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Neumann problem in divergence form modeled on the $p(x)$-Laplace equation

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abstract: We consider a Neumann problem in divergence form with variable growth, modeled on the p(x)-Laplace equation. We establish the existence of solutions under appropriate hypotheses.

Key Words: p(x)-laplacian, variational method, Critical point.

Contents

1. Introduction

In recent decades, the study of differential equations and variational problems involving variable exponent conditions has been an interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and the mathematical models of stationary thermo-rheological viscous flows of non- Newtonian fluids. For more information on modeling physical phenomena by equations involving $p(x)$growth condition we refer to [1,10]. This paper was motivated by [4,8,11].

The aim of this paper is to discuss the existence of solutions of the following problem which involves a general elliptic operator in divergence form

$$(\mathcal{P}) \begin{cases} -div((a(x,\nabla u))+ \mid u \mid^{p(x)-2} u = f(x,u) \ in \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \ on \ \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ν is the outward normal vector on $\partial\Omega$, $1 < p^- = \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) = p^+$,

$N \geq 2, p \in C(\overline{\Omega})$, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a potential with the assumption as below

(A_1) $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function with a continuous derivative with respect to second variable ξ where $a = DA = A'$,

(A_2) $A(x,0) = 0, \forall x \in \Omega$,

(A_3) $a(x,\xi) \leq C_1[1+ \mid \xi \mid^{p(x)-1}]$, $C_1 > 0$,

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(A_4) A is $p(x)$-uniformly convex: there exists a constant $k > 0$ such that

$$A(x, \frac{\xi + \eta}{2}) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k \mid \xi - \eta \mid^{p(x)}, \forall x \in \Omega, \xi, \eta \in \mathbb{R}^N,$$

(A_5) A satisfies elliptic condition, i.e. there exists $C_2 > 0$ such that

$$A(x, \xi) \geq C_2 \mid \xi \mid^{p(x)}, \ \forall x \in \Omega, \xi \in \mathbb{R}^N,$$

(A_6) A is $p(x)$-subhomogeneous: for all $x \in \Omega, \xi, \eta \in \mathbb{R}^N$,

$$0 \leq a(x, \xi)\xi \leq p(x)A(x, \xi),$$

and
$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function with $F(x, t) = \int_0^t f(x, s)ds$ such that
(F_1) $\lim\limits_{|s| \to 0} \dfrac{f(x, s)}{\mid s \mid^{p(x)-1}} = 0,$
(F_2) $\lim\limits_{|s| \to +\infty} \dfrac{f(x, s)}{\mid s \mid^{p(x)-1}} = 0,$
(F_3) there exists $u_* > 0$ such that $F(x, u_*) > 0$ for a.e $x \in \Omega$,
(F_4) there exists $\lambda > 0$ such that $f(x, t) \geq \lambda \mid t \mid^{q(x)}, \ q \in C_+(\Omega)$ with

$$q^+ = \sup_{\Omega} q(x) < p^- \ and \ C_+(\Omega) = \{h \in C(\Omega) : \ h(x) > 1\}.$$

Theorem 1.1. *Let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a potential verifies $(A_1) - (A_6)$ and under the assumptions $(F_1) - (F_3)$, then the problem $(\mathcal{P})$ has at least two nontrivial solutions.*

Theorem 1.2. *Assume that the potential $a(x, .), f$ are odd with respect to the second argument:*

$$f(x, -s) = -f(x, s) \ and \ a(x, -s) = -a(x, s),$$

and the conditions of Theorem 1.1 with (F_4) are satisfied. Then the problem $(\mathcal{P})$ has infinitely many solutions.

The model case leading to problem $(\mathcal{P})$ is the Neumann problem for the $p(x)$-Laplacian operator

$$\begin{cases} -\Delta_{p(x)}u + \mid u \mid^{p(x)-2} u = f(x, u) \ in \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \ on \ \partial\Omega, \end{cases}$$

where $a(x, s) = s$. The operator $\Delta_{p(x)}u = div(\mid \nabla u \mid^{p(x)-2} u)$ is called $p(x)$-Laplacian, which becomes p-Laplacian when $p(x) = p$ (a constant). These related problems has been investigated by many authors (cf. [2,3,5,9]).

