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Some remarks on statistical summability of order $\tilde{\alpha}$ defined by generalized De la Vallée-Poussin Mean

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ABSTRACT: In this article we define $(\lambda, \mu)-$statistical summability and $(V, \lambda, \mu)-$summability of order $\tilde{\alpha}$ for double sequences and obtain some relations between these summability methods. We demonstrate examples which shows our method of summability is more general for double sequences.

Key Words: Statistical Convergence, $\lambda-$statistical convergence, Double sequences.

Contents

1. Introduction

Fast [6] introduced the notion of statistical convergence as a generalized summability method in order to assign limits to those sequences which are not convergent in usual sense. He used the concept of natural density of subsets of $\mathbb{N}$, the set of positive integers. The natural density of a set $K \subset \mathbb{N}$, is denoted by $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k)$$

provided the limit exists, where χ_K denotes the characteristic function of K. As the sum on the right side of the above expression denotes the cardinality of the set $\{k \le n : k \in K\}$ so Fast [6] defined statistical convergence as follows.

Definition 1.1. *[6] A sequence $x = (x_k)$ of numbers is said to be statistically convergent to a number L provided that, for every $\epsilon > 0$,*

$$\delta\left(\{k \le n : |x_k - L| > \epsilon\}\right) = 0.$$

In this case, we write $S - \lim_{k\to\infty} x_k = L$.

Let $S(x)$ denotes the set of all statistically convergent sequences.

Although, statistical convergence was introduced in the mid of last century but a rapid development on statistical convergence starts with the papers of Šalát [20],

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Fridy [7] and Connor [5]. For more details and related concepts, we refer to [12], [19], [21,22,23,24] and [28].

In [13], Mursaleen presented an interesting extention of statistical convergence namely $\lambda-$statistical convergence and show how it is related with $(V, \lambda)-$summability.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ of numbers is said to be $(V, \lambda)-$summable to a number L (see [11]) if $t_n(x) \to L$ as $n \to \infty$.

Definition 1.2. *[13] A sequence $x = (x_k)$ of numbers is said to be $\lambda-$statistically convergent to a number L provided that for every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called $\lambda-$statistical limit of the sequence $x = (x_k)$ and we write $S_\lambda - \lim_{k\to\infty} x_k = L$. We denote the set of all $\lambda-$statistically convergent sequences by $S_\lambda(x)$.

Further, an interesting generalization of statistical convergence was introduced by Çolak [2] under the name of "statistical convergence of order α" for some $\alpha \in (0, 1]$. This new idea was further investigated by Çolak and Bektaş in [4] via $(V, \lambda)-$summability and obtained some interesting results. Before we go further we quote the following definition.

Definition 1.3. *[3] Let $\lambda = (\lambda_n)$ be a sequence of real numbers as defined above and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent of order α if there is a number L such that*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda^\alpha - \lim_{k\to\infty} x_k = L$. The set of all $\lambda-$statistically convergent sequences of order α is denoted by $S_\lambda^\alpha(x)$.

We next give some ideas and developments on double sequences which have been frequently appeared in literature.

A double sequence $x = (x_{ij})$ of real numbers is said to be convergent in Priengsheim's sense or $P-$convergent (See [18]) if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j \geq n$. The number L is called Priengsheim limit of $x = (x_{ij})$ and we write $P - \lim x = L$.

Double sequences were initially discussed by Bromwich [1] and Hardy [8]. Later, many authors including Móricz [16], Patterson [17], Tripathy and Sarma [25,26,27], Kumar [9] and Kumar and Mursaleen [10] etc. have shown their interest to study double sequences and related convergence problems. Mursaleen and Edely [15] and Mursaleen *et al*. [14] respectively extended Definition 1.1 and Definition 1.2 on double sequences and obtained some analogous results. However, Çolak and Altin [4] introduced statistical convergence of order $\tilde{\alpha}$ for these kind of sequences.

Definition 1.4. *[15] A double sequence $x = (x_{ij})$ of real numbers is said to be statistically convergent to L if for every $\epsilon > 0$*

$$P - \lim_{n,m\to\infty} \frac{1}{nm} |\{(i,j) \in \mathbb{N}\times\mathbb{N}, i \le n, j \le m : |x_{ij} - L| \ge \epsilon\}| = 0.$$

In this case, we write $S_2 - \lim_{i,j\to\infty} x_{ij} = L$ and $S_2(x)$ denotes the set of all statistically convergent double sequences.

Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to ∞ with $\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1$ and $\mu_{m+1} \le \mu_m + 1, \mu_1 = 1$. The generalized de la Vallée-Poussin mean of $x = (x_{ij})$ is defined by

$$t_{mn}(x) = \frac{1}{\lambda_n \mu_m} \sum_{(i,j)\in I_n\times I_m} x_{ij},$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$. Moreover, a double sequence $x = (x_{ij})$ is said to be $(V, \lambda, \mu)-$summable to a number L provided that $t_{mn}(x) \to L$ as $m, n \to \infty$.

Definition 1.5. *[14] A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)-$statistically convergent to a number L provided for every $\epsilon > 0$,*

$$P - \lim_{n,m\to\infty} \frac{1}{\lambda_n \mu_m} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\}| = 0.$$

In this case, the number L is called $(\lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ and we write $S_{(\lambda,\mu)} - \lim_{i,j\to\infty} x_{ij} = L$.

Let, $S_{(\lambda,\mu)}(x)$ denotes the set of all $(\lambda, \mu)-$statistically convergent double sequences of numbers.

In this article, we aim to define $(\lambda, \mu)-$statistical convergence and $(V, \lambda, \mu)-$summability of order $\tilde{\alpha}$ and obtain some relevant connections. Throughout we take $a, b, c, d \in (0, 1]$ as otherwise indicated. We will write $\tilde{\alpha}$ as an alternative of (a, b) and $\tilde{\beta}$ as an alternative of (c, d). Also we define: $\tilde{\alpha} \preceq \tilde{\beta} \iff a \le c \,\text{and}\, b \le d; \tilde{\alpha} \prec \tilde{\beta} \iff a < c \,\text{and}\, b < d; \tilde{\alpha} \cong \tilde{\beta} \iff a = c \,\text{and}\, b = d; \tilde{\alpha} \in (0, 1] \iff a, b \in (0, 1]; \tilde{\beta} \in (0, 1] \iff c, d \in (0, 1]; \tilde{\alpha} \cong 1 \,\text{in case}\, a = b = 1; \tilde{\beta} \cong 1 \,\text{in case}\, c = d = 1$ and $\tilde{\alpha} \succ 1 \,\text{in case}\, a > 1, b > 1$.

2. Main Results

In this section, we present our main results. We begin with the following definition:

Definition 2.1. *Let* $\lambda = (\lambda_n)$ *and* $\mu = (\mu_m)$ *be two non-decreasing sequences of positive real numbers tending to* ∞ *with*
$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1; \mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$
and $\tilde{\alpha} \in (0, 1]$ *be given.*
A double sequence $x = (x_{ij})$ *of numbers is said to be* $(\lambda, \mu)-$*statistically convergent of order* $\tilde{\alpha}$ *if there exists a number* L *such that for every* $\epsilon > 0$

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0,$$

where $\lambda^a = (\lambda_n^a) = (\lambda_1^a, \lambda_2^a, \lambda_3^a, \cdots\cdots)$; $\mu^b = (\mu_m^b) = (\mu_1^b, \mu_2^b, \mu_3^b, \cdots\cdots)$ *and* $\lambda_n^a \mu_m^b$ *denotes the usual multiplication of the corresponding entries of the sequences* λ^a *and* μ^b. *In this case, the number* L *is called* $(\lambda, \mu)-$*statistical limit of the sequence* $x = (x_{ij})$ *of order* $\tilde{\alpha}$ *and we write* $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = L$.

Let $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$ denotes the set of all $(\lambda, \mu)-$statistically convergent double sequences of order $\tilde{\alpha}$.

For $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.1 coincides with $(\lambda, \mu)-$statistical convergence of double sequences of [14]. For the choice $\lambda = (n)$ and $\mu = (m)$, Definition 2.1 coincides with statistical convergence of double sequences of order $\tilde{\alpha}$ of [3]. Moreover, if we take $\lambda = (n)$; $\mu = (m)$ and $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.1 coincides with statistical convergence of double sequences of [15].

Theorem 2.2. *For* $\tilde{\alpha} \in (0, 1]$, *if* $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = x_0$, *then* x_0 *is unique.*

Proof: Easy, so omitted. □

We next provide an example to show that the Definition 2.1 is well defined for $\tilde{\alpha} \in (0, 1]$ but not for $\tilde{\alpha} \succ 1$ in general.

