Bol. Soc. Paran. Mat. (3s.) **v. 32** 2 (2014): 95–107.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i2.18812

Asymptotic modeling of thin plastic oscillating layer

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ABSTRACT: In this paper we study the asymptotic behavior of solutions to a elasticity problem, of a containing structure a plastic thin oscillating layer of thickness and rigidity depending of small parameters ε. We use the epi-convergence method to find the limit problems with interface conditions.

Key Words: Limit behavior, plasticity problem, thin layer, epi-convergence method.

Contents

1. Introduction

The inclusion of a very thin layer of very rigid material into a given elastic body has been widely considered, and in the classic literature. For more details, we can refer to [6], [7], [10] and [11]. In general, the computation of solution using numerical methods is very difficult. In one hand, this is because the thickness of the adhesive requires a fine mesh, which in turn implies an increase of the degrees of flexible than the adherents, and this produces numerical instabilities in the stiffness matrix. To overcame this difficulties, thanks to Goland and Reissner [12] find a limit problem in which the adhesive is treated on this theoretical approach, see for example A. Ait Moussa and J. Messaho [1], Acerbi, Buttazzo and Perceivable [2], Licht and Michail [4] and A. Ait Moussa and L. Zlaïji [8].

In this present work, we consider a structure containing a plastic thin oscillating layer of thickness, rigidity, and periodicity parameter depending on ε, where ε is

2000 *Mathematics Subject Classification:* 335B40, 35L70 a parameter intended to tend towards 0. In a such structure, we have treated the scalar case for a thermal conductivity problem in [3]. The aim of this work is to study the limit behavior of an elasticity problem with a convex energy functional posed in a such structure.

This paper is organized in the following way. In section 2, we express the problem to study, and we give some notation and we define functional spaces for this study in the section 3. In the section 4, we study the problem (4.1). The section 5 is reserved to the determination of the limits problems and our main result.

2. Statement of the problem

We consider a structure, occupying a bonded domain Ω in $\mathbb{R}^3$ with Lipschitzian boundary $\partial\Omega$. It is constituted of two elastic bodies joined together by a rigid thin layer with oscillating boundary, the latter obeys to nonlinear elastic law of power type. More precisely, the stress field is related to the displacement's field by

$$\sigma^\varepsilon = \frac{1}{\varepsilon}\,|e(u^\varepsilon)|^{-1}\,e(u^\varepsilon),\ \varepsilon > 0.$$

The structure occupies the regular domain $\Omega = B_\varepsilon \cup \Omega_\varepsilon$, where B_ε is given by $B_\varepsilon = \{x = (x', x_3)\ /\ |x_3| < \frac{\varepsilon\varphi_\varepsilon}{2}\}$, and $\Omega_\varepsilon = \Omega \setminus B_\varepsilon$ represent the regions occupied by the thin plate and the two elastic bodies, see Figure 1, ε being a positive parameter intended to approach 0, and $\Sigma = \{x = (x', x_3)\ /\ |x_3| = 0\}$. The structure is subjected to a density of forces of volume $f : \Omega \to \mathbb{R}^3$, and it is fixed on the boundary $\partial\Omega$. Equations which relate the stress field σ^ε, $\sigma^\varepsilon : \Omega \to \mathbb{R}^9_S$, and the field of displacement u^ε, $u^\varepsilon : \Omega \to \mathbb{R}^3$ are

$$\begin{cases} div(\sigma^\varepsilon) + f = 0 & \text{in} \quad \Omega \\ \sigma^\varepsilon_{ij} = a_{ijkh}e_{kh}(u^\varepsilon) & \text{in} \quad \Omega_\varepsilon \\ \sigma^\varepsilon = \frac{1}{\varepsilon}|e(u^\varepsilon)|^{-1}e(u^\varepsilon) & \text{in} \quad B_\varepsilon \\ u^\varepsilon = 0 & \text{on} \quad \partial\Omega \end{cases} \qquad (\mathbb{P}_\epsilon)$$

where a_{ijkh} are the elasticity coefficients, and $\mathbb{R}^9_S$ the vector space of the square symmetrical matrices of order three, $e_{ij}(u)$ are the components of the linearized tensor of deformation $e(u)$. φ_ε is a bounded real function and $]0, \varepsilon[^2$-periodic. In the sequel, we assume that the elasticity coefficients a_{ijkh} satisfy to the following hypotheses :

$$a_{ijkh} \in L^\infty(\Omega) \tag{2.1}$$

$$a_{ijkh} \;=\; a_{jikh} \;=\; a_{khij} \tag{2.2}$$

$$a_{ijkh}\tau_{ij}\tau_{kh} \geq C\tau_{ij}\tau_{ij},\ \forall\tau \in \mathbb{R}^9_S \tag{2.3}$$