This paper is organized as three sections. In section 2, we introduce some basic properties of the variable exponent Lebesgue-Sobolev spaces. In section 3, we give the existence of two nontrivial weak solutions for problem $(\mathcal{P})$ and infinitely of solutions under additional conditions.

Definition 1.3. *A weak solution of* $(\mathcal{P})$ *is a function* $u \in W^{1,p(x)}(\Omega)$ *if*

$$\int_\Omega a(x, \nabla u).\nabla v dx + \int_\Omega | u |^{p(x)-2} uv dx - \int_\Omega f(x, u) v dx = 0,$$

$\forall v \in W^{1,p(x)}(\Omega)$.

2. Preliminaries

We introduce the setting of our problem with some auxiliary results. For convenience, we only recall some basic facts which will be used later. Define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega | u |^{p(x)} dx < \infty\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$| u |_{p(x)} = \inf\{\lambda > 0 : \int_\Omega | \frac{u}{\lambda} |^{p(x)} dx \leq 1\}$$

becomes a Banach space separable and reflexive.

Let $X = W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega)\}$ equipped with the norm

$$\| u \| = \inf\{\lambda > 0 : \int_\Omega (| \frac{\nabla u}{\lambda} |^{p(x)} + | \frac{u}{\lambda} |^{p(x)}) dx \leq 1\},$$

is a separable reflexive Banach space.

Proposition 2.1. *[7] Set,* $\rho(u) = \int_\Omega | u |^{p(x)} dx$.

If $u \in W^{1,p(x)}(\Omega)$ *we have*

(1) $\| u \| \geq 1 \Rightarrow \| u \|^{p^-} \leq \rho(u) \leq \| u \|^{p^+}$,

(2) $\| u \| \leq 1 \Rightarrow \| u \|^{p^+} \leq \rho(u) \leq \| u \|^{p^-}$.

Proposition 2.2. *[7] For any* $u \in L^{p(x)}(\Omega)$, $v \in L^{p'(x)}(\Omega)$, *we have*

$$| \int_\Omega uv \, dx | \leq 2 | u |_{p(x)} | v |_{p'(x)},$$

with

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Proposition 2.3. *[7] If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then the embedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} \text{ if } p(x) < N, \\ +\infty \text{ if } p(x) \geq N. \end{cases}$$

The energy functional $\phi : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is given by

$$\phi(u) = \int_\Omega A(x, \nabla u)dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx - \int_\Omega F(x, u)dx$$

is well defined and of class C^1. Here, the derivative is

$$\phi'(u).v = \int_\Omega a(x, \nabla u).\nabla v dx + \int_\Omega \mid u \mid^{p(x)-2} uv dx - \int_\Omega f(x, u)v dx$$

for all $v \in X$. Therefore, the critical points of ϕ are weak solutions of $(\mathcal{P})$.

Definition 2.4. *An operator $a : X \to X^*$ verifies the (S_+) condition if for any sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \rightharpoonup x$ weakly and $\limsup_{n\to+\infty} \langle a(x_n), x_n - x\rangle \leq 0$, we have $u_n \to u$ strongly.*

3. Proof of the main results

To prove Theorem 1.1 we will use the minimization and a Mountain Pass type argument to find nonzero critical point of ϕ. So we need the following lemma.

