) = \Phi(u,v) - \lambda\Psi(u,v),$$

for all $(u, v) \in X$, where

$$\Phi(u,v) = \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx+\int_\Omega \frac{1}{q(x)}|\triangle v|^{q(x)}dx \quad \text{and} \quad \Psi(u,v) = \int_\Omega F(x,u,v)dx.$$

The functionals $\Phi, \Psi : X \to \mathbb{R}$ are well defined, Gâteaux differentiable functionals whose Gâteaux derivatives at $(u, v) \in X$ are given by

$$\begin{aligned} \langle \Phi'(u,v),(\varphi,\psi)\rangle &= \int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta\varphi dx + \int_\Omega |\Delta v|^{q(x)-2}\Delta v\Delta\psi dx, \\ \langle \Psi'(u,v),(\varphi,\psi)\rangle &= \int_\Omega F_u(x,u,v)\varphi dx + \int_\Omega F_v(x,u,v)\psi dx, \end{aligned}$$

for all $(\varphi, \psi) \in X$.

In view of (2.2) and proposition 2.5, we see that $\Phi \in C^1(X, \mathbb{R})$ and $(u, v) \in X$ is a weak solution of (1.1) if and only if (u, v) is a critical point of the functional J_λ.

Since X is compactly embedded in $C(\overline{\Omega}) \times C(\overline{\Omega})$, we can see that $\Phi, \Psi : X \to \mathbb{R}$ are sequentially weakly lower semi-continuous. Moreover Φ is coercive.

Our main results are the following two theorems.

Theorem 3.2. *Assume that*

(i1) $F(x, s, t) \geq 0$ *for every* $(x, s, t) \in \Omega \times [0, +\infty)^2$;

(i2) *There exist* $x_0 \in \Omega$, $0 < R_1 < R_2$ *as considered in* (3.1) *such that, if we put*

$$\alpha := \liminf_{b\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\le b} F(x,s,t)dx}{b^{\min\{p^-,q^-\}}}, \quad \beta := \limsup_{s,t\to+\infty} \frac{\int_{B(x_0,R_1)} F(x,s,t)dx}{\frac{s^{p^+}}{p^-}+\frac{t^{q^+}}{q^-}},$$

one has

$$\alpha < L\beta, \tag{3.2}$$

where $L := \min\{L_{p^+}, L_{q^+}\}$.

Then, for every

$$\lambda \in \Lambda := \frac{1}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big]\frac{1}{L\beta}, \frac{1}{\alpha}\Big[$$

problem (1.1) *admits an unbounded sequence of weak solutions.*

Theorem 3.3. *Assume that* (i1) *holds and*

(i3) $F(x,0,0) = 0$ *for every* $x \in \Omega$.

(i4) *There exist* $x_0 \in \Omega$, $0 < R_1 < R_2$ *as considered in* (3.1) *such that, if we put*

$$\alpha^0 := \liminf_{b\to 0^+} \frac{\int_\Omega \sup_{|s|+|t|\le b} F(x,s,t)dx}{b^{\min\{p^-,q^-\}}}, \quad \beta^0 := \limsup_{s,t\to 0^+} \frac{\int_{B(x_0,R_1)} F(x,s,t)dx}{\frac{s^{p^+}}{p^-}+\frac{t^{q^+}}{q^-}},$$

one has

$$\alpha^0 < L\beta^0. \tag{3.3}$$

where $L := \min\{L_{p^+}, L_{q^+}\}$.

Then, for every

$$\lambda \in \Lambda := \frac{1}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big]\frac{1}{L\beta^0}, \frac{1}{\alpha^0}\Big[,$$

problem (1.1) *admits a sequence* (u_n) *of weak solutions such that* $u_n \rightharpoonup 0$.

