Bol. Soc. Paran. Mat. (3s.) **v. 32** 2 (2014): 83–93.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i2.17522

Existence and multiplicity of solutions for class of Navier boundary p-biharmonic problem near resonance

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ABSTRACT: This paper studies the existence and multiplicity of weak solutions for the following elliptic problem
$\Delta(\rho(x)|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x)$ in Ω, $u = \Delta u = 0$ on $\partial\Omega$.
By using Ekeland's variational principle, Mountain pass theorem and saddle point theorem, the existence and multiplicity of weak solutions are established.

Key Words: p-biharmonic, resonance, Ekeland's principle, Mountain pass theorem, saddle point theorem.

Contents

1. Introduction and main results

In this article, we are concerned with the following elliptic problem of p-biharmonic type

$$\begin{cases} \Delta(\rho(x)|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u + f(x,u) + h(x) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$ is a bounded smooth domain, $p > 1$, $\rho \in C(\overline{\Omega})$ with $\inf_{\overline{\Omega}} \rho(x) > 0$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $m \in C(\overline{\Omega})$ is nonnegative weight functions.

The investigation of existence and multiplicity of solutions for problems involving p-biharmonic operator has drawn the attention of many authors, see reference. In [4], Li and Tang considered the following Navier boundary value problem

$$\begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > \max\left\{1, \frac{N}{2}\right\}$ and $\lambda, \mu \geq 0$. Under suitable assumptions the existence of at least three weak solutions is established. In [6], Ma and Pelicer study a multiplicity for the perturbed p-Laplacian equation

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x,u) + h(x) \quad \text{in } \mathbb{R}^N,$$

2000 *Mathematics Subject Classification:* 35J35, 35J40 where λ is near λ_1, the principal eigenvalue of the weighted problem

$$-\Delta_p u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

they proved the existence of one or three solutions.

In the present paper, we study problem (1.1) that result was extended to the p-biharmonic operator in bounded domains, with the weight functions. We were inspired by Ma and Pelicer [6] in which problems involving the p-laplacian operator is studied. Our technical approach is based on Ekeland's variational principle, Mountain pass theorem and saddle point theorem. We assume that f satisfies the following conditions

(F_1) There exists a real $a > 0$ and a function $b \in L^{(p^\star)'}(\Omega)$ such that

$$|f(x,t)| \leq a|t|^{\sigma-1} + b(x) \;\; a.e \; in \; \Omega \;\; \text{for all}\, t \in \mathbb{R},$$

with $1 < \sigma < p$.

(F_2) There exist $\alpha > 0$ and $\beta(x) \in L^\infty(\Omega)$ satisfying

$$pF(x,u) - f(x,u)u \geq \alpha|u|^\mu + \beta(x) \;\; a.e \; in \; \Omega \;\; \text{for all}\, u \in \mathbb{R},$$

where $1 < \mu \leq \sigma < p$ and $F(x,u) = \int_0^u f(x,s)ds$.

We introduce the space $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, which is a reflexive Banach space endowed with the norm

$$||u|| = \left(\int_\Omega \rho|\Delta u|^p dx\right)^{1/p}, \;\; \text{(see, e.g., [1, 10])}.$$

Consider the following problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Let λ_1 denote the first eigenvalue of problem (1.3). According to the work of M.Talbi and N.Tsouli [10], since $m \in C(\overline{\Omega})$ and $m \geq 0$, λ_1 is positive, simple, isolated and is given by

$$\lambda_1 = \inf\left\{\int_\Omega \rho|\Delta u|^p dx : u \in X, \int_\Omega m(x)|u|^p dx = 1\right\}. \tag{1.4}$$

Therefore

$$\int_\Omega \rho|\Delta u|^p dx \geq \lambda_1 \int_\Omega m(x)|u|^p dx \;\text{ for all}\, u \in X. \tag{1.5}$$

Let φ_1 normalized eigenfunction associated to λ_1, which can be chosen positive. Let

$$\lambda_2 := \inf\left\{\lambda : \lambda \text{ is an eigenvalue of (1.3) with } \lambda > \lambda_1\right\}. \tag{1.6}$$

The fact that λ_1 is isolated implies that $\lambda_1 < \lambda_2$. It can also be shown (see Lemma 2.1) that there exists $\overline{\lambda} \in (\lambda_1, \lambda_2]$ such that

$$\int_\Omega \rho|\Delta u|^p dx \geq \overline{\lambda} \int_\Omega m(x)|u|^p dx, \tag{1.7}$$

for all $u \in X$ with $\int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 u dx = 0$.