Example 2.3. *Let* $x = (x_{ij})$ *be defined as follows:*

$$x_{ij} = \begin{cases} 1 & \text{if } i+j \text{ even} \\ 0 & \text{if } i+j \text{ odd} \end{cases}$$

Then for $\tilde{\alpha} \succ 1$,

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - 1| \geq \epsilon\}| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0$$

and

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}| \leq \lim_{n,m\to\infty} \frac{[\lambda_n \mu_m] + 1}{2\lambda_n^a \mu_m^b} = 0.$$

This shows that $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = 0$ *and* $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim_{i,j} x_{ij} = 1$ *which leads to a contradiction to Theorem* 2.2.

We state the following result without proof.

Theorem 2.4. *Let $x = (x_{ij})$ and $y = (y_{ij})$ be two double sequences of complex numbers and $\tilde{\alpha} \in (0, 1]$.*
(i) If $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim x_{ij} = L$ and $c \in \mathbb{C}$, then $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim(cx_{ij}) = cL$.
(ii) If $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim x_{ij} = L$ and $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim y_{ij} = M$, then $S^{\tilde{\alpha}}_{(\lambda,\mu)} - \lim(x_{ij} + y_{ij}) = L + M$.

Definition 2.5. *Let $\tilde{\alpha}$ be any real number such that $\tilde{\alpha} \in (0, 1]$ and p be a positive real number. A double sequence $x = (x_{ij})$ is said to be strongly $(V, \lambda, \mu)-$ summable of order $\tilde{\alpha}$ to a number L provided that*

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = 0,$$

where $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$. In this case, the number L is called strong $(V, \lambda, \mu)-$statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$.

Let $[w_p^2]_{\tilde{\alpha}}(x)$ denote the set of all strongly $(V, \lambda, \mu)-$summable double sequences of order $\tilde{\alpha}$.

For $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.5 coincides with strong $(V, \lambda, \mu)-$summability of double sequences of [14]. For $\lambda = (n)$ and $\mu = (m)$, Definition 2.5 coincides with strong p-Cesàro summability of double sequences of order $\tilde{\alpha}$ of [3]. However, if we take $\lambda = (n)$; $\mu = (m)$ and $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.5 coincides with strong p-Cesàro summability of double sequences of [15].

Theorem 2.6. *Let $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$. Then $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S^{\tilde{\beta}}_{(\lambda,\mu)}(x)$ and the inclusion is strict for some $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$.*

Proof: Let $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$. Since, $\tilde{\alpha} \preceq \tilde{\beta}$ so $a \le c$ and $b \le d$; which for any $\epsilon > 0$ gives the inequality

$$\frac{1}{\lambda_n^c \mu_m^d} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\}| \le \frac{1}{\lambda_n^a \mu_m^b} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\}|;$$

and therefore the result follows immediately from the fact that $x = (x_{ij}) \in S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$. For rest part of the Theorem we consider the following example. Define $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} ij, & \text{if } n - [\sqrt{\lambda_n}] + 1 \le i \le n \quad \text{and} \quad m - [\sqrt{\mu_m}] + 1 \le j \le m] \\ 0, & \text{otherwise} \end{cases} \text{; then}$$

$$\begin{aligned} &\frac{1}{\lambda_n^c \mu_m^d} |\{(i, j) \in I_n \times I_m : |x_{ij} - 0| \ge \epsilon\}| \\ &\quad = \frac{1}{\lambda_n^c \mu_m^d} \left| \left\{ (i, j) \in I_n \times I_m : {\scriptstyle n-[\sqrt{\lambda_n}]+1 \le i \le n \text{ and} \atop m-[\sqrt{\mu_m}]+1 \le j \le m} \right\} \right| \le \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n^c \mu_m^d}. \end{aligned}$$

It follows, for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (*i.e.* for $\frac{1}{2} < c \le 1$ and $\frac{1}{2} < d \le 1$), we have

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^c \mu_m^d} \left|\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \ge \epsilon\}\right| \le \lim_{n,m\to\infty} \frac{[\sqrt{\lambda_n}\sqrt{\mu_m}]}{\lambda_n^c \mu_m^d} = 0.$$