4. Proofs of main results

Proof: [Proof of Theorem 3.2] To apply proposition 2.4, we set

$$\varphi(r) := \inf_{(w,z)\in\Phi^{-1}(]-\infty,r[)} \frac{\Big(\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v)\Big) - \Psi(w,z)}{r - \Phi(w,z)}$$

Note that $\Phi(0,0) = 0$, and by (i1), $\Psi(0,0) \ge 0$. Therefore, for every $r > 0$,

$$\begin{aligned}
\varphi(r) &= \inf_{(w,z)\in\Phi^{-1}(]-\infty,r[)} \frac{\Big(\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v)\Big) - \Psi(w,z)}{r - \Phi(w,z)} \\
&\le \frac{\sup_{(u,v)\in\Phi^{-1}(]-\infty,r[)} \Psi(u,v)}{r} \\
&= \frac{\sup_{\Phi(u,v)<r} \int_\Omega F(x,u,v)dx}{r}.
\end{aligned} \tag{4.1}$$

Let (b_n) a sequence of positive numbers such that $\lim_{n\to+\infty} b_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\le b_n} F(x,s,t)dx}{b_n^{\min\{p^-,q^-\}}} = \alpha < +\infty. \tag{4.2}$$

Put

$$r_n := \Big(\frac{b_n}{K((p^+)^{1/p^-} + (q^+)^{1/q^-})}\Big)^{\min\{p^-,q^-\}}.$$

Let $(u,v) \in \Phi^{-1}(]-\infty, r_n[)$, so we have

$$\int_\Omega \frac{1}{p(x)} |\triangle u|^{p(x)} dx + \int_\Omega \frac{1}{q(x)} |\triangle v|^{q(x)} dx < r_n,$$

then

$$\frac{1}{p^+} \int_\Omega |\triangle u|^{p(x)} dx + \frac{1}{q^+} \int_\Omega |\triangle v|^{q(x)} dx < r_n,$$

so, by proposition 2.3, we have

$$\frac{1}{p^+} \min\{\|u\|_{p(x)}^{p^+}, \|u\|_{p(x)}^{p^-}\} + \frac{1}{q^+} \min\{\|v\|_{q(x)}^{q^+}, \|v\|_{q(x)}^{q^-}\} < r_n,$$

thus

$$\frac{1}{p^+} \min\{\|u\|_{p(x)}^{p^+}, \|u\|_{p(x)}^{p^-}\} < r_n \quad \text{and} \quad \frac{1}{q^+} \min\{\|v\|_{q(x)}^{q^+}, \|v\|_{q(x)}^{q^-}\} < r_n.$$

When $\|u\|_{p(x)} \le 1$, we have $\frac{1}{p^+}\|u\|_{p(x)}^{p^+} < r_n$, so $\|u\|_{p(x)} < (p^+ r_n)^{\frac{1}{p^+}}$.
When $\|u\|_{p(x)} > 1$, we have $\frac{1}{p^+}\|u\|_{p(x)}^{p^-} < r_n$, so $\|u\|_{p(x)} < (p^+ r_n)^{\frac{1}{p^-}}$.
Hence, for n large enough ($r_n > 1$),

$$\|u\|_{p(x)} < (p^+ r_n)^{\frac{1}{p^-}} \quad \text{and} \quad \|v\|_{q(x)} < (q^+ r_n)^{\frac{1}{q^-}}. \tag{4.3}$$

Using (2.1) and (4.3), we obtain, for all $x \in \Omega$

$$|u(x)| < K(p^+ r_n)^{\frac{1}{p^-}} \quad \text{and} \quad |v(x)| < K(q^+ r_n)^{\frac{1}{q^-}}.$$

Therefore, for n large enough ($r_n > 1$),

$$|u(x)| + |v(x)| < K((p^+)^{\frac{1}{p^-}} + (q^+)^{\frac{1}{q^-}}) r_n^{\frac{1}{\min\{p^-,q^-\}}} = b_n.$$

Then

$$\begin{aligned}\varphi(r_n) &\le \frac{\sup_{\{(u,v)\in X: |u(x)|+|v(x)|<b_n,\, \forall x\in\Omega\}} \int_\Omega F(x,u,v)dx}{\big(\frac{b_n}{K((p^+)^{1/p^-}+(q^+)^{1/q^-})}\big)^{\min\{p^-,q^-\}}} \\ &\le \Big(K((p^+)^{1/p^-} + (q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} \frac{\int_\Omega \sup_{|s|+|t|<b_n} F(x,s,t)dx}{b_n^{\min\{p^-,q^-\}}}.\end{aligned} \tag{4.4}$$

Let

$$\gamma := \liminf_{r\to+\infty} \varphi(r).$$

It follows from (4.2) and (4.4) that

$$\begin{aligned}\gamma &\le \liminf_{n\to+\infty} \varphi(r_n)\\ &\le \Big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|<b_n} F(x,s,t)}{b_n^{\min\{p^-,q^-\}}}\\ &= \alpha\Big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} < +\infty.\end{aligned} \tag{4.5}$$

From (4.5), it is clear that $\Lambda \subseteq]0, \frac{1}{\gamma}[$.