Definition 1.1. *We say that $u \in X$ is a weak solution of problem (1.1) if*

$$\int_\Omega \rho|\Delta u|^{p-2}\Delta u \Delta\varphi dx - \int_\Omega m(x)|u|^{p-2}u\varphi dx - \int_\Omega f(x,u)\varphi dx - \int_\Omega h(x)\varphi dx = 0,$$

for all $\varphi \in X$.

The corresponding energy functional of problem (1.1) is given by

$$I(u) = \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda}{p}\int_\Omega m(x)|u|^p dx - \int_\Omega F(x,u)dx - \int_\Omega h(x)u dx, \tag{1.8}$$

it is well known that $I \in \mathcal{C}^1(X, \mathbb{R})$, with derivative at point $u \in X$ is given by

$$\langle I'(u), \varphi\rangle = \int_\Omega \rho|\Delta u|^{p-2}\Delta u \Delta\varphi dx - \lambda\int_\Omega m|u|^{p-2}u\varphi dx - \int_\Omega f(x,u)\varphi dx - \int_\Omega h\varphi dx,$$

for every $\varphi \in X$. Consequently, the critical points of the functional I correspond to the weak solutions of the problem (1.1).

Let here recall the weak version of Mountain pass theorem (see [2], [3]) and the saddle point theorem (see [7]).

Theorem 1.2. *let X be a real Banach space and $I : X \to \mathbb{R}$ be a C^1 functional satisfying the Palais-Smale condition. Furthermore assume that $I(0) = 0$ and that the following conditions hold:*
(i) there exits a number $r > 0$ such that $I|_{\partial B_r} \geq 0$
(ii) there is an element $e \in X \backslash \overline{B_r}$ with $I(e) \leq 0$.
Then the real number c, characterized as

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$$

where

$$\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}$$

is a critical value of I with $c \geq 0$. If $c = 0$, there exists a critical point of I on ∂B_r corresponding to the critical value 0.

Theorem 1.3. *Let X be a Banach space. Let $I : X \to \mathbb{R}$ be a C^1 functional that satisfies the Palais-Smale condition, and suppose that $X = V \oplus W$, with V a finite dimensional subspace of X. If there exists $R > 0$ such that*

$$\max_{v\in V, ||v||=R} I(v) < \inf_{w\in W} I(w),$$

then I has a least a critical point on X.

Now we are ready to state our main result.

Theorem 1.4. *Assume that* (F_1) *holds. If in addition*

$$\lim_{|t|\to\infty} F(x, t\varphi_1) = +\infty, \quad \text{uniformly in } x \in \Omega, \tag{1.9}$$

then for any $h \in L^{(p^\star)'}(\Omega)$, *with* $(p^\star)' = \frac{p^\star}{p^\star - 1}$, *satisfying*

$$\int_\Omega h(x)\varphi_1 dx = 0, \tag{1.10}$$

problem (1.1) has at least three solutions when λ *is sufficiently close to* λ_1 *from left.*

Theorem 1.5. *Assume that* (F_1) *and* (F_2) *hold. If in addition* $\lambda_1 \leq \lambda < \overline{\lambda}$, *then for any* $h \in L^{(p^\star)'}$, *problem (1.1) has at least one solution.*

2. Preliminaries and proofs of Theorems

Let denote $V = \langle \varphi_1 \rangle$ the linear spans of φ_1 and

$$W = \left\{ u \in X : \int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 u dx = 0 \right\}. \tag{2.1}$$

Then we can decompose X as a direct sum of V and W. In fact, let $u \in X$, writing

$$u = \alpha\varphi_1 + w,$$

where $w \in X$, and $\alpha = \lambda_1 \int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 u dx$.
Since

$$\int_\Omega \rho|\Delta\varphi_1|^p dx = 1,$$

$$\int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 w dx = 0.$$

Therefore $w \in W$, hence

$$X = V \oplus W.$$

We begin by establishing the existence of $\overline{\lambda}$ for which (1.7) holds.