This shows that $x = (x_{ij}) \in S^{\tilde{\beta}}_{(\lambda,\mu)}(x)$, but one can easily verify that $x \notin S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (*i.e.* for $0 < a \le \frac{1}{2}$ and $0 < b \le \frac{1}{2}$). □

Corollary 2.7. *Let* $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$,

(i) If $\tilde{\beta} \cong 1$, *then* $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S^1_{(\lambda,\mu)} = S_(\lambda,\mu)$ *and the inclusion is strict.*

(ii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) = S^{\tilde{\beta}}_{(\lambda,\mu)}(x) \iff \tilde{\alpha} \cong \tilde{\beta}$.

(iii) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) = S_{(\lambda,\mu)}(x) \iff \tilde{\alpha} \cong 1$.

Theorem 2.8. *Let* $\lambda = (\lambda_n)$, $\mu = (\mu_m)$ *be two sequences as defined above and* $\tilde{\alpha} \in (0, 1]$, *then*

(i) $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S_2(x)$ *for all* λ, μ *and* $\tilde{\alpha} \in (0, 1]$.

(ii) $S_2(x) \subseteq S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$, *if and only if,* $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$ *and* $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$.

Proof: (i) By the nature of the sequences (λ_n), (μ_m) and from the expression $\frac{\lambda_n \mu_m}{nm} \le 1$, the result follows.

(ii) Let, $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$; $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$ and $x = (x_{ij}) \in S_2(x)$. For given $\epsilon > 0$, we have,

$$\{(i,j), i \le n \, and \, j \le m : |x_{ij} - L| \ge \epsilon\} \supset \{(i,j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\},$$

it follows that,

$$\begin{aligned}\frac{1}{nm}\left|\{(i,j), i \le n \, and \, j \le m : |x_{ij} - L| \ge \epsilon\}\right| &\ge \frac{1}{nm}\left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\}\right| \\ &= \left(\frac{\lambda_n^a}{n}\right)\left(\frac{\mu_m^b}{m}\right)\frac{1}{\lambda_n^a \mu_m^b}\left|\{(i,j) \in I_n \times I_m : |x_{ij} - L| \ge \epsilon\}\right|.\end{aligned}$$

Taking limit as $n, m \to \infty$ we have, $S_2(x) \subseteq S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$.

Conversely, suppose that either $\liminf_{n\to\infty} \frac{\lambda_n^a}{n}$ or $\liminf_{m\to\infty} \frac{\mu_m^b}{m}$ or both are zero. Then we can choose two subsequences (n_p) and (m_q) such that $\frac{\lambda_{n_p}^a}{n_p} < \frac{1}{p}$ and $\frac{\mu_{m_q}^b}{m_q} < \frac{1}{q}$. Define double sequence $x = (x_{ij})$ as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i \in I_{n_p} \quad \text{and} \quad j \in I_{m_q} \qquad (p, q = 1, 2, 3, \ldots) \\ 0 & \text{otherwise,} \end{cases}$$

Then clearly $x \in S_2(x)$, but $x \notin S_{(\lambda,\mu)}(x)$. From Corollary 2.7, since $S^{\tilde{\alpha}}_{(\lambda,\mu)}(x) \subseteq S_{(\lambda,\mu)}(x)$, we have $x \notin S^{\tilde{\alpha}}_{(\lambda,\mu)}(x)$. Hence, $\liminf_{n\to\infty} \frac{\lambda_n^a}{n} > 0$ and $\liminf_{m\to\infty} \frac{\mu_m^b}{m} > 0$. □

Theorem 2.9. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. Then $[w_p^2]_{\tilde{\alpha}}(x) \subseteq [w_p^2]_{\tilde{\beta}}(x)$ and the inclusion is strict for some $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \prec \tilde{\beta}$.*

Proof: Let $x = (x_{ij}) \in [w_p^2]_{\tilde{\alpha}}(x)$, then for $\tilde{\alpha} \in (0,1]$ and a positive real number p