For $\lambda \in \Lambda$, we claim that the functional J_λ is unbounded from below. Indeed, since $\frac{1}{\lambda} < \Big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} L\beta$, we can consider a sequence (η_n) of positive numbers and $\delta > 0$ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\lambda} < \delta < L\Big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} \frac{\int_{B(x^0,R_1)} F(x,\eta_n,\eta_n)dx}{\frac{\eta_n^{p^+}}{p^-}+\frac{\eta_n^{q^+}}{q^-}}, \tag{4.6}$$

Now we consider the function u_n defined by

$$u_n(x) = \begin{cases} 0, & x \in \Omega\backslash B(x_0,R_2),\\ \eta_n, & x \in B(x_0,R_1),\\ \frac{\eta_n}{R_2^2-R_1^2}\big(|x-x_0|^2-R_2^2\big), & x\in B(x_0,R_2)\backslash B(x_0,R_1),\end{cases} \tag{4.7}$$

then $(u_n, u_n) \in X$ and

$$\frac{\partial u_n(x)}{\partial x_i} = \begin{cases} 0, & x \in \big(\Omega\backslash B(x_0,R_2)\big)\bigcup B(x_0,R_1),\\ \frac{2\eta_n}{R_2^2-R_1^2}\big(x_i-x_0\big), & x\in B(x_0,R_2)\backslash B(x_0,R_1),\end{cases}$$

$$\frac{\partial^2 u_n(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \big(\Omega\backslash B(x_0,R_2)\big)\bigcup B(x_0,R_1),\\ \frac{2\eta_n}{R_2^2-R_1^2}, & x\in B(x_0,R_2)\backslash B(x_0,R_1),\end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 u_n(x)}{\partial x_i^2} = \begin{cases} 0, & x \in \big(\Omega\backslash B(x_0,R_2)\big)\bigcup B(x_0,R_1),\\ \frac{2\eta_n N}{R_2^2-R_1^2}, & x\in B(x_0,R_2)\backslash B(x_0,R_1).\end{cases} \tag{4.8}$$

Then, for n large enough

$$\begin{aligned}
\Phi(u_n,u_n) &\le \frac{1}{p^-}\int_\Omega |\triangle u_n|^{p(x)}dx + \frac{1}{q^-}\int_\Omega |\triangle u_n|^{q(x)}dx \\
&\le \frac{1}{p^-}\int_{B(x_0,R_2)\setminus B(x_0,R_1)} |\triangle u_n|^{p(x)}dx + \frac{1}{q^-}\int_{B(x_0,R_2)\setminus B(x_0,R_1)} |\triangle u_n|^{q(x)}dx \\
&\le \frac{\pi^{\frac{N}{2}}}{p^-\Gamma(1+\frac{N}{2})}\Big(\frac{2N\eta_n}{R_2^2-R_1^2}\Big)^{p^+}(R_2^N-R_1^N) \\
&\quad + \frac{\pi^{\frac{N}{2}}}{q^-\Gamma(1+\frac{N}{2})}\Big(\frac{2N\eta_n}{R_2^2-R_1^2}\Big)^{q^+}(R_2^N-R_1^N) \\
&= \frac{1}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big(\frac{\eta_n^{p^+}}{p^-L_{p^+}}+\frac{\eta_n^{q^+}}{q^-L_{q^+}}\Big).
\end{aligned} \tag{4.9}$$

By (i1), we have

$$\Psi(u_n,u_n) = \int_\Omega F(x,u_n,u_n)dx \ge \int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx. \tag{4.10}$$

