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Global relative controllability of fractional stochastic dynamical systems with distributed delays in control

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abstract: This paper is concerned with the global relative controllability of linear and nonlinear fractional stochastic dynamical systems with distributed delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using the Banach fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffler matrix function. An example is provided to illustrate the theory.

Key Words: Distributed delays, Relative controllability, Stochastic systems, Fractional differential equations, Mittag-Leffler function

Contents

1. Introduction

The notion of controllability has played a central role throughout the history of modern control theory. Conceived by Kalman, controllability study was started systematically at the beginning of the sixties. Since then various researches have been carried out extensively in the context of finite-dimensional linear systems, nonlinear systems and infinite-dimensional systems using different kinds of approaches (e.g., [3,7,23]).

Recently there has been a great interest to differential equations with fractional order, that is fractional models are more accurate than integer models. Fractional calculus provide an excellent instrument for the description of systems with memory and hereditary properties. Many books, monographs and papers are devoted to the subject, for more details we refer the reader to [1,8,9,15,28,31]

Stochastic differential equations (SDEs) are used to model diverse phenomena such as fluctuating stock prices or physical systems subject to thermal fluctuations. In the literature, there are different definitions of controllability for SDEs, both for linear and nonlinear dynamical systems [7].

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For linear systems, results were obtained about three types of stochastic controllability: approximate, complete, and S-controllability in Banach spaces and Hilbert spaces, respectively, in [25,26]. With the help of backward SDEs and dual technique, Goreac [14] characterized the approximate controllability of linear SDEs. Sirbu and Tessitore [35] were concerned with the exact null controllability of infinite dimensional linear SDEs in Hilbert space. In particular, Klamka [21] generalized the results in [22] from the deterministic case to the stochastic one, and investigated the controllability of linear SDEs with delay in control.

In the setting of nonlinear SDEs, Arapostathis et al. [4] obtained sufficient conditions that guarantee weak and strong controllability. Assuming the corresponding linear SDEs are controllable, Mahmudov and Zorlu [24] studied the controllability of nonlinear SDEs. Later, Mahmudov [27] gave a characterization of weaker concept-approximate controllability for nonlinear SDEs. And recently, results in [17] were generalized by Balachandran et al. [5] about controllability on nonlinear SDEs with distributed delays in control.

In the theory of dynamical systems with delays in control, it is necessary to distinguish between two fundamental concepts of controllability, namely relative controllability and controllability, see [5,17,21] for more details. Controllability problems for fractional dynamical systems have drawn considerable attention recently. However, to the best of our knowledge, there are no relevant reports on the global relative controllability of fractional stochastic dynamical systems with delay in control as treated in the current paper. Very recently, Sakthivel et al. [30] discussed the approximate controllability for a class of dynamic control systems described by nonlinear fractional differential equation in Hilbert space by means of fixed point technique, under the assumptions that the corresponding linear system is approximately controllable. in [6] Balachandran et al., investigated the controllability of linear and nonlinear fractional dynamical systems in finite dimensional spaces. the authors obtained sufficient conditions for controllability by Schauder's fixed point theorem and the controllability Grammian matrix which is defined by the mittag-Leffler matrix function. Our goal in this article is to study the global relative controllability for both linear and nonlinear fractional stochastic dynamical systems with distributed delays in control. The rest of the paper is organized as follows: In Section 2, some well known fractional operators and special functions, along with a set of properties are defined which will be of use as we proceed in our discussion. In Section 3, the linear fractional stochastic system with distributed delay in control is considered and the controllability condition is established using the controllability Grammian matrix which is defined by means of Mittag Leffler matrix function. The corresponding nonlinear fractional system is also considered and the controllability results are examined with the natural assumption that the linear fractional system is relatively controllable. The results are established by using using the Banach fixed point theorem and the fractional calculus. Finally, Section 4 ends up with an example to illustrate the theory.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). Let $\alpha, \beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$ and $n \in \mathbb{N}$, D is the usual differential operator. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$, and suppose $f \in L^1(\mathbb{R}_+)$. The following definitions and properties are well known, for $\alpha, \beta > 0$ and f as a suitable function (see, for instance, [15]):