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = 0.$$

Also for given $\tilde{\alpha}$ and $\tilde{\beta}$ such that $\tilde{\alpha} \preceq \tilde{\beta}$, one can write

$$\lim_{n,m\to\infty} \frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p \leq \lim_{n,m\to\infty} \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = 0$$

which implies $x = (x_{ij}) \in [w_p^2]_{\tilde{\beta}}(x)$. Hence, $[w_p^2]_{\tilde{\alpha}}(x) \subseteq [w_p^2]_{\tilde{\beta}}(x)$. The following example will show that the inclusion is strict. Define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} 1, & \text{if } n - \sqrt{\lambda_n} + 1 \leq i \leq n \quad \text{and} \quad m - \sqrt{\mu_m} + 1 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

Then for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (that is for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$),

$$\frac{1}{\lambda_n^c \mu_m^d} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - 0|^p \leq \frac{\sqrt{\lambda_n}\sqrt{\mu_m}}{\lambda_n^c \mu_m^d} = \frac{1}{\lambda_n^{c-\frac{1}{2}} \mu_m^{d-\frac{1}{2}}}.$$

Since $\frac{1}{\lambda_n^{c-\frac{1}{2}} \mu_m^{d-\frac{1}{2}}} \to 0$ as $n, m \to \infty$, therefore $x = (x_{ij}) \in [w_p^2]_{\tilde{\beta}}(x)$, but for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (that is for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$)

$$\frac{(\sqrt{\lambda_n} - 1)(\sqrt{\mu_m} - 1)}{\lambda_n^a \mu_m^b} \leq \frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - 0|^p$$

and $\frac{(\sqrt{\lambda_n}-1)(\sqrt{\mu_m}-1)}{\lambda_n^a \mu_m^b} \to \infty$ as $n, m \to \infty$, which implies $x = (x_{ij}) \notin [w_p^2]_{\tilde{\alpha}}(x)$. Hence the inclusion is strict. $\square$

Corollary 2.10. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. Then*
(i) $[w_p^2]_{\tilde{\alpha}}(x) = [w_p^2]_{\tilde{\beta}}(x) \Leftrightarrow \tilde{\alpha} \cong \tilde{\beta}$.
(ii) $[w_p^2]_{\tilde{\alpha}}(x) \subseteq w_p^2$ for each $\tilde{\alpha} \in (0,1]$ and $0 < p < \infty$.

Theorem 2.11. *Let $\tilde{\alpha}, \tilde{\beta} \in (0,1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$ and p be a positive real number. If a sequence $x = (x_{ij})$ is strongly $(V, \lambda, \mu)-$summable to L of order $\tilde{\alpha}$, then it is $(\lambda, \mu)-$statistically convergent to L of order $\tilde{\beta}$, i.e., $[w_p^2]_{\tilde{\alpha}}(x) \subset S_{(\lambda,\mu)}^{\tilde{\beta}}(x)$.*

Proof: For any sequence $x = (x_{ij})$ and $\epsilon > 0$

$$\sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p = \sum_{\substack{(i,j)\in I_n\times I_m \\ |x_{ij}-L|\geq\epsilon}} |x_{ij} - L|^p + \sum_{\substack{(i,j)\in I_n\times I_m \\ |x_{ij}-L|<\epsilon}} |x_{ij} - L|^p$$

$$\geq \sum_{(i,j)\in I_n\times I_m |x_{ij}-L|\geq\epsilon} |x_{ij} - L|^p \geq |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p,$$

which implies

$$\frac{1}{\lambda_n^a \mu_m^b} \sum_{(i,j)\in I_n\times I_m} |x_{ij} - L|^p \geq \frac{1}{\lambda_n^a \mu_m^b} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p$$

$$\geq \frac{1}{\lambda_n^c \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| .\epsilon^p.$$

It follows that if $x = (x_{ij})$ is strong $(V, \lambda, \mu)-$summable to L of order $\tilde{\alpha}$, then it is $(\lambda, \mu)-$statistically convergent to L of order $\tilde{\beta}$. □

For particular choice of $\tilde{\alpha} \cong \tilde{\beta}$ in above Theorem we have the following result.

Corollary 2.12. *Let $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$ such that $\tilde{\alpha} \preceq \tilde{\beta}$,*
(i) If $\tilde{\alpha} \cong \tilde{\beta}$ then $[w_p^2]_{\tilde{\alpha}}(x) \subset S_{(\lambda,\mu)}^{\tilde{\alpha}}(x)$.
(ii) For $\tilde{\beta} \cong 1$, $[w_p^2]_{\tilde{\alpha}}(x) \subset S_{(\lambda,\mu)}(x)$.

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