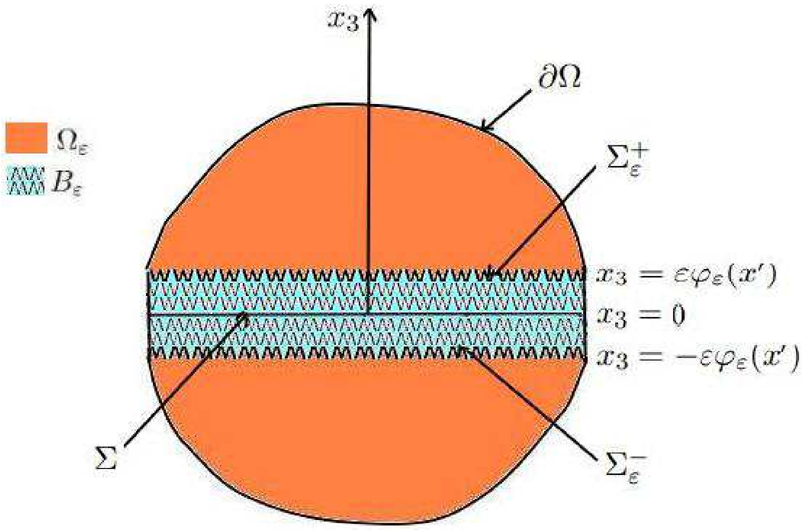

Figure 1: Domain

3. Notations and Functions Setting

3.1. Notations

We begin by introducing some notation which is used throughout the paper $x = (x', x_3)$, where $x' = (x_1, x_2)$, $\tau \otimes \zeta = (\tau_i \zeta_i)_{1 \leq i,j \leq 3}$ and

$$\tau \otimes_s \zeta = \frac{\tau \otimes \zeta + \zeta \otimes \tau}{2}, \ \forall \tau, \zeta \in \mathbb{R}^3; \ e^*(.) = (\nabla' + \nabla'^t)(.); \text{ or } \nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}).$$

We set $Y =]0,1[\times]0,1[$, $\varphi : \mathbb{R}^2 \to [a_1, a_2]$, such that $a_2 \geq a_1 > 0$, and φ is Y-periodic, $\varphi_\varepsilon(x') = \varphi(\frac{x'}{\varepsilon})$, and

$$m(\varphi) = \frac{1}{\int_Y dx'} \int_Y \varphi(x') dx'.$$

In the following C will denote any constant with respect to ε and $[v]$ is the jump of displacement field v through Σ.

3.2. Functions

First, we introduce the following space :

$$\begin{aligned} V^\varepsilon \ = \ & \{u \in L^1(\Omega)/e(u) \in L^2(\Omega_\varepsilon, \mathbb{R}_s^9), \ u \in LD_0(B_\varepsilon, \mathbb{R}_s^3), \\ & [u]^\varepsilon = 0 \text{ in } \Sigma_\varepsilon^\pm, \text{ and } u = 0 \text{ on } \partial\Omega \ \} \end{aligned}$$

where $[u]^\varepsilon$ is the jump of u on $\Sigma_\varepsilon^\pm$ defined by

$$[u]^\varepsilon = \pm u_{|_{\Omega_\varepsilon^\pm}} \mp u_{|_{B_\varepsilon^\pm}},$$

and

$$LD_0(\Omega) = \{u \in L^1(\Omega, \mathbb{R}^3)/e(u) \in L^1(\Omega, \mathbb{R}_s^9), \text{ and } u = 0 \text{ on } \partial\Omega\}$$

we easily show that V^ε is a Banach space with respect to the norm

$$u \to \|e(u)\|_{L^2(\Omega_\varepsilon, \mathbb{R}_s^9)} + \|e(u)\|_{L^1(B_\varepsilon, \mathbb{R}_s^9)}.$$

Our goal in this work is to study the problem (P_ε), and its limit behavior when ε tends to zero.

4. Study of Problem

The problem $\mathbb{P}_\varepsilon$ is equivalent of the minimization problem

$$inf_{v\in V^\varepsilon}\{\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk}e_{ij}(v)e_{hk}(v)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon}|e(v)| - \int_\Omega fvdx\} \tag{4.1}$$

To study problem $\mathbb{P}_\varepsilon$, we will study the minimization problem (4.1). The existence and uniqueness of solutions to (4.1) is given in the following proposition.