(a) Riemann-Liouville fractional operators:

$$\begin{aligned}(I^\alpha_{0+}f)(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\\ (D^\alpha_{0+}f)(x) &= D^n(I^{n-\alpha}_{a+}f)(x).\end{aligned}$$

(b) Caputo fractional derivative:

$$({}^cD^\alpha_{0+}f)(x) = (I^{n-\alpha}_{0+}D^n f)(x),$$

in particular $I^\alpha_{0+}\,{}^cD^\alpha_{0+}f(t) = f(t) - f(0)$, $(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D^\alpha_{a+}f)(x) = ({}^cD^\alpha_{a+}f)(x) + \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(x-a)^{k-\alpha}, \quad n = \mathcal{R}(\alpha)+1.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^cD^\alpha_{0+}f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^cD^\alpha_{0+}f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+)s^{\alpha-1-k}.$$

The Riemann-Liouville fractional derivatives have singularity at zero and the fractional differential equations in the RiemannŨLiouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo introduced a new definition of fractional derivative but in general, both the Riemann-Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in [15].

(c) Linear Sequential Derivative:

For $n \in \mathbb{N}$ the sequential fractional derivative for suitable function f is defined by

$$f^{(k\alpha)} := (\mathbf{D}^{k\alpha}f)(x) = (\mathbf{D}^\alpha\mathbf{D}^{(k-1)\alpha}f)(x),$$

where $k = 1, \ldots, n$, $(\mathbf{D}^{\alpha} f)(x) = f(x)$, and $\mathbf{D}^{\alpha}$ is any fractional differential operator, here we mention it as ${}^{c}D^{\alpha}_{0+}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\alpha+\beta)}, \quad \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} y) dt = \frac{1}{1-y}, \quad |y| < 1.$$

The Laplace transform of $E_{\alpha,\beta}(y)$ follows from the integral

$$\int_0^{\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha}) dt = \frac{s^{\alpha-\beta}}{(s \mp a)}.$$

That is

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^{\alpha})\} = \frac{s^{\alpha-\beta}}{(s \mp a)},$$

for $\Re(s) > |a|^{1/\alpha}$ and $\Re(\beta) > 0$. In particular, for $\beta = 1$,

$$E_{\alpha,1}(\lambda y^{\alpha}) = E_{\alpha}(\lambda y^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^k y^{k\alpha}}{\Gamma(\alpha k+1)}, \quad \lambda, y \in \mathbf{C}$$

have the interesting property ${}^{c}D^{\alpha} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha})$ and

$$\mathcal{L}\{E_{\alpha}(\pm a t^{\alpha})\} = \frac{s^{\alpha-1}}{(s \mp a)}, \quad \text{for } \beta = 1.$$

For brevity of notation let us take I^q_{0+} as I^q and ${}^{c}D^q_{0+}$ as ${}^{c}D^q$ and the fractional derivative is taken as Caputo sense.

Let us consider the linear fractional stochastic differential equation of the form

$$\begin{aligned} {}^{c}D^q x(t) &= Ax(t) + \sigma(t)\frac{dw(t)}{dt}, \quad t \in [0,T], \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, A is an $n \times n$ matrix, $w(t)$ is a given l-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0,T] \to \mathbb{R}^{n\times l}$ is appropriate function. In order to find the solution, apply Laplace transform on both sides and use the Laplace transform of Caputo derivative, we get

$$s^q X(s) - s^{q-1} x(0) = AX(s) + \Sigma(s)\frac{dw(s)}{ds}.$$

Apply inverse Laplace transform on both sides (see [6]) we have

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{s^{q-1}(s^q I - A)^{-1}\}x_0 + \mathcal{L}^{-1}\{\Sigma(s)\tfrac{dw(s)}{ds}\} * \mathcal{L}^{-1}\{(s^q I -$$

$A)^{-1}\}$.
Finally, substituting Laplace transformation of the Mittag-Leffler function, we get the solution of the given system

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds$$

where $E_q(At^q)$ is the matrix extension of the mentioned Mittag-Leffler functions with the following representation:

$$E_q(At^q) = \sum_{k=0}^{\infty} \frac{A^k t^{kq}}{\Gamma(1+kq)}$$

with the property ${}^cD^qE_q(At^q) = AE_q(At^q)$.

