



Biharmonic Constant Π_1 –Slope Curves according to Type-2 Bishop Frame in Heisenberg Group Heis^3

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ABSTRACT: In this paper, we introduce constant Π_1 – slope curves according to type-2 Bishop frame in the Heisenberg group Heis^3 . We characterize the biharmonic constant Π_1 – slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group Heis^3 . Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, type-2 Bishop frame, Heisenberg group.

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1. Introduction

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 dv_h,$$

where $\mathcal{T}(\phi) := \text{tr} \nabla^\phi d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr} R(\mathcal{T}(\phi), d\phi) d\phi,$$

and called the bitension field of ϕ . Non-harmonic biharmonic maps are called proper biharmonic maps, [5,6,7].

This study is organised as follows: Firstly, we introduce constant Π_1 – slope curves according to type-2 Bishop frame in the Heisenberg group Heis^3 . We characterize the biharmonic constant Π_1 – slope curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Heisenberg group Heis^3 . Additionally, we illustrate our main theorem.

2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2}\bar{x}y + \frac{1}{2}x\bar{y}) \quad (2.1)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}, \quad (2.2)$$

for which we have the Lie products

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

We obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1} \mathbf{e}_1 &= \nabla_{\mathbf{e}_2} \mathbf{e}_2 = \nabla_{\mathbf{e}_3} \mathbf{e}_3 = 0, \\ \nabla_{\mathbf{e}_1} \mathbf{e}_2 &= -\nabla_{\mathbf{e}_2} \mathbf{e}_1 = \frac{1}{2} \mathbf{e}_3, \\ \nabla_{\mathbf{e}_1} \mathbf{e}_3 &= \nabla_{\mathbf{e}_3} \mathbf{e}_1 = -\frac{1}{2} \mathbf{e}_2, \\ \nabla_{\mathbf{e}_2} \mathbf{e}_3 &= \nabla_{\mathbf{e}_3} \mathbf{e}_2 = \frac{1}{2} \mathbf{e}_1. \end{aligned}$$

3. Biharmonic Constant Π_1 -Slope Curves according to New Type-2 Bishop Frame in Heisenberg Group Heis^3

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along γ . Then, the Frenet frame satisfies the following Frenet-Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= -\tau \mathbf{N}, \end{aligned} \quad (3.1)$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \quad g(\mathbf{N}, \mathbf{N}) = 1, \quad g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T},\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}g(\mathbf{T}, \mathbf{T}) &= 1, \quad g(\mathbf{M}_1, \mathbf{M}_1) = 1, \quad g(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0.\end{aligned}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\mathfrak{U}(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \mathfrak{U}'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned}k_1 &= \kappa(s) \cos \mathfrak{U}(s), \\ k_2 &= \kappa(s) \sin \mathfrak{U}(s).\end{aligned}$$

Theorem 3.1. $\gamma : I \longrightarrow \text{Heis}^3$ is a unit speed biharmonic curve with Bishop frame if and only if

$$\begin{aligned}k_1^2 + k_2^2 &= \text{constant} = C \neq 0, \\ k_1'' - Ck_1 &= k_1 \left[\frac{1}{4} - (M_2^3)^2 \right] - k_2 M_1^3 M_2^3, \\ k_2'' - Ck_2 &= k_1 M_1^3 M_2^3 + k_2 \left[\frac{1}{4} - (M_1^3)^2 \right].\end{aligned}\tag{3.3}$$

Let γ be a unit speed regular curve in Heis^3 and (3.1) be its Frenet–Serret frame. Let us express a relatively parallel adapted frame:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{\Pi}_1 &= -\epsilon_1\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{\Pi}_2 &= -\epsilon_2\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \epsilon_1\mathbf{\Pi}_1 + \epsilon_2\mathbf{\Pi}_2,\end{aligned}\tag{3.4}$$

where

$$\begin{aligned}g(\mathbf{B}, \mathbf{B}) &= 1, \quad g(\mathbf{\Pi}_1, \mathbf{\Pi}_1) = 1, \quad g(\mathbf{\Pi}_2, \mathbf{\Pi}_2) = 1, \\ g(\mathbf{B}, \mathbf{\Pi}_1) &= g(\mathbf{B}, \mathbf{\Pi}_2) = g(\mathbf{\Pi}_1, \mathbf{\Pi}_2) = 0.\end{aligned}$$