Lemma 2.1. *There exists* $\overline{\lambda} \in (\lambda_1, \lambda_2]$ *such that*

$$\int_\Omega \rho|\Delta u|^p dx \geq \overline{\lambda} \int_\Omega m(x)|u|^p dx, \tag{2.2}$$

for all $u \in W$.

Proof: Let

$$\lambda = \inf \left\{ \int_\Omega \rho |\Delta u|^p dx : u \in W, \int_\Omega m(x)|u|^p dx = 1 \right\}.$$

This value is attained in W. To see why this is so, let (u_n) be a sequence in W, satisfying $\int_\Omega m(x)|u_n|^p dx = 1$ for all n, and $\int_\Omega \rho|\Delta u_n|^p dx \to \lambda$. It follows that (u_n) is bounded in X and therefore, up to subsequence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } X \quad \text{and} \quad u_n \to u \text{ strongly in } L^p(\Omega).$$

From the strong convergence of the sequence in $L^p(\Omega)$ we obtain

$$\int_\Omega m(x)|u|^p dx = \lim_{n\to\infty} \int_\Omega m(x)|u_n|^p dx = 1$$

and

$$\int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 u dx = \lim_{n\to\infty} \int_\Omega m(x)|\varphi_1|^{p-2}\varphi_1 u_n dx = 0,$$

so that $u \in W$. By the weakly lower semicontinuity of the norm $||.||$, we get

$$\lambda \leq \int_\Omega \rho|\Delta u|^p dx \leq \liminf_{n\to\infty} \int_\Omega \rho|\Delta u_n|^p dx = \lambda,$$

and hence λ is attained at u.

Now we claim that $\lambda > \lambda_1$. It follows from (1.4) that $\lambda \geq \lambda_1$. If $\lambda = \lambda_1$, by simplicity of λ_1 there is $\alpha \in \mathbb{R}$ such that $u = \alpha\varphi_1$. Since $u \in W$,

$$\alpha \int_\Omega m(x)|\varphi_1|^p dx = 0,$$

which implies $\alpha = 0$. This contradicts the fact that $\int_\Omega m(x)|u|^p dx = 1$. So, choose $\overline{\lambda} = \min\{\lambda, \lambda_2\}$. It is clear that $\overline{\lambda}$ satisfies (2.2) and the proof of lemma is complete. □

Lemma 2.2. *Assume that $(F1)$ holds. Then, for $\lambda < \lambda_1$ the functional I is coercive in X, and bounded from below on W. Moreover there exists a constant m independent of λ such that $\inf_W I(u) \geq m$.*

Proof: From $(F1)$, we have

$$\int_\Omega |F(x,u)|dx \leq \frac{a}{\sigma} \int_\Omega |u|^\sigma dx + \int_\Omega b(x)|u|dx$$

By Hölder's and Sobolev's inequalities, it follows from (1.5) that

$$\begin{aligned} I(u) &\geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda}{p}\int_\Omega m(x)|u|^p dx \\ &\quad -\frac{a}{\sigma}\int_\Omega |u|^\sigma dx - \int_\Omega b(x)|u|dx - \int_\Omega h(x)u dx \\ &\geq \frac{1}{p}||u||^p - \frac{\lambda}{p\lambda_1}||u||^p - C_1||u||^\sigma - C_2||b||_{(p^\star)'}||u|| - C_3||h||_{(p^\star)'}||u|| \\ &= \frac{1}{p}\left(1-\frac{\lambda}{\lambda_1}\right)||u||^p - C_1||u||^\sigma - C_2||b||_{(p^\star)'}||u|| - C_3||h||_{(p^\star)'}||u||, \qquad (2.3) \end{aligned}$$

where C_1, C_2 and C_3 are the embedding constants of Sobolev. Since $\lambda < \lambda_1$ and $\sigma < p$, I is coercive.

Similarly, let $u \in W$, by Lemma 2.1, for $\lambda < \lambda_1$, we have

$$\begin{aligned} I(u) &\geq \frac{1}{p}\left(1-\frac{\lambda}{\overline{\overline{\lambda}}}\right)||u||^p - C_1||u||^\sigma - C_2||b||_{(p^\star)'}||u|| - C_3||h||_{(p^\star)'}||u|| \\ &\geq \frac{1}{p}\left(1-\frac{\lambda_1}{\overline{\lambda}}\right)||u||^p - C_1||u||^\sigma - C_2||b||_{(p^\star)'}||u|| - C_3||h||_{(p^\star)'}||u|| \qquad (2.4) \end{aligned}$$

Hence I is bounded from below on W. Moreover, we can find a constant m independent of λ such that $\inf_W I(u) \geq m$.