3. Controllability results

Let $L^2_{\mathcal{F}_t}(J\times\Omega, \mathbb{R}^n)$ be the Banach space of all $\mathcal{F}_t$-measurable square integrable processes $x(t)$ with norm $\|x\|^2_{L^2} = \sup_{t\in J}\mathbb{E}\|x(t)\|^2$, where $\mathbb{E}(.)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $C = C([0,T]; L^2_{\mathcal{F}_t})$ be the Banach space of continuous maps from $[0,T]$ into $L^2_{\mathcal{F}_t}(J\times\Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in J}\mathbb{E}\|x(t)\|^2 < \infty$. Consider the linear fractional stochastic dynamical system with distributed delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^qx(t) &= Ax(t) + \int_{-h}^0 d_\tau B(t,\tau)u(t+\tau) + \sigma(t)\frac{dw(t)}{dt}, \quad t\in J := [0,T] \\ x(0) &= x_0, \end{aligned} \tag{3.1}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, and the second integral term is in the Lebesgue-Stieltjes sense with respect to τ. Let $h > 0$ be given. For function $u : [-h,T] \to \mathbb{R}^m$ and $t \in J$, we use the symbol u_t to denote the function on $[-h,0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h,0)$. A is an $n\times n$ matrix, $B(t,\tau)$ is an $n\times m$ matrix continuous in t for fixed τ and is of bounded variation in τ on $[-h,0]$ for each $t \in J$ and continuous from left in τ on the interval $(-h,0)$. Here $w(t)$ is a given m-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$, $0 \le s \le t$ and $\sigma : [0,T] \to \mathbb{R}^{n\times m}$.

The following definitions of complete state of the system (2) at time t and relative controllability are assumed.

Definition 3.1. *The set $\phi(t) = \{x(t), u_t\}$ is the complete state of the system* (2) *at time t.*

Definition 3.2. *System* (2) *is said to be globally relatively controllable on J if for every complete state $\phi(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system* (2) *satisfies $x(T) = x_1$.*

Note that the solution of system (2) ca be expressed in the following form

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_{-h}^0 d_\tau B(s,\tau)u(s+\tau)\right]ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds, \end{aligned}$$

where $E_q(A(t)^q)$ is the Mittag Leffler matrix function. Now using the well known result of unsymmetric Fubini theorem [10] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_{-h}^0 dB_\tau\left[\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)u(s+\tau)B(s,\tau)ds\right] \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &= E_q(A(t)^q)x_0 + \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\right. \\ &\qquad \left. B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_{-h}^0 dB_\tau\left[\int_0^{t+\tau} (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u(s)ds\right] \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &= E_q(A(t)^q)x_0 + \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\right. \\ &\qquad \left. B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta)dw(\theta)\right)E_{q,q}(A(t-s)^q)ds, \end{aligned} \tag{3.2}$$

where

$$B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases}$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t,\tau)$.

For brevity, let us introduce the following notations:

$$\varphi(t,s) = \int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau), \tag{3.3}$$

and

$$\chi(t) = \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(t-s)^q) ds. \tag{3.4}$$

Recall the controllability Grammian matrix

$$\psi_0^T = \int_0^T \varphi(T,s)\varphi^\star(T,s) ds$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and the $\star$ denotes the matrix transpose.

Theorem 3.3. *The linear stochastic control system* (2) *is relatively controllable on* $[0,T]$ *if and only if the controllability Grammian matrix* ψ_0^T *is positive definite for some* $T > 0$.