We shall call this frame as Type-2 Bishop Frame. In order to investigate this new frame's relation with Frenet–Serret frame, first we write

$$\tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}.\tag{3.5}$$

The relation matrix between Frenet–Serret and type-2 Bishop frames can be expressed

$$\begin{aligned}\mathbf{T} &= \sin \mathfrak{A}(s) \mathbf{\Pi}_1 - \cos \mathfrak{A}(s) \mathbf{\Pi}_2, \\ \mathbf{N} &= \cos \mathfrak{A}(s) \mathbf{\Pi}_1 + \sin \mathfrak{A}(s) \mathbf{\Pi}_2, \\ \mathbf{B} &= \mathbf{B}.\end{aligned}$$

So by (3.5), we may express

$$\begin{aligned}\epsilon_1 &= -\tau \cos \mathfrak{A}(s), \\ \epsilon_2 &= -\tau \sin \mathfrak{A}(s).\end{aligned}$$

By this way, we conclude

$$\mathfrak{A}(s) = \arctan \frac{\epsilon_2}{\epsilon_1}.$$

The frame $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}\}$ is properly oriented, and τ and $\mathfrak{A}(s) = \int_0^s \kappa(s) ds$ are polar coordinates for the curve γ . We shall call the set $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{B}, \epsilon_1, \epsilon_2\}$ as type-2 Bishop invariants of the curve γ , [19].

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned}\mathbf{\Pi}_1 &= \pi_1^1 \mathbf{e}_1 + \pi_1^2 \mathbf{e}_2 + \pi_1^3 \mathbf{e}_3, \\ \mathbf{\Pi}_2 &= \pi_2^1 \mathbf{e}_1 + \pi_2^2 \mathbf{e}_2 + \pi_2^3 \mathbf{e}_3. \\ \mathbf{B} &= B^1 \mathbf{e}_1 + B^2 \mathbf{e}_2 + B^3 \mathbf{e}_3,\end{aligned}\tag{3.6}$$

Theorem 3.2. *Let $\gamma : I \rightarrow \text{Heis}^3$ be a unit speed non-geodesic biharmonic constant $\mathbf{\Pi}_1$ -slope curves according to type-2 Bishop frame in the Heis^3 . Then, the parametric equations of γ*

$$\begin{aligned}x(s) &= \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\ &\quad - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_3,\end{aligned}$$

$$\begin{aligned}y(s) &= \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\ &\quad + \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_4,\end{aligned}$$

$$\begin{aligned}
z(s) = & \frac{\sin^2 \mathfrak{A}}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} [-\mathcal{L}_1(\mathcal{L}_1 - \kappa)\sin[2\kappa s] + \kappa((-\mathcal{L}_1 + \kappa) \\
& \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \sin \mathfrak{A} \cos \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((\mathcal{L}_1 - \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)])] \\
& - \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \sin^2 \mathfrak{A} [-\mathcal{L}_1(\mathcal{L}_1 - \kappa)\sin[2\kappa s] + \kappa((-\mathcal{L}_1 \\
& + \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& + \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \sin \mathfrak{A} \cos \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((\mathcal{L}_1 - \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)])] \\
& - \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \cos \mathfrak{A} \sin \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((-\mathcal{L}_1 + \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)])] \\
& + \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \cos^2 \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] - \kappa((\mathcal{L}_1 \\
& - \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(-2\mathcal{L}_1 s + 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \cos \mathfrak{A} \sin \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((-\mathcal{L}_1 + \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)])] \\
& + \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \cos^2 \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] - \kappa((\mathcal{L}_1 \\
& - \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(-2\mathcal{L}_1 s + 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& + \mathcal{L}_3 \sin \mathfrak{A} \frac{\sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \mathcal{L}_3 \sin \mathfrak{A} \frac{\sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\
& + \mathcal{L}_3 \cos \mathfrak{A} \frac{\cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \mathcal{L}_3 \cos \mathfrak{A} \frac{\cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\
& - \frac{1}{\kappa} \cos[\kappa s] \cos \mathfrak{A} + \frac{1}{\kappa} \sin[\kappa s] \sin \mathfrak{A} + \mathcal{L}_4,
\end{aligned}$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ are constants of integration.