Proposition 4.1. *Under the hypotheses* (2.1), (2.2), (2.3) *and for $f \in L^\infty$, problem* (4.1) *admits an unique solution.*

Proof: From (2.1) and (2.3), we show easily that the energy functional in (4.1) is weakly lower semicontinuous, strictly convex and coercive over V^ε, Since V^ε is not reflexive, so we may not apply directly result given in Dacorogna [13], but we can follow our proof by using the compact imbedding of Sobolev for the $LD_0(\Omega)$ space in the reflexivity space $L^q(\Omega)$, or $q \in]1, \frac{3}{2}]$ for more information see Temam ([5] p.117).
On the other hand, let u_n be a minimizing sequence for (4.1), to simplify the writing let

$$F^\varepsilon(v) = \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk}e_{ij}(v)e_{hk}(v)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon}|e(v)| - \int_\Omega fvdx$$

so, we have $F^\varepsilon(u_n) \to \inf_{v\in V^\varepsilon} F(v)$. Using the coercivity of F^ε, we may then deduce that there exists a constant $C > 0$, independent of n, such that

$$\|u_n\|_{V^\varepsilon} \leq C,$$

then u_n bounded in L^q, therefore a subsequence of u_n, still denoted by u_n, there exists $u_0 \in V^\varepsilon$ such that $u_n \rightharpoonup u_0$ in V^ε. The weak lower semi-continuity and the strict convexity of F^ε imply then the result. □

Lemma 4.2. *Assuming that for any sequence* $(u^\varepsilon)_\varepsilon \subset V^\varepsilon$, *there exists a constant* $C > 0$ *such that* $F^\varepsilon(u^\varepsilon) \le C$, *under* (2.1), (2.3) *and for* $f \in L^\infty(\Omega, \mathbb{R}^3)$, $(u^\varepsilon)_{\varepsilon>0}$ *satisfies*

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_s)} \le C \tag{4.2}$$

$$\frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)| \le C. \tag{4.3}$$

moreover u^ε is bounded in $W_0^{1,1}(\Omega, \mathbb{R}^3)$.

Proof: Since $F^\varepsilon(u^\varepsilon) \le C$, we have

$$\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk} e_{ij}(u^\varepsilon) e_{hk}(u^\varepsilon) dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)| - \int_\Omega f u^\varepsilon dx \le C$$

according to (2.3), we have

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_s)} + \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)| \le C + C\int_\Omega f u^\varepsilon dx. \tag{4.4}$$

Then, taking advantage of the fact that u^ε vanishes on $\partial\Omega$:

$$\int_\Omega f u^\varepsilon dx = \int_{\Omega_\varepsilon} f u^\varepsilon dx + \int_{B_\varepsilon} f u^\varepsilon dx.$$

Where

$$\int_{\Omega_\varepsilon} f u^\varepsilon dx = \int_\Omega \chi_{\Omega_\varepsilon} f u^\varepsilon dx \le C\|e(u^\varepsilon)\|_{L^2_{(\Omega_\varepsilon, \mathbb{R}^9_s)}} \tag{4.5}$$

otherwise since $LD_0 \hookrightarrow L^q(\Omega, \mathbb{R}^3)$ for all $q \in [1, \frac{3}{2}]$, in particular for $q_0 = \frac{3}{2}$, we denote by q_0' the conjugate of q_0, by Hölder inequality, we obtain

$$\int_{B_\varepsilon} f u^\varepsilon \le \|f\|_{L^{q_0'}(B_\varepsilon, \mathbb{R}^3)} \|u^\varepsilon\|_{L^{q_0}(B_\varepsilon, \mathbb{R}^3)}$$

since $u^\varepsilon = 0$ on ∂B_ε, one has, according to Pioncaré's type inequality see [5],

$$\begin{aligned} \|f\|_{L^{q_0'}(B_\varepsilon, \mathbb{R}^3)} \|u^\varepsilon\|_{L^{q_0}(B_\varepsilon, \mathbb{R}^3)} &\le C(\varepsilon\varphi_\varepsilon)^{\frac{1}{q_0'}} \int_\Omega |e(u^\varepsilon)| \\ &\le C(\varepsilon\varphi_\varepsilon)^{\frac{1}{q_0'}} \left(\int_{\Omega_\varepsilon} |e(u^\varepsilon)| + \int_{B_\varepsilon} |e(u^\varepsilon)|\right) \end{aligned} \tag{4.6}$$

such as φ_ε is Y-periodic and for a small enough ε, than we have :

$$\varphi_\varepsilon < \frac{1}{\varepsilon(1+C)^{q_0'}}, \quad \text{let} \quad c_\varepsilon = \frac{C}{\varepsilon(1+C)}.$$