Proof: Since ψ is positive definite, it is non-singular and therefore its inverse is well defined. Define the control function as,

$$\begin{aligned} u(t) &= \varphi^\star(T,t)\psi^{-1}\Bigg(x_1 - E_q(At^q)x_0 - \int_{-h}^0 dB_\tau \Bigg[(T-(s-\tau))^{q-1} \\ &\qquad E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds \Bigg] - \chi(T) \Bigg), \end{aligned} \tag{3.5}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (6) in (3) and using (4) we get

$$\begin{aligned} x(T) &= E_q(A(T)^q)x_0 + \int_{-h}^0 dB_\tau \Bigg[\int_\tau^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) \\ &\qquad B(s-\tau,\tau)u_0(s)ds \Bigg] \\ &+ \int_0^T \left[\int_{-h}^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T B(s-\tau,\tau) \right] \\ &\times \left[\int_{-h}^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T B(s-\tau,\tau) \right]^\star \psi^{-1} \\ &\quad \Bigg(x_1 - E_q(AT^q)x_0 - \int_{-h}^0 dB_\tau \Bigg[(T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) \\ &\qquad B(s-\tau,\tau)u_0(s)ds \Bigg] - \chi(T) \Bigg) d\tau \\ &+ \int_0^T (T-s)^{q-1} \left(\int_0^\tau \sigma(\theta) dw(\theta) \right) E_{q,q}(A(T-s)^q) ds \\ &= x_1. \end{aligned} \tag{3.6}$$

Thus the control $u(t)$ transfers the initial state $\phi(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T. Hence the system (2) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero ϕ such that $\phi^\star\psi\phi = 0$, that is

$$\phi^\star \int_0^T \varphi(T,s)\varphi^\star(T,s)\phi ds = 0$$
$$\phi^\star\varphi(T,s) = 0, \quad \text{on } [0,T].$$

Let $x_0 = [E_q(AT^q)]^{-1}\phi$. By assumption, there exists a control u such that it steers the complete initial state $\phi(0) = \{x(0), u_0(s)\}$ to the origin in the interval $[0,T]$. It follows that

$$\begin{aligned}
x(T) &= E_q(A(T)^q)x_0 + \int_{-h}^0 dB_\tau \Bigg[\int_\tau^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) \\
&\qquad B(s-\tau,\tau)u_0(s)ds \Bigg] \\
&+ \int_0^T \left[\int_{-h}^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T B(s-\tau,\tau) \right] u(s)ds \\
&+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta)dw(\theta) \right) E_{q,q}(A(t-s)^q)ds \\
&= \phi + \int_{-h}^0 dB_\tau \Bigg[\int_\tau^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) \\
&\qquad B(s-\tau,\tau)u_0(s)ds \Bigg] \\
&+ \int_0^T \left[\int_{-h}^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T B(s-\tau,\tau) \right] u(s)ds \\
&+ \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta)dw(\theta) \right) E_{q,q}(A(t-s)^q)ds \\
&= 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
0 &= \phi^\star\phi + \int_0^T \phi^\star\varphi(T,s)u(s)ds \\
&+ \phi^\star \Bigg(\int_{-h}^0 dB_\tau \Bigg[\int_\tau^0 (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) \\
&\qquad B(s-\tau,\tau)u_0(s)ds \Bigg] + \chi(T) \Bigg).
\end{aligned}$$

Then, taking into account that both of the terms

$$\int_0^T \phi^\star \varphi(T,s)u(s)ds$$

and

$$\phi^\star\left(\int_{-h}^0 dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)\right.\right.$$
$$\left.\left.B(s-\tau,\tau)u_0(s)ds\right]+\chi(T)\right)$$

are zero leading to the conclusion $\phi^\star\phi = 0$. This is a contradiction to $\phi \neq 0$. Thus ψ is positive definite. Hence the desired result. □