Proof: The vector Π_1 is a unit vector, we have the following equation

$$\Pi_1 = \sin \mathfrak{A} \cos[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_1 + \sin \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_2 + \cos \mathfrak{A} \mathbf{e}_3, \quad (3.7)$$

where $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{R}$.

Then by type-2 Bishop formulas (3.4) and (2.1), we have

$$\Pi_2 = \cos \mathfrak{A} \cos[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_1 + \cos \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_2 - \sin \mathfrak{A} \mathbf{e}_3.$$

Applying above equation and (3.9), we get

$$\mathbf{B} = -\sin[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_1 + \cos[\mathcal{L}_1 s + \mathcal{L}_2] \mathbf{e}_2.$$

Then, a combination of these equations with the second equation of (5.1) would give us

$$\begin{aligned} \mathbf{T} = & [\sin[\kappa s] \sin \mathfrak{A} \cos[\mathcal{L}_1 s + \mathcal{L}_2] - \cos[\kappa s] \cos \mathfrak{A} \cos[\mathcal{L}_1 s + \mathcal{L}_2]] \mathbf{e}_1 \\ & + [\sin[\kappa s] \sin \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2] - \cos[\kappa s] \cos \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2]] \mathbf{e}_2 \\ & + [\sin[\kappa s] \cos \mathfrak{A} + \cos[\kappa s] \sin \mathfrak{A}] \mathbf{e}_3. \end{aligned} \quad (3.8)$$

From (2.2) and (3.8), we have

$$\begin{aligned} x(s) = & \frac{\sin \mathfrak{A} \cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\ & - \frac{\cos \mathfrak{A} \sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\cos \mathfrak{A} \sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_3, \end{aligned}$$

where $\mathcal{L}_3$ is constant of integration.

Also,

$$\begin{aligned} y(s) = & \frac{\cos \mathfrak{A} \cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\ & + \frac{\sin \mathfrak{A} \sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_4, \end{aligned}$$

where $\mathcal{L}_4$ is constant of integration.

Again, by combining (2.2) and (3.8) we have

$$\begin{aligned} \frac{dz}{ds} = & \frac{1}{2(\mathcal{L}_1 - \kappa)} \sin^2 \mathfrak{A} \cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2] \sin[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & - \frac{1}{2(\mathcal{L}_1 - \kappa)} \sin \mathfrak{A} \cos \mathfrak{A} \cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2] \cos[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & - \frac{1}{2(\mathcal{L}_1 + \kappa)} \sin^2 \mathfrak{A} \cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2] \sin[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & + \frac{1}{2(\mathcal{L}_1 + \kappa)} \sin \mathfrak{A} \cos \mathfrak{A} \cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2] \cos[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & - \frac{1}{2(\mathcal{L}_1 - \kappa)} \cos \mathfrak{A} \sin \mathfrak{A} \sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2] \sin[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & + \frac{1}{2(\mathcal{L}_1 - \kappa)} \cos^2 \mathfrak{A} \sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2] \cos[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & - \frac{1}{2(\mathcal{L}_1 + \kappa)} \cos \mathfrak{A} \sin \mathfrak{A} \sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2] \sin[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & + \frac{1}{2(\mathcal{L}_1 + \kappa)} \cos^2 \mathfrak{A} \sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2] \cos[\kappa s] \sin[\mathcal{L}_1 s + \mathcal{L}_2] \\ & + \mathcal{L}_3 [\sin[\kappa s] \sin \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2] - \cos[\kappa s] \cos \mathfrak{A} \sin[\mathcal{L}_1 s + \mathcal{L}_2]] \\ & + [\sin[\kappa s] \cos \mathfrak{A} + \cos[\kappa s] \sin \mathfrak{A}]. \end{aligned}$$

Integrating both sides, we have theorem. Thus, the proof of theorem is completed. $\square$