□

Lemma 2.3. *Assume that* $(F1)$ *and (1.9) hold. Then, for* $\lambda < \lambda_1$ *sufficiently close to* λ_1*, there exist* $t^- < 0 < t^+$ *such that*

$$I(t^+\varphi_1) < m \quad \textit{and} \quad I(t^-\varphi_1) < m,$$

where m *is given by Lemma 2.2.*

Proof: By definition of λ_1 and (1.10), we have

$$\begin{aligned} I(t\varphi_1) &= \frac{|t|^p}{p}\int_\Omega \rho|\Delta\varphi_1|^p dx - \lambda\frac{|t|^p}{p}\int_\Omega m(x)|\varphi_1|^p dx \\ &\quad -\int_\Omega F(x,t\varphi_1)dx - t\int_\Omega h(x)\varphi_1 dx \\ &= \frac{|t|^p}{p}\int_\Omega \rho|\Delta\varphi_1|^p dx - \frac{\lambda|t|^p}{p\lambda_1}\int_\Omega \rho|\Delta\varphi_1|^p dx - \int_\Omega F(x,t\varphi_1)dx \\ &= \frac{|t|^p}{p}\left(1-\frac{\lambda}{\lambda_1}\right) - \int_\Omega F(x,t\varphi_1)dx. \qquad (2.5) \end{aligned}$$

From (1.9), for $t > 0$ large enough, we have

$$F(x, t^+\varphi_1) \geq 0, \quad \text{a.e. } x \in \Omega,$$

by Fatou's Lemma, we get

$$\begin{aligned}\liminf_{t\to+\infty}\int_\Omega F(x,t^+\varphi_1)dx &\geq \int_\Omega \liminf_{t\to+\infty} F(x,t^+\varphi_1)dx\\ &= \int_\Omega \lim_{t\to+\infty} F(x,t^+\varphi_1)dx\\ &= +\infty,\end{aligned}$$

so, there exists $t^+>0$ such that

$$\int_\Omega F(x,t^+\varphi_1)dx > -m+1. \tag{2.6}$$

For $\lambda_1 - \frac{p\lambda_1}{(t^+)^p} < \lambda < \lambda_1$, (2.5) and (2.6) imply

$$I(t^+\varphi_1) < m.$$

Similarly, we get $I(t^-\varphi_1) < m$, for some $t^- < 0$. □

Proof: (**Theorem 1.4**) First we show that I satisfies the (PS) condition in X, that is for every sequence such that

$$I(u_n)\to c, \quad I'(u_n)\to 0, \tag{2.7}$$

possesses a convergent subsequence.

Let $(u_n)\subset X$ be a (PS) sequence. Since I is coercive, (u_n) is bounded in X, so up to subsequence, we may assume that $u_n \rightharpoonup u$ weakly in X. Therefore

$$\langle I'(u_n), u_n-u\rangle = o_n(1). \tag{2.8}$$

By Hölder's inequality, we have

$$\left|\int_\Omega m(x)|u_n|^{p-2}u_n(u_n-u)dx\right| \leq ||m||_\infty \left(\int_\Omega |u_n|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |u_n-u|^p dx\right)^{\frac{1}{p}}. \tag{2.9}$$

Since $u_n\to u$ in $L^p(\Omega)$,

$$\lim_{n\to\infty}\int_\Omega m(x)|u_n|^{p-2}u_n(u_n-u)dx = 0. \tag{2.10}$$

Since $(\sigma-1)(p^\star)' < p^\star$, $u_n\to u$ strongly in $L^{(\sigma-1)(p^\star)'}(\Omega)$, and hence there exists $g\in L^{(\sigma-1)(p^\star)'}(\Omega)$ such that

$$|u_n|\leq g \quad \text{a.e. in } \Omega.$$

Thus

$$\begin{aligned}|f(x,u_n)|^{(p^\star)'} &\leq 2^{(p^\star)'}\left(a^{(p^\star)'}|u_n|^{(\sigma-1)(p^\star)'} + |b(x)|^{(p^\star)'}\right)\\ &\leq 2^{(p^\star)'}\left(a^{(p^\star)'}g^{(\sigma-1)(p^\star)'} + |b(x)|^{(p^\star)'}\right).\end{aligned}$$