Lemma 3.1. *ϕ is sequentially weakly lower semi continuous.*

Proof:
Using (F_1) and (F_2), we can see that $\mid f(x, s) \mid \leq C_5(1+ \mid s \mid^{p(x)-1})$, $\forall s \in \mathbb{R}$ with $C_5 > 0$. By the compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we deduce that $u \to \int_\Omega F(x, u)dx$ is sequentially lower semi continuous. Besides,

$$u \to \int_\Omega A(x, \nabla u)dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx$$

is convex uniformly, which assures that it is sequentially lower semi continuous. □

Proof: [Proof of Theorem 1.1:]
We start to check that ϕ is coercive and satisfies the $(P.S)$ condition. From (F_2) for $\varepsilon > 0$ small enough, there exists $\delta > 0$ such that

$$\mid f(x, s) \mid \leq \varepsilon \mid s \mid^{p(x)-1}, \ \forall |s| \geq \delta,$$

and thus we get

$$\mid F(x, t) \mid \leq \frac{\varepsilon}{p(x)} \mid t \mid^{p(x)} + \max_{|s|\leq\delta} \mid f(x, s) \mid \mid t \mid , \forall t \in \mathbb{R},$$

for a.e $x \in \Omega$. Consequently, for $\parallel u \parallel > 1$ we obtain

$$\begin{aligned}
\phi(u) &\geq \int_\Omega A(x,\nabla u)dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx - \varepsilon \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx \\
&\quad - \max_{|t|\leq\delta} \mid f(x,t) \mid \mid u \mid dx \\
&\geq C_2 \int_\Omega \mid \nabla u \mid^{p(x)} dx + (1-\varepsilon) \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx \\
&\quad - \max_{|t|\leq\delta} \mid f(x,t) \mid \mid u \mid dx \\
&\geq \min\{C_2, \frac{1-\varepsilon}{p^+}\} \parallel u \parallel^{p^-} - \max_{|t|\leq\delta} \mid f(x,t) \mid \int_\Omega \mid u \mid dx \\
&\geq \min\{C_2, \frac{1-\varepsilon}{p^+}\} \parallel u \parallel^{p^-} - C' \mid meas(\Omega) \mid_{p'(x)} \max_{|t|\leq\delta} \mid f(x,t) \mid \parallel u \parallel,
\end{aligned}$$

where C' is a positive constant and $\varepsilon \to 0$.

Therefore, ϕ is coercive and has a global minimizer u_1 which is nontrivial because

$$\phi(u_1) \leq \phi(u_*) < 0.$$

In the sequel, we claim that ϕ satisfies the Palais Smale condition. In fact, let $(u_n)_n \subset X$ be a Palais Smale sequence, that is

$$\phi'(u_n) \to 0 \ in \ X^*, \ \phi(u_n) \to l \in \mathbb{R}.$$

First we show that $(u_n)_n$ is bounded. One has, from (A_4)

$$\begin{aligned}
\phi(u_n) &= \int_\Omega A(x,\nabla u_n)dx + \int_\Omega \frac{1}{p(x)} \mid u_n \mid^{p(x)} dx - \int_\Omega F(x,u_n)dx \\
&\geq C_2 \int_\Omega \mid \nabla u_n \mid^{p(x)} dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx - \int_\Omega F(x,u_n)dx.
\end{aligned}$$

On the other hand, from (A_5), we get

$$\begin{aligned}
\langle \phi'(u_n), u_n \rangle &= \int_\Omega a(x,\nabla u_n).\nabla u_n dx + \int_\Omega \mid u_n \mid^{p(x)} dx - \int_\Omega f(x,u_n)u_n dx \\
&\leq \int_\Omega p(x)A(x,\nabla u_n)dx + \int_\Omega \mid u_n \mid^{p(x)} dx - \int_\Omega f(x,u_n)u_n dx.
\end{aligned}$$

Thus,

$$\begin{aligned}\phi(u_n) - \frac{1}{2p^+}\langle\phi'(u_n), u_n\rangle &\geq \int_\Omega (1 - \frac{1}{p^+}p(x))A(x, \nabla u_n)dx\\ &+ \int_\Omega (\frac{1}{p(x)} - \frac{1}{2p^+}) \mid u_n \mid^{p(x)} dx\\ &- \int_\Omega F(x, u_n)dx + \int_\Omega \frac{1}{2p^+} f(x, u_n)u_n dx\\ &\geq \frac{C_2}{2}\int_\Omega \mid \nabla u_n \mid^{p(x)} dx + \frac{1}{2p^+}\int_\Omega \mid u_n \mid^{p(x)} dx\\ &+ \int_\Omega [\frac{1}{2p^+} f(x, u_n)u_n - F(x, u_n)]dx.\end{aligned}$$