According to (4.4), and using (4.5), (4.6), then we obtain

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon, \mathbb{R}^9_s)} + \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)| \le C + C\|e(u^\varepsilon)\|_{L^2(\Omega_\varepsilon, \mathbb{R}^9_s)} + c_\varepsilon \int_{B_\varepsilon} |e(u^\varepsilon)|$$

Using Young inequality,

$$\leq C + \frac{1}{2}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_s)} + c_\varepsilon \int_{B_\varepsilon} |e(u^\varepsilon)|$$

so that

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_s)} + (\frac{1}{\varepsilon} - c_\varepsilon) \int_{B_\varepsilon} |e(u^\varepsilon)| \leq C$$

we obtain :

$$\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_s)} + \frac{1}{\varepsilon}(1 - \frac{C}{1+C}) \int_{B_\varepsilon} |e(u^\varepsilon)| \leq C.$$

Therefore, we will have (4.2) and (4.3). According to (4.2), (4.3) and for a small enough ε the sequence u^ε is bounded in $LD_0(\Omega)$. □

We give some lemmas that will be used in the sequel.

Lemma 4.3. *Let $g \in C^\infty(\Sigma, \mathbb{R}^9)$ and $u \in \mathcal{D}(\Sigma, \mathbb{R}^3)$, so we have*

$$\int_\Sigma \tau e(u) = -\int_\Sigma div_T(\tau) u$$

with $div_T(\tau) = div(\frac{\tau+\tau^T}{2})$.

Lemma 4.4. *Let u be a regular function defined in a neighborhood of Σ, then*

$$\delta_j(\int_0^{\varepsilon\varphi_\varepsilon} u) = \varepsilon u(x', \varepsilon\varphi_\varepsilon)\delta_j\varphi_\varepsilon + \int_0^{\varepsilon\varphi_\varepsilon} \delta_j u.$$

This lemme is a consequence of ([2] Proposition 2).

To apply the epi-convergence method, we need to characterize the topological spaces containing any cluster point of the solution of the problem (4.1) with respect to the used topology, therefore the weak topology to use is insured by the Lemma 4.2. So the topological spaces characterization is given in the following proposition.

Let us

$$w^\varepsilon = \frac{1}{2\varepsilon\varphi_\varepsilon} \int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon$$

Proposition 4.5. *The solution u^ε of the problem (4.1) possess a cluster point u^* in $LD_0(\Omega)$, with respect to the weak topology and $u^*_{|\Sigma}$ is a weak cluster point of w^ε in $LD_0(\Sigma, \mathbb{R}^3)$.*

Proof: According to a (4.2), (4.3) and for a small ε, the solution u^ε is bounded in $LD_0(\Omega)$, then It's relatively compact in $L^1(\Omega)$, this is consequence of ([5], Theorem 1.4 p.117), and $e(u^\varepsilon)$ so for a subsequences of u^ε, still denoted by u^ε, there exists there exists $u^* \in L^1(\Omega)$, such that

$$u^\varepsilon \to u^* \text{ in } L^1(\Omega),$$

we have

$$u^\varepsilon \rightharpoonup u^* \ \text{ in } LD_0(\Omega)$$

other hand

$$\begin{aligned}\int_\Sigma |w^\varepsilon - u^\varepsilon_{|\Sigma}|dx' &\le \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |u^\varepsilon(x) - u^\varepsilon(x',0)|dx_3dx' \\ &\le \frac{C}{\varepsilon}\int_\Sigma\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |\int_0^{x_3} \frac{\partial u^\varepsilon}{\partial x_3}(x',t)dt|dx_3dx' \\ &\le \frac{C}{\varepsilon}\int_\Sigma (\varepsilon\varphi_\varepsilon)^2\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |\frac{\partial u^\varepsilon}{\partial x_3}(x',x_3)|dx_3dx' \\ &\le C\varepsilon\int_{B_\varepsilon} \nabla u^\varepsilon dx \le C\varepsilon\int_\Omega e(u^\varepsilon)dx\end{aligned}$$

Thanks to Lemma 4.2 and the Young's inequality, so we have

$$\begin{aligned}\int_\Sigma |w^\varepsilon - u^\varepsilon_{|\Sigma}| &\le C\varepsilon(\int_{\Omega_\varepsilon} e(u^\varepsilon) + \int_{B_\varepsilon} e(u^\varepsilon)) \\ &\le C\varepsilon(C+C).\end{aligned}$$

Then

$$\lim_{\varepsilon\to 0}\int_\Sigma |w^\varepsilon - u^\varepsilon_{|\Sigma}| = 0$$

since $u^\varepsilon_{|\Sigma} \rightharpoonup u^*_{|\Sigma}$ in $LD_0(\Sigma)$, so $w^\varepsilon \rightharpoonup u^*_{|\Sigma}$ in $LD_0(\Sigma)$. □

Remark 4.6. *The Proposition 4.5 remains true for any weak cluster point u of a sequence u^ε in $LD_0(\Omega,\mathbb{R}^3)$ satisfies (4.2) and (4.3).*

To study the limit behavior of the solution of the problem (4.1), we will use the epi-convergence method, (see Annex, definition).