Theorem 3.3. *Let $\gamma : I \rightarrow \text{Heis}^3$ be a unit speed non-geodesic biharmonic constant Π_1 -slope curves according to type-2 Bishop frame in the Heis^3 . Then, the position vector of γ is*

$$\begin{aligned}
\gamma(s) = & \left[\frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \right. \\
& - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_3 \Big] \mathbf{e}_1 \\
& + \left[\frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \right. \\
& + \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_4 \Big] \mathbf{e}_2 \\
& + \left[- \left[\frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \frac{\sin \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \right. \right. \\
& - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\cos \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_3 \Big] \\
& \left[\frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \frac{\cos \mathfrak{A} \cos [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \right. \\
& + \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{R}_1 - \kappa)} - \frac{\sin \mathfrak{A} \sin [(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} + \mathcal{L}_4 \Big] \\
& + \frac{\sin^2 \mathfrak{A}}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} [-\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] + \kappa((-\mathcal{L}_1 + \kappa) \\
& \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \sin \mathfrak{A} \cos \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((\mathcal{L}_1 - \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \sin^2 \mathfrak{A} [-\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] + \kappa((-\mathcal{L}_1 \\
& + \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(2\mathcal{L}_1 s - 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& + \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \sin \mathfrak{A} \cos \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((\mathcal{L}_1 - \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& - \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \cos \mathfrak{A} \sin \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((-\mathcal{L}_1 + \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& + \frac{1}{2(\mathcal{L}_1 - \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \cos^2 \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] - \kappa((\mathcal{L}_1 \\
& - \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(-2\mathcal{L}_1 s + 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(-\mathcal{L}_1 + \kappa)\kappa} \cos \mathfrak{A} \sin \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \cos[2\kappa s] \\
& + \kappa((-\mathcal{L}_1 + \kappa) \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1 \cos[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)])] \\
& + \frac{1}{2(\mathcal{L}_1 + \kappa)8\mathcal{L}_1(\mathcal{L}_1 - \kappa)\kappa} \cos^2 \mathfrak{A} [\mathcal{L}_1(\mathcal{L}_1 - \kappa) \sin[2\kappa s] - \kappa((\mathcal{L}_1 \\
& - \kappa) \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s)] + \mathcal{L}_1(-2\mathcal{L}_1 s + 2\kappa s + \sin[2(\mathcal{L}_2 + \mathcal{L}_1 s - \kappa s)]))] \\
& + \mathcal{L}_3 \sin \mathfrak{A} \frac{\sin[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} - \mathcal{L}_3 \sin \mathfrak{A} \frac{\sin[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\
& + \mathcal{L}_3 \cos \mathfrak{A} \frac{\cos[(\mathcal{L}_1 - \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 - \kappa)} + \mathcal{L}_3 \cos \mathfrak{A} \frac{\cos[(\mathcal{L}_1 + \kappa)s + \mathcal{L}_2]}{2(\mathcal{L}_1 + \kappa)} \\
& - \frac{1}{\kappa} \cos[\kappa s] \cos \mathfrak{A} + \frac{1}{\kappa} \sin[\kappa s] \sin \mathfrak{A} + \mathcal{L}_4] \mathbf{e}_3,
\end{aligned}$$

where $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ are constants of integration.

Proof: Substituting (2.2) in Theorem 3.2, we have above equation. This completes the proof. $\square$

We can use Mathematica in Theorem 3.2, yields

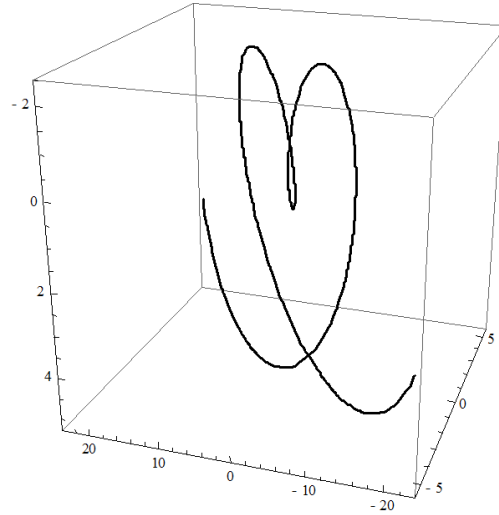


Fig. 1

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