According to (F_1) and (F_2), for $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that

$$\mid f(x,t) \mid \leq \varepsilon \mid t \mid^{p(x)-1}$$

for all $\mid t \mid \leq \delta$ and for all $\mid t \mid \geq \eta$. Hence, one has

$$\mid F(x,t) \mid \leq \frac{\varepsilon}{p(x)} \mid t \mid^{p(x)} + C$$

for all $\mid t \mid \leq \delta$ and for all $\mid t \mid \geq \eta$ with C is a constant.

Furthermore,

$$\mid f(x,t)t \mid \leq \varepsilon \mid t \mid^{p(x)}, \forall \mid t \mid \leq \delta \text{ and } \forall \mid \text{t} \mid \geq \eta.$$

It yields,

$$\begin{aligned}\phi(u_n) - \frac{1}{2p^+}\langle\phi'(u_n), u_n\rangle &\geq \frac{1}{2}\min\{C_2, \frac{1}{p^+}\}\int_\Omega (\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)})dx\\ &\quad -\varepsilon(\frac{1}{2p^+} + \frac{1}{p^-})\int_\Omega \mid u_n \mid^{p(x)} dx + c_1\\ &\geq [\frac{1}{2}\min\{C_2, \frac{1}{p^+}\} - \varepsilon(\frac{1}{2p^+} + \frac{1}{p^-})]\int_\Omega (\mid \nabla u_n \mid^{p(x)}\\ &\quad + \mid u_n \mid^{p(x)})dx + c_1,\end{aligned}$$

with $c_1 > 0$. For $\varepsilon > 0$ small enough with $M = \frac{1}{2}\min\{C_2, \frac{1}{p^+}\} - \varepsilon(\frac{1}{2p^+} + \frac{1}{p^-} > 0$, then we get

$$\int_\Omega (\mid \nabla u_n \mid^{p(x)} + \mid u_n \mid^{p(x)})dx \leq \frac{1}{M}\phi(u_n) - \frac{1}{2p^+}\langle\phi'(u_n), u_n\rangle - c_1.$$

Since $\phi(u_n)$ is bounded and $\langle\phi'(u_n), u_n\rangle \to 0$. Then $(u_n)_n$ is bounded in X, passing to a subsequence, so $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^{p(x)}(\Omega)$. Now, we are ready to prove that $u_n \to u$ in X.

We have

$$\begin{aligned}\langle \phi'(u_n), u_n - u\rangle &= \int_\Omega a(x,\nabla u_n)(u_n-u)dx + \int_\Omega \mid u_n \mid^{p(x)-2} u_n(u_n-u)dx \\ &\quad - \int_\Omega f(x,u_n)(u_n-u)dx,\end{aligned}$$

accordingly,

$$\begin{aligned}\mid \int_\Omega a(x,\nabla u_n)(u_n-u)dx \mid &= \mid \langle \phi'(u_n), u_n-u\rangle - \int_\Omega \mid u_n \mid^{p(x)-2} u_n(u_n-u)dx \\ &\quad + \int_\Omega f(x,u_n)(u_n-u)dx \\ &\leq \mid \langle \phi'(u_n), u_n-u\rangle \mid + \mid u_n \mid^{p(x)-1} \mid u_n-u \mid dx \\ &\quad + \int_\Omega \mid f(x,u_n) \mid\mid u_n-u \mid dx \\ &\leq \parallel \phi'(u_n) \parallel_{X^*} \parallel u_n-u \parallel \\ &\quad + c_2 \mid\mid u_n \mid^{p(x)-1} \mid_{p'(x)} \mid u_n-u \mid_{p(x)} \\ &\quad + \int_\Omega \mid f(x,u_n) \mid \mid u_n-u \mid dx.\end{aligned}$$