5. Limit Behavior

In this section, we are interested to the asymptotic behavior of the solution of the problem (4.1) when ε close to zero. In the sequel, we consider the following functionals

$$F_\varepsilon(v) = \begin{cases} \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk}e_{ij}(v)e_{hk}(v)dx + \frac{1}{\varepsilon}\int_{B_\varepsilon}|e(v)| & \text{if } v\in V^\varepsilon \\ +\infty & \text{if } v\notin V^\varepsilon \end{cases} \tag{5.1}$$

$$G(v) = -\int_\Omega fvdx,\ \forall v\in V^\varepsilon$$

We design by τ_f the weak topology on the space. In the sequel, we shall characterize, the epi-limit of the energy functional given by (5.1) in the following theorem :

Theorem 5.1. *Under* (2.1)*,* (2.2)*,* (2.3) *and for* $f \in L^\infty(\Omega, \mathbb{R}^3)$*, there exists a functional* $F : W^{1,1}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ *such that*

$$\tau_f - \lim_\varepsilon F^\varepsilon = F \quad \textit{in } W_0^{1,1}(\Omega)$$

where F *is given by*

$$F(u) = \begin{cases} \frac{1}{2}\int_\Omega a_{ijhk} e_{ij}(u) e_{hk}(u) dx + m(\varphi)\int_\Sigma |e^*(u_{|\Sigma})| & \textit{if } u \in W_0^{1,1}(\Omega) \\ +\infty & \textit{if } u \notin W_0^{1,1}(\Omega) \end{cases}.$$

Proof: • – (a) We are now in position to determine the upper epi-limit.
Let $u \in LD_0(\Omega)$, as $C^\infty(\Omega)$ is dense in $LD_0(\Omega)$ see ([5], p.116), so there exists a sequence (u^n) in $C^\infty(\Omega)$ such that $(u^n) \rightharpoonup u$ weakly in $LD_0(\Omega)$.
Let us consider the sequence

$$u^{\varepsilon,n} = \theta_\varepsilon(x) u^n_{|\Sigma} + (1 - \theta_\varepsilon(x)) u^n$$

where θ is a regular function satisfies :

$$\theta(x) = 1 \text{ if } |x| \le 1, \; \theta(x) = 0 \text{ if } |x| \ge 2 \text{ and } |\theta'(x)| \le 2, \; \forall x \in \mathbb{R}$$

we set

$$\theta_\varepsilon(x) = \theta(\frac{x}{\varepsilon\varphi_\varepsilon})$$

we have

$$F^\varepsilon(u^{\varepsilon,n}) = \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk} e_{ij}(u^{\varepsilon,n}) e_{hk}(u^{\varepsilon,n}) dx + \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^{\varepsilon,n})|$$

which implies that

$$\begin{aligned} F^\varepsilon(u^{\varepsilon,n}) = & \frac{1}{2}\int_{|x_3| > 2\varepsilon\varphi_\varepsilon} a_{ijhk} e_{ij}(u^{\varepsilon,n}) e_{hk}(u^{\varepsilon,n}) dx \\ & + \frac{1}{2}\int_{\varepsilon\varphi_\varepsilon < |x_3| < 2\varepsilon\varphi_\varepsilon} a_{ijhk} e_{ij}(u^{\varepsilon,n}) e_{hk}(u^{\varepsilon,n}) dx \\ & + \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^{\varepsilon,n})| \\ = & \int_{|x_3| > 2\varepsilon\varphi_\varepsilon} a_{ijhk} e_{ij}(u^{\varepsilon,n}) e_{hk}(u^{\varepsilon,n}) dx \\ & + \frac{1}{2}\int_{\varepsilon\varphi_\varepsilon < |x_3| < 2\varepsilon\varphi_\varepsilon} a_{ijhk} e_{ij}(u^{\varepsilon,n}) e_{hk}(u^{\varepsilon,n}) dx \\ & + \frac{1}{\varepsilon}\int_\Sigma \varepsilon\varphi_\varepsilon |e^*(u^{\varepsilon,n}_{|\Sigma})|. \end{aligned}$$