Combining (4.6), (4.9) and (4.10), we obtain

$$\begin{aligned}
J_\lambda(u_n,u_n) &= \Phi(u_n,u_n) - \lambda\Psi(u_n,u_n) \\
&\le \frac{1}{\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big(\frac{\eta_n^{p^+}}{p^-L_{p^+}}+\frac{\eta_n^{q^+}}{q^-L_{q^+}}\Big) \\
&\quad - \lambda\int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx \\
&\le \frac{1}{L\,\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big(\frac{\eta_n^{p^+}}{p^-}+\frac{\eta_n^{q^+}}{q^-}\Big) \\
&\quad - \lambda\int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx \\
&< \frac{1-\lambda\delta}{L\,\big(K((p^+)^{1/p^-}+(q^+)^{1/q^-})\big)^{\min\{p^-,q^-\}}}\Big(\frac{\eta_n^{p^+}}{p^-}+\frac{\eta_n^{q^+}}{q^-}\Big),
\end{aligned} \tag{4.11}$$

for n large enough, so

$$\lim_{n\to+\infty} I_\lambda(u_n,u_n) = -\infty,$$

and hence the claim follows.

The alternative of proposition 2.4 case (b) assures the existence of unbounded sequence (u_n) of critical points of the functional J_λ and the proof of Theorem 3.2 is complete. □

Proof: [Proof of Theorem 3.3] First, note that

$$\min_X \Phi = \Phi(0,0) = 0. \tag{4.12}$$

Let (b_n) be a sequence of positive numbers such that $b_n \to 0^+$ and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|\le b_n} F(x,s,t)dx}{b_n^{\min\{p^-,q^-\}}} = \alpha^0 < +\infty. \tag{4.13}$$

Put

$$r_n = \Big(\frac{b_n}{K((p^+)^{1/p^-} + (q^+)^{1/q^-})}\Big)^{\min\{p^-,q^-\}}, \quad \delta := \liminf_{r\to 0^+} \varphi(r).$$

It follows from (4.1) and (4.13) that

$$\begin{aligned} \delta &\le \liminf_{n\to+\infty} \varphi(r_n) \\ &\le \Big(K((p^+)^{1/p^-} + (q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|s|+|t|<b_n} F(x,s,t)}{b_n^{\min\{p^-,q^-\}}} \\ &= \alpha^0 \Big(K((p^+)^{1/p^-} + (q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} < +\infty. \end{aligned} \tag{4.14}$$

By (4.14), we see that $\Lambda \subseteq]0, \frac{1}{\delta}[$.

Now, for $\lambda \in \Lambda$, we claim that J_λ has not a local minimum at zero. Indeed, since $\frac{1}{\lambda} < \Big(K((p^+)^{1/p^-} + (q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} L\beta^0$, we can consider a sequence (η_n) of positive numbers and $\delta > 0$ such that $\eta_n \to 0^+$ and

$$\frac{1}{\lambda} < \delta < L\Big(K((p^+)^{1/p^-} + (q^+)^{1/q^-})\Big)^{\min\{p^-,q^-\}} \frac{\int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx}{\frac{\eta_n^{p^+}}{p^-} + \frac{\eta_n^{q^+}}{q^-}}, \tag{4.15}$$

for n large enough. Let (u_n) be the sequence defined in (4.7). By combining (4.9), (4.10) and (4.15), and taking into account $(i3)$, we have

$$\begin{aligned} J_\lambda(u_n,u_n) &= \Phi(u_n,u_n) - \lambda\Psi(u_n,u_n) \\ &\le \frac{1}{(K((p^+)^{1/p^-} + (q^+)^{1/q^-}))^{\min\{p^-,q^-\}}} \Big(\frac{\eta_n^{p^+}}{p^- L_{p^+}} + \frac{\eta_n^{q^+}}{q^- L_{q^+}}\Big) \\ &\quad - \lambda \int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx \\ &\le \frac{1}{L\,(K((p^+)^{1/p^-} + (q^+)^{1/q^-}))^{\min\{p^-,q^-\}}} \Big(\frac{\eta_n^{p^+}}{p^-} + \frac{\eta_n^{q^+}}{q^-}\Big) \\ &\quad - \lambda \int_{B(x_0,R_1)} F(x,\eta_n,\eta_n)dx \\ &< \frac{1-\lambda\delta}{L\,(K((p^+)^{1/p^-} + (q^+)^{1/q^-}))^{\min\{p^-,q^-\}}} \Big(\frac{\eta_n^{p^+}}{p^-} + \frac{\eta_n^{q^+}}{q^-}\Big) \\ &< 0 = J_\lambda(0,0) \end{aligned} \tag{4.16}$$

for n large enough. This together with the fact that $\|(u_n, u_n)\| \to 0$ show that J_λ has not a local minimum at zero, and the claim follows.