Consider a nonlinear fractional stochastic dynamical system with distributed delays in control represented by the fractional stochastic differential equation of the form

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + \int_{-h}^0 d_\tau B(t,\tau)u(t+\tau) + f(t,x(t)) + \sigma(t,x(t))\frac{dw(t)}{dt}, \\ & \quad t \in J := [0,T] \\ x(0) &= x_0, \end{aligned} \tag{3.7}$$

where $0 < q < 1$, $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B are as above, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : J \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$, and $w(t)$ is a given m-dimensional Wiener process with the filtration $\mathcal{F}_t$ generated by $w(s)$. Then the solution of the system (8) ca be expressed in the following form [12]

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+ \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\left[\int_{-h}^0 d_\tau B(t,\tau)u(t+\tau)\right]ds. \end{aligned}$$

Using the well known result of unsymmetric Fubini theorem [10] and change of order of integration to the last term, we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+ \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds \end{aligned} \tag{3.8}$$

where

$$B_t(s,\tau) = \begin{cases} B(s,\tau), & s \le t \\ 0, & s > t \end{cases}$$

and dB_τ denotes the integration of Lebesgue Stieltjes sense with respect to the variable τ in the function $B(t,\tau)$.

For brevity, let us introduce the notation:

$$\begin{aligned} \Upsilon(\phi(0),x_1;x) &= x_1 - E_q(A(T)^q)x_0 - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &- \int_{-h}^0 dB_\tau\Bigg[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q) \\ &\qquad B(s-\tau,\tau)u_0(s)ds\Bigg]. \end{aligned} \tag{3.9}$$

Define the control function

$$u(t) = \varphi^\star\psi^{-1}\Upsilon(\phi(0),x_1;x), \tag{3.10}$$

where the complete state $\phi(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $\star$ denotes the matrix transpose.

Now, we impose the following conditions on data of the problem:

i. The linear fractional stochastic dynamical system (2) is globally relatively controllable.
ii. f and σ satisfy Lipschitz and linear growth conditions. That is, there exists some constants $N, \tilde{N}, L, \tilde{L} > 0$ such that

$$\begin{aligned} \|f(t,x)-f(t,y)\|^2 &\le N\|x-y\|^2, \quad \|f(t,x)\|^2 \le \tilde{N}(1+\|x\|^2) \\ \|\sigma(t,x)-\sigma(t,y)\|^2 &\le L\|x-y\|^2, \quad \|\sigma(t,x)\|^2 \le \tilde{L}(1+\|x\|^2). \end{aligned}$$

For our convenience, let us introduce the following notations.

$$\begin{array}{llll} a_1 & = \max\{\|E_q(At^q)\|^2; t\in J\}, & a_2 & = \max\{\|E_{q,q}(A(t-s)^q)\|^2; t\in J\} \\ a_3 & = \max\{\|E_{q,q}(A(t-(s-\tau))^q)\|^2; t\in J\}, & c_1 & = \max\{\|u_0(t)\|^2; t\in J\} \\ c_2 & = \int_{-h}^{0}(t-(s-\tau))^{2(q-1)}ds, & c_3 & = \int_{-\tau}^{0}(t-(s-\tau))^{2(q-1)}ds \\ M_B & = \max\{\|B(s-\tau,\tau)\|^2; 0\le\tau<s\le T\}, & M & = \max\{\|\varphi(t,s)\|^2; 0\le s<t\le T\} \end{array}$$

We claim that if **i**. holds, the operator ψ_0^T is strictly positive definite and thus the inverse linear operator $(\psi_0^T)^{-1}$ is bounded, say, by l, (see [21] for more details).

Theorem 3.4. *Under the conditions* **i**. *and* **ii**., *the nonlinear system* (8) *is globally relatively controllable on* J.