As φ_ε is bounded, therefore

$$\lim_{\varepsilon\to 0}\{\ \frac{1}{2}\int_{\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon} a_{ijhk}e_{ij}(u^{\varepsilon,n})e_{hk}(u^{\varepsilon,n})dx\ \} = 0.$$

Otherwise, $\varphi_\varepsilon \to m(\varphi)$ in $L^1(\Omega)$ see Annex, so by passing to the upper limit, we obtain :

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n}) &= \limsup_{\varepsilon\to 0}\{\ \frac{1}{2}\int_{|x_3|>2\varepsilon\varphi_\varepsilon} a_{ijhk}e_{ij}(u^{\varepsilon,n})e_{hk}(u^{\varepsilon,n})dx + \int_\Sigma \varphi_\varepsilon|e^*(u^{\varepsilon,n}_{|\Sigma})|\ \}\\ &= \frac{1}{2}\int_\Omega a_{ijhk}e_{ij}(u^n)e_{hk}(u^n)dx + m(\varphi)\int_\Sigma |e^*(u^n_{|\Sigma})|.\end{aligned}$$

Since $u^n \to u$ in $C^\infty(\Omega)$, there fore according to the classic result diagonalization's Lemma see [9], there exists a real function $n(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$, such that $u^{\varepsilon,n(\varepsilon)} \rightharpoonup u$ in $C^\infty(\Omega)$ when $\varepsilon \to 0$.
Consequently, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n(\varepsilon)}) &\le \limsup_{n\to 0}\limsup_{\varepsilon\to 0} F^\varepsilon(u^{\varepsilon,n})\\ \le \frac{1}{2}\int_\Omega a_{ijhk}e_{ij}(u)e_{hk}(u)dx &+ m(\varphi)\int_\Sigma |e^*(u_{|\Sigma})|\end{aligned}$$

• – (b) We are now in position to determine the lower epi-limit.
Let as $(u^\varepsilon)_\varepsilon \subset V^\varepsilon$ such as $u^\varepsilon \rightharpoonup u$ in $LD_0(\Omega)$, If $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) = +\infty$ there is nothing to prove, because

$$\frac{1}{2}\int_\Omega a_{ijhk}e_{ij}(u)e_{hk}(u)dx + m(\varphi)\int_\Sigma |e^*(u_{|\Sigma})| \le +\infty.$$

Otherwise, we suppose $\liminf_{\varepsilon\to 0} F^\varepsilon(u^\varepsilon) < +\infty$, there exists a subsequence of $F^\varepsilon(u^\varepsilon)$, still denoted by $F^\varepsilon(u^\varepsilon)$ and a constant $C > 0$, such that
$F^\varepsilon(u^\varepsilon) \le C$, which implies that

$$\begin{aligned}\|e(u^\varepsilon)\|^2_{L^2(\Omega_\varepsilon,\mathbb{R}^9_s)} &\le C\\ \frac{1}{\varepsilon}\int_{B_\varepsilon} |e(u^\varepsilon)| &\le C.\end{aligned}$$

Then $\chi_{\Omega_\varepsilon}e(u^\varepsilon)$ is bounded in $L^2(\Omega)$, so for a subsequence of $\chi_{\Omega_\varepsilon}e(u^\varepsilon)$, still denoted by $\chi_{\Omega_\varepsilon}e(u^\varepsilon)$ we then show easily, like in the proof of the above proposition, that

$$\chi_{\Omega_\varepsilon}e(u^\varepsilon) \rightharpoonup e(u) \text{ in } L^2(\Omega)$$

From the subdifferentiability's inequality of $u \to \frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk}e_{ij}(u)e_{hk}(u)dx$ and passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_{\Omega_\varepsilon} a_{ijhk}e_{ij}(u^\varepsilon)e_{hk}(u^\varepsilon)dx \ge \frac{1}{2}\int_\Omega a_{ijhk}e_{ij}(u)e_{hk}(u)dx. \tag{5.2}$$

Let us set, for $\eta < \frac{\varepsilon}{2}$

$$B_\eta = \{x \in \Omega : |x_3| < \eta\}.$$

According to the diagonalization's Lemma ([9], Lemma 1.15 p.32), there exists a function $\eta(\varepsilon) : \mathbb{R}^+ \to \mathbb{R}^+$ decreasing to 0 when $\varepsilon \to 0$ such that

$$\liminf_{\varepsilon\to 0} \int_{B^{\eta(\varepsilon)}} |e(u^\varepsilon)| \geq \liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \int_{B_\eta} |e(u^\varepsilon)| \tag{5.3}$$

since

$$\int_{B^\eta} |e(u^\varepsilon)| \geq \int_{B_\eta} \phi(e(u^\varepsilon) - e(u)) + \int_{B^\eta} \phi(e(u)) \ \forall \phi \in C_0^\infty(B_\eta, \mathbb{R}_s^9)$$

it follow that

$$\liminf_{\varepsilon\to 0} \int_{B^\eta} |e(u^\varepsilon)| \geq \int_{B^\eta} \phi(e(u)) \ \forall \phi \in C_0^\infty(B_\eta, \mathbb{R}_s^9)$$

therefore,

$$\liminf_{\varepsilon\to 0} \int_{B^\eta} |e(u^\varepsilon)| \geq \int_{B^\eta} e(u).$$