The alternative of proposition 2.4 case (c) ensures the existence of sequence (u_n) of pairwise distinct critical points (local minima) of J_λ which weakly converges to 0. This completes the proof of Theorem 3.3. □

Example 4.1. *Let $\Omega = (]-1;1[)^2$, p, q two functions defined on Ω by:*

$$p(x,y) = x^2 + y^2 + 3, \;\; q(x,y) = x^2 + y^2 + 4,$$

and $F : \mathbb{R}^2 \to \mathbb{R}$ a function defined by:

$$F(s,t) = \begin{cases} (a_{n+1})^7 e^{1-\frac{1}{1-[(s-a_{n+1})^2+(t-a_{n+1})^2]}} & \text{if } (s,t) \in \bigcup\limits_{n\geq 1} B((a_{n+1}, a_{n+1}), 1) \\ 0 & \text{otherwise}, \end{cases} \tag{4.17}$$

where

$$a_1 := 2, \;\; a_{n+1} := n(a_n)^{\frac{7}{3}} \;\; \forall n \in \mathbb{N}^*$$

and $B((a_{n+1}, a_{n+1}), 1)$ is an open unit ball of center (a_{n+1}, a_{n+1}).

It is easy to verify that F is non-negative and $F \in C^1(\mathbb{R}^2)$. for all $n \in \mathbb{N}^*$, the restriction of F on $B((a_{n+1}, a_{n+1}), 1)$ attains its maximum in (a_{n+1}, a_{n+1}) and

$$F(a_{n+1}, a_{n+1}) = (a_{n+1})^7,$$

hence

$$\limsup_{n\to+\infty} \frac{F(a_{n+1}, a_{n+1})}{\frac{a_{n+1}^5}{3} + \frac{a_{n+1}^6}{4}} = +\infty.$$

Therefore

$$\begin{aligned} \beta : &= \limsup_{s,t\to+\infty} \frac{\int_{B(x_0,R_1)} F(s,t)dx}{\frac{s^5}{3} + \frac{t^6}{4}} \\ &= |B(x_0, R_1)| \limsup_{s,t\to+\infty} \frac{F(s,t)}{\frac{s^5}{3} + \frac{t^6}{4}} \\ &= +\infty. \end{aligned}$$

On the other hand, we have

$$\sup_{|s|+|t|\leq a_{n+1}-1} F(s,t) = a_n^7 \text{ for all } n \in \mathbb{N}^*.$$

So

$$\lim_{n\to+\infty} \frac{\sup_{|s|+|t|\leq a_{n+1}-1} F(s,t)}{(a_{n+1}-1)^3} = 0,$$

accordingly

$$\liminf_{\sigma\to+\infty}\frac{\sup_{|s|+|t|\leq\sigma}F(s,t)}{\sigma^3}=0.$$

Thus

$$\begin{aligned}\alpha: &= \liminf_{\sigma\to+\infty}\frac{\int_\Omega \sup_{|s|+|t|\leq\sigma}F(s,t)dx}{\sigma^3}\\ &= |\Omega|\liminf_{\sigma\to+\infty}\frac{\sup_{|s|+|t|\leq\sigma}F(s,t)}{\sigma^3}\\ &= 0\\ &< L\beta.\end{aligned}$$

From Theorem 3.2, for each $\lambda>0$ *the problem*

$$\begin{gathered}\Delta(|\Delta u|^{x^2+y^2+1}\Delta u)=\lambda F_u(x,u,v)\quad \text{in } \Omega,\\ \Delta(|\Delta v|^{x^2+y^2+2}\Delta v)=\lambda F_v(x,u,v)\quad \text{in } \Omega,\\ u=v=\Delta u=\Delta v=0\quad \text{on } \partial\Omega,\end{gathered}$$

admits an unbounded sequence of weak solutions.

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