Since the right side of the last inequality belongs to $L^1(\Omega)$, it follows from Lebesgue theorem that

$$f(x,u_n) \to f(x,u) \quad \text{in } L^{(p^\star)'}(\Omega).$$

By using the fact that $u_n \rightharpoonup u$ in $L^{p^\star}(\Omega)$, we deduce that

$$\lim_{n\to\infty} \int_\Omega (f(x,u_n)+h)(u_n-u)dx = 0. \tag{2.11}$$

Combining (2.8), (2.10) and (2.11) we obtain

$$\lim_{n\to\infty} \int_\Omega \rho|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n-u)dx = 0.$$

In the same way, we obtain

$$\lim_{n\to\infty} \int_\Omega \rho|\Delta u|^{p-2}\Delta u \Delta(u_n-u)dx = 0.$$

Therefore, the Hölder inequality imply that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_\Omega (\rho|\Delta u_n|^{p-2}\Delta u_n - \rho|\Delta u|^{p-2}\Delta u)\Delta(u_n-u)dx \\
&\geq \lim_{n\to\infty} \Bigg[||u_n||^p - \left(\int_\Omega \rho|\Delta u_n|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \rho|\Delta u|^p dx\right)^{1/p} \\
&- \left(\int_\Omega \rho|\Delta u|^p dx\right)^{\frac{p-1}{p}} \left(\int_\Omega \rho|\Delta u_n|^p dx\right)^{1/p} + ||u||^p\Bigg] \\
&= \lim_{n\to\infty} \left[||u_n||^p - ||u_n||^{p-1}||u|| - ||u||^{p-1}||u_n|| + ||u||^p\right] \\
&= \lim_{n\to\infty} (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \geq 0,
\end{aligned}$$

hence $||u_n|| \to ||u||$. By the uniform convexity of X, it follows that $u_n \to u$ strongly in X and I satisfies the (PS) condition.

Next, let

$$\Lambda^\pm = \{u \in X : u = \pm t\varphi_1 + w,\ t > 0,\ w \in W\}. \tag{2.12}$$

Let $(u_n) \subset \Lambda^+$ such that $I(u_n) \to c < m$ and $I'(u_n) \to 0$ as $n \to \infty$. Then $u_n \to u$ strongly in X. Noting that $\partial\Lambda^+ = W$. So, if $u \in \partial\Lambda^+$, it follows from $\inf_W I \geq m$ that

$$I(u_n) \to c = I(u) \geq m,$$

which is impossible. Therefore $u \in \Lambda^+$, and hence I satisfies the $(PS)_{c,\Lambda^+}$ for all $c < m$. Similarly, I satisfies the $(PS)_{c,\Lambda^-}$ for all $c < m$.

In view of Lemma 2.3 for $\lambda < \lambda_1$ sufficiently close to λ_1 , we have

$$-\infty < \inf_{\Lambda^+} I < m. \tag{2.13}$$

By Ekeland's variational principle in $\overline{\Lambda^+}$, there exists a sequence $(u_n) \subset \Lambda^+$ such that

$$I(u_n) \to \inf_{\Lambda^+} I \quad \text{and} \quad I'(u_n) \to 0.$$

Since I satisfies the $(PS)_{c,\Lambda^+}$ for all $c < m$, there exists $u^+ \in \Lambda^+$ such that $I(u^+) = \inf_{\Lambda^+} I$. Similarly, we find $u^- \in \Lambda^-$ such that $I(u^-) = \inf_{\Lambda^-} I$. Hence I has two distinct critical points u^+ and u^-.

Now, we prove the existence of the third solution. To fix ideas, suppose that $I(u^+) \le I(u^-)$ and Putting

$$J(u) := I(u + u^-) - I(u^-), \quad e = u^+ - u^-.$$

So, $J(0) = 0,\ J(e) \le 0$. We can find $r > 0$ such that $\overline{B(u^-, r)} \subset \Lambda^-$, thus $\inf\limits_{||u-u^-||=r} I(u) \ge I(u^-)$ and hence $\inf\limits_{||u||=r} J(u) \ge 0$. Let

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.14}$$

where

$$\Gamma = \left\{\gamma \in C([0,1], X) : \gamma(0) = u^-,\ \gamma(1) = u^+\right\}.$$