According to a Lemma 4.4, and let w^ε be the sequence define before the Proposition 4.5, we have

$$\liminf_{\varepsilon\to 0} \int_{B^\eta} |e(u^\varepsilon)| \geq \int_\Sigma e^*(w^\varepsilon) - \int_{B_\eta} \varepsilon\varphi_\varepsilon \delta\varphi_\varepsilon \otimes_s U^\varepsilon \tag{5.4}$$

where

$$U^\varepsilon = [u^\varepsilon(x', \varepsilon\varphi_\varepsilon) + u^\varepsilon(x', -\varepsilon\varphi_\varepsilon)]$$

according to the Lemma 4.3, let $g \in \mathcal{D}(\Sigma, \mathbb{R}^9)$ we have

$$\int_\Sigma g e^*(w^\varepsilon) = -\int_\Sigma div_T g\left(\frac{1}{\varepsilon\varphi_\varepsilon} \int_{\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} u^\varepsilon\right)$$

thanks to a Proposition 4.5 and $\varphi_\varepsilon \to m(\varphi)$ in $L^1(\Sigma)$(see Lemma 7.1 Annex), so passing to limit, we obtain

$$\int_\Sigma g e^*(w^\varepsilon) = -m(\varphi) \int_\Sigma div_T g u_{|\Sigma} = m(\varphi) \int_\Sigma g e^*(u_{|\Sigma}) \ \forall g \in \mathcal{D}(\Sigma, \mathbb{R}^9). \tag{5.5}$$

By passing to the limit $(\eta \to 0)$ in (5.4) we have

$$\liminf_{\eta\to 0} \liminf_{\varepsilon\to 0} \int_{B^\eta} |e(u^\varepsilon)| \geq m(\varphi) \int_\Sigma e^*(u_{|\Sigma}).$$

From the definition of B_η with (5.3), we deduce that

$$\liminf_{\varepsilon\to 0} \int_{B^\varepsilon} |e(u^\varepsilon)| \geq m(\varphi) \int_\Sigma e^*(u_{|\Sigma}) \tag{5.6}$$

after (5.2) and (5.6),

$$\liminf_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon) \geq \frac{1}{2} \int_\Omega a_{ijhk} e_{ij}(u) e_{hk}(u) dx + m(\varphi) \int_\Sigma e^*(u_{|\Sigma})$$

As, for $u \in LD_0(\Omega)$ and $u^\varepsilon \in V^\varepsilon$, such that $u^\varepsilon \rightharpoonup u$ in $LD_0(\Omega)$, Assume that

$$\liminf_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon) < +\infty.$$

So there exists a constant $C > 0$ and a subsequence of $F^\varepsilon(u^\varepsilon)$, still de noted by $F^\varepsilon(u^\varepsilon)$, such that

$$F^\varepsilon(u^\varepsilon) < C.$$

So u^ε verifies the following evaluation (4.2) and (4.3), as $u^\varepsilon \rightharpoonup u$ in $LD_0(\Omega)$ thanks to the Remark 4.1 we have $u \in \mathbb{H}_0^1(\Omega)$, what contradicts the fact that $u \in LD_0(\Omega) \setminus \mathbb{H}_0^1$, consequently we have

$$\liminf_{\varepsilon \to 0} F^\varepsilon(u^\varepsilon) = +\infty.$$

Hence the proof of the Theorem 5.1 is complete. □

In the sequel, we determine the limit problem linked to (4.1), when ε approaches to zero. Thanks to the epi-convergence results, see Annex Theorem 7.3, Proposition 7.2 and the Theorem 5.1, according to the τ_f-continuity of the functional G in $W_0^{1,1}(\Omega)$, we have $F^\varepsilon + G$ τ_f-epiconverges to $F + G$ in $LD_0(\Omega)$