By (F_1) and (F_2), there exists $C>0$ such that

$$\mid f(x,s) \mid \leq C(1+ \mid s \mid^{p(x)-1}), \forall s \in \mathbb{R}.$$

Which yields

$$\begin{aligned}\int_\Omega \mid f(x,u_n)(u_n-u) \mid dx &\leq C\int_\Omega \mid u_n-u \mid dx + C\int_\Omega \mid u_n \mid^{p(x)-1} \mid u_n-u \mid dx \\ &\leq C' \mid u_n-u \mid_{p(x)} \\ &\quad + C \mid \mid u_n \mid^{p(x)-1} \mid_{p'(x)} \mid u_n-u \mid_{p(x)} .\end{aligned}$$

Since $\parallel \phi'(u_n) \parallel_{X^*} \to 0$ and $\mid u_n-u \mid_{p(x)} \to 0$ we get

$$\limsup_{n\to\infty} \int_\Omega a(x,\nabla u_n).(u_n-u)dx \leq 0.$$

From proposition 2.1 in [4] we know that $a(x,\xi)$ is of S_+ type, it follows that $u_n \to u$ in X. Next, we verify the geometric condition of Mountain Pass Theorem. Indeed, by (F_2), there exists $\delta>0$ such that $\mid F(x,s) \mid \leq \frac{\varepsilon}{p(x)} \mid s \mid^{p(x)},\ for\ all \mid s \mid < \delta$. Using (F_3), there exists $K(\delta)>0$ such that

$$\mid F(x,s) \mid \leq K(\delta) \mid s \mid^{p(x)} \leq K(\delta) \mid s \mid^{q(x)},$$

for every $\mid s \mid > \delta$ with $p^+ < q^-$ and $q(x) < p^*(x)$. Thereby, for $\| u \| = r$ small enough,(till $\| u \| \leq \min\{1, \| u_1 \|\}$)

$$\begin{aligned}\phi(u) \quad &\geq \quad \int_\Omega A(x, \nabla u)dx + \int_\Omega \frac{1}{p(x)} \mid u \mid^{p(x)} dx - \int_{[|u| \leq \delta]} \frac{\varepsilon}{p(x)} \mid u \mid^{p(x)} dx \\ & \quad -K(\delta) \int_{|u| > \delta} \mid u \mid^{q(x)} dx \\ &\geq \quad \min\{C_2, \frac{1}{p^+}\} \| u \|^{p^+} - \frac{\varepsilon}{p^-} \| u \|^{p^-} - K(\delta) \| u \|^{q^-} = h(r).\end{aligned}$$

Since $q^- > p^+ \geq p^-$, there exists $r > 0$ small enough and $a > 0$ such that $h(r) \geq a > 0$. Besides, $\phi(0) = 0$. Then, by the Mountain Pass Lemma, there exists $u_2 \in X$ such that $\phi'(u_2) = 0$. □

Proof: [Proof of Theorem 1.2:]

The functional ϕ is even. Since ϕ is coercive then ϕ satisfies condition $(P.S)_c$ for any $c \neq 0$. Obviously, $\phi(0) = 0$ and $\inf_{u \in X} \phi > -\infty$.

By the Ljusternik-Schnirelman theorem (see [12]), to prove Theorem 1.2, it is sufficient to prove that for every positive integer n, there exists a symmetric closed set $A_n \subset X$ such that $\gamma(A_n) \geq n$ and $sup_{u \in A_n} \phi(u_n) < 0$, where γ is the Krasnoselskii genus.

Now let any n be given, since it is known that $C_0^\infty(\Omega)$ is an infinite-dimensional subspace of X, so we can take an n-dimensional subspace $Y_n \subset C_0^\infty \subset X$. By the proof of Theorem 4.3 in [6] we have

$$\gamma(A_n) = n \ and \ \sup_{u \in A_n} \phi(u) < 0,$$

where $A_n = kS_{n-1},\ k > 0$ and $S_{n-1} = \{u \in Y_n \mid \| u \| = 1\}$. Hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, \}$ of $(\mathcal{P})$ such that $\phi(\pm u_k) < 0$. □

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