Since J also satisfies the (PS) condition and $J' = I'$, it follows from the Mountain pass theorem 1.2 that c is a critical value of I. Noting that all paths joining u^- to u^+ pass through W, so $c \ge m$. Therefore the third solution is obtained, and the proof of theorem is complete. □

Proof: (**Theorem 1.5**) The proof will be divided in some steps.
Step 1 (the growth of F).
We prove that for some $C_1,\ C_2 > 0$,

$$\int_\Omega F(x, t\varphi_1)dx \ge C_1 ||t\varphi_1||^\mu - C_2. \tag{2.15}$$

From (F_2), we have

$$\frac{d}{du}\left(\frac{F(x,u)}{|u|^p}\right) \le -\alpha |u|^{\mu-p-2}u - \beta(x)|u|^{-p-2}u, \ \ (u > 0).$$

Noting that $\frac{F(x,u)}{|u|^p} \to 0$ as $u \to \infty$, thus after integration from $u > 0$ to $+\infty$, we see that

$$F(x,u) \ge \frac{\alpha}{p-\mu}|u|^\mu + \frac{\beta(x)}{p}$$

Since this inequality holds for $u < 0$, we get

$$\begin{aligned}
\int_\Omega F(x, t\varphi_1)dx &\ge \frac{\alpha |t|^\mu}{p-\mu}\int_\Omega |\varphi_1|^\mu dx + \frac{1}{p}\int_\Omega \beta(x)dx \\
&\ge \frac{\alpha |t|^\mu}{p-\mu}\int_\Omega |\varphi_1|^\mu dx - \frac{1}{p}||\beta||_\infty |\Omega| \\
&\ge C_1|t|^\mu - C_2
\end{aligned}$$

and (2.15) follows.

Step 2 (the Palais-Smale condition). Let (u_n) be a sequence satisfying (2.7), we note that

$$\begin{aligned}\langle I'(u_n), u_n\rangle - pI(u_n) &= \int_\Omega pF(x,u_n)dx - \int_\Omega f(x,u_n)u_n dx + (p-1)\int_\Omega hu_n dx \\ &\geq \alpha\int_\Omega |u_n|^\mu dx + \int_\Omega \beta(x)dx + (p-1)\int_\Omega hu_n dx \\ &\geq \alpha C_3||u_n||^\mu - C_4||h||_{L^{(p^\star)'}}||u_n|| + C_5. \qquad (2.16)\end{aligned}$$

From the boundedness of $\langle I'(u_n), u_n\rangle - pI(u_n)$, we deduce that (u_n) is bounded in X. By a similar argument as in the proof of Theorem 1.4, we conclude that (u_n) possesses a convergent subsequence in X.

Step 3 (the saddle point theorem). Using again Lemma 2.1, we get

$$\begin{aligned}I(u) &\geq \frac{1}{p}\int_\Omega \rho|\Delta u|^p dx - \frac{\lambda}{p}\int_\Omega m(x)|u|^p dx \\ &\quad -\frac{a}{\sigma}\int_\Omega |u|^\sigma dx - \int_\Omega b(x)|u|dx - \int_\Omega h(x)udx \\ &\geq \frac{1}{p}\left(1-\frac{\lambda}{\overline{\lambda}}\right)||u||^p - C_1||u||^\sigma - C_2||b||_{(p^\star)'}||u|| - C_3||h||_{(p^\star)'}||u||.\end{aligned}$$

Since $\lambda < \overline{\lambda}$,

$$\inf_{w\in W} I(w) > -\infty. \qquad (2.17)$$

On the other hand, by (2.15) we see that

$$I(t\varphi_1) \leq -\left(\frac{\lambda-\lambda_1}{p\lambda_1}\right)||t\varphi_1||^p - C_1||t\varphi_1||^\mu + C||h||_{(p^\star)'}||t\varphi_1|| + C_2.$$

It follows from $\lambda \geq \lambda_1$ and $1 < \mu < p$ that

$$\lim_{v\in V, ||v||\to\infty} I(v) = -\infty. \qquad (2.18)$$

By (2.17) and (2.18), there exists $R > 0$ such that

$$\max_{v\in V, ||v||=R} I(v) < \inf_{w\in W} I(w).$$

Hence, I satisfies the hypotheses of Theorem 1.3, and there exists a critical point of I, that is a solution of (1.1). □

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