Proposition 5.2. *For any $f \in L^1(\Omega, \mathbb{R}^3)$, there exists $u^* \in LD_0(\Omega, \mathbb{R}^3)$ satisfies*

$$u^\varepsilon \rightharpoonup u^* \text{ in } LD_0(\Omega, \mathbb{R}^3),$$

$$F(u^*) + G(u^*) = \inf_{u \in LD_0(\Omega)} \{F(u) + G(u)\}.$$

Proof: Thanks to Lemma 4.2, the family $(u^\varepsilon)_\varepsilon$ is bounded in $L^1(\Omega)$, therefore it passes a τ_f-cluster point u^* in $L^1(\Omega)$. And thanks to a classical epi-convergence method, Theorem 7.3, it follows that u^* is a solution of the problem : Find

$$\inf_{u \in LD_0(\Omega)} \{F(u) + G(u)\} \tag{5.7}$$

Since $F = +\infty$ on $LD_0(\Omega) \setminus \mathbb{H}_0^1(\Omega)$, so (5.6) became

$$\inf_{u \in \mathbb{H}_0^1(\Omega)} \{F(u) + G(u)\}.$$

According to the uniqueness of solutions of problem (5.6), so u^ε admits an unique τ_f-cluster point u^*, and therefore $u^\varepsilon \rightharpoonup u^*$ in $LD_0(\Omega)$ □

6. Conclusion

Using the epi-convergence method, we showed that the question of finding the limit problem, composed of a classical linear elasticity problem posed over Σ, contains an interface condition which depends on the displacement field jump. We found the same result of A. Ait Moussa and J. Messaho, with p=1 in [1].

7. Annex

Definition 7.1. *([9] Definition 1.9). Let $(\mathbb{X}, \tau)$ be a metric space, $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted $\tau - \lim_e F^\varepsilon = F$, if the following assertions are satisfied :*
• *$\forall x$ in $\mathbb{X}$, there exists x^0_ε, $x^0_\varepsilon \to^\tau x$, such that $\limsup_{\varepsilon\to 0} F^\varepsilon(x^0_\varepsilon) \leq F(x)$.*
• *$\forall x, x_\varepsilon$ with $x_\varepsilon \to^\tau x$, $\liminf_{\varepsilon\to 0} F^\varepsilon(x^0_\varepsilon) \geq F(x)$.*
We have the following stability result for epi-convergence.

Proposition 7.2. *([9] p.40)*
Suppose that F^ε epi-converge to F, in $(\mathbb{X}, \tau)$ and that $G : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, is τ-continuous. Then $(F^\varepsilon + G)$ epi-converges to $F + G$ in $(\mathbb{X}, \tau)$.

This epi-convergence is a special case of the Γ-convergence introduced by De Giorgi (1979), for more detail [9]. It is well suited to the asymptotic analysis of sequences of minimization problems since one has the following fundamental result.

Theorem 7.3. *([9] p.27)*
Suppose that :
(1) F^ε admits a minimizer on $\mathbb{X}$.
(2) The sequence u^ε is τ-relatively compact.
(3) The sequence F^ε epi-converges to F in this topology τ.
Then every cluster point u of the sequence u^ε minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon'\to 0} F^{\varepsilon'} = F(u),$$

where $(u^{\varepsilon'})_{\varepsilon'}$ denotes any subsequence of $(u^\varepsilon)_\varepsilon$ which converges to u.

Lemma 7.4. *Let $\varphi \in L^\infty(\Sigma)$, a Y-periodic, $Y =]0,1[\times]0,1[$. Let*

$$\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon}), \textit{for a small enough } \varepsilon > 0.$$

So that

$$\varphi_\varepsilon \to m(\varphi) \textit{ in } L^s(\Sigma) \textit{ for } 1 \leq s \leq \infty, \varphi_\varepsilon \rightharpoonup^* m(\varphi) \textit{ in } L^\infty(\Sigma).$$

Proof: Since φ_ε is a εY-periodic, so one has

$$\varphi_\varepsilon \rightharpoonup m(\varphi) \text{ in } L^s(\Sigma) \text{ for } 1 \leq s \leq \infty, \varphi_\varepsilon \rightharpoonup^* m(\varphi) \text{ in } L^\infty(\Sigma).$$

Since φ is bounded in Σ, so for evry $s \geq 1$, there existes a constant $C > 0$, such that

$$\int_{\Sigma} |\varphi_\varepsilon - m(\varphi)|^s \leq C \int_{\Sigma} |\varphi_\varepsilon - m(\varphi)|$$
$$\leq C[\int_{\varphi \geq m(\varphi)} (\varphi_\varepsilon - m(\varphi)) - \int_{\varphi \leq m(\varphi)} (m(\varphi) - \varphi_\varepsilon)]. \tag{7.1}$$

Passing to the limit in (7.1), one has $\varphi_\varepsilon \to m(\varphi)$ in $L^s(\Sigma)$ for $1 \leq s \leq \infty$.

□

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