Proof: Firstly, from the definition of the control function (11), we can write u as

$$\begin{aligned} u(t) &= \varphi^\star(T,t)\psi^{-1}\Upsilon(\phi(0),x_1;x) \\ &= \varphi^\star(T,t)\psi^{-1}\Bigg(x_1 - E_q(AT^q)x_0 - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,x(s))ds \\ &- \int_0^T (T-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(T-s)^q)ds \\ &- \int_{-h}^0 dB_\tau\left[\int_\tau^0 (T-(s-\tau))^{q-1}E_{q,q}(A(T-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right]\Bigg) \end{aligned}$$

Secondly, we define the operator $\mathcal{P}: C\to C$ by

$$\begin{aligned} \mathcal{P}(x)(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds \\ &+ \int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds \\ &+ \int_{-h}^0 dB_\tau\left[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)B(s-\tau,\tau)u_0(s)ds\right] \\ &+ \int_0^t\left[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)d_\tau B_t(s-\tau,\tau)\right]u(s)ds. \end{aligned}$$

In order to prove the global relative controllability of the system (8) it is enough to show that $\mathcal{P}$ has a fixed point in C. To do this, we can employ the contraction mapping principle. To apply the principle, first we show that $\mathcal{P}$ maps C into itself.

We have

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 &\leq 5a_1\mathbb{E}\|x_0\|^2 + 5\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,x(s))ds\right\|^2\\ &+ 5\mathbb{E}\left\|\int_0^t (t-s)^{q-1}\left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta)\right)E_{q,q}(A(t-s)^q)ds\right\|^2\\ &+ 5\mathbb{E}\left\|\int_{-h}^0 dB_\tau\Big[\int_\tau^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\right.\\ &\left.\qquad B(s-\tau,\tau)u_0(s)ds\Big]\right\|^2\\ &+ 5\mathbb{E}\left\|\int_0^t \Big[\int_{-h}^0 (t-(s-\tau))^{q-1}E_{q,q}(A(t-(s-\tau))^q)\right.\\ &\left.\qquad d_\tau B_t(s-\tau,\tau)\Big]u(s)ds\right\|^2\end{aligned}$$

It follows from Lemma 2.5, in [30], and the above notation that:

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 &\leq 5a_1\mathbb{E}\|x_0\|^2 + 5a_2\frac{t^{2q-1}}{2q1}\int_0^t \mathbb{E}\|f(s,x(s))\|^2ds\\ &+ 5L_\sigma a_2\frac{t^{2q-1}}{2q-1}\int_0^t\left(\int_0^\tau \mathbb{E}\|\sigma(\theta,x(\theta))\|^2d\theta\right)ds + 5MM_Ba_3c_1c_3\\ &+ 5M\int_0^t \mathbb{E}\|u(s)\|^2ds.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 &\leq 5a_1\mathbb{E}\|x_0\|^2 + 5a_2\frac{t^{2q-1}}{2q1}\tilde{N}\int_0^t (1+\mathbb{E}\|x(s)\|^2)ds\\ &+ 5a_2L_\sigma\frac{t^{2q-1}}{2q-1}\tilde{L}\int_0^t\left(\int_0^\tau (1+\mathbb{E}\|x(\theta)\|^2)d\theta\right)ds + 5MM_Ba_3c_1c_3\\ &+ 5M^2l^2\Bigg[\mathbb{E}\|x_1\|^2 + a_1\mathbb{E}\|x_0\|^2 + a_2\frac{T^{2q-1}}{2q1}\tilde{N}\int_0^T (1+\mathbb{E}\|x(s)\|^2)ds\\ &+ a_2L_\sigma\frac{T^{2q-1}}{2q-1}\tilde{L}\int_0^T\left(\int_0^\tau (1+\mathbb{E}\|x(\theta)\|^2)d\theta\right)ds + 5MM_Ba_3c_1c_3\Bigg].\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\|\mathcal{P}(x)(t)\|^2 &\leq 5M^2l^2\mathbb{E}\|x_1\|^2 + 5a_1\mathbb{E}\|x_0\|^2(1+M^2l^2)\\ &+ 5MM_Ba_3c_1c_3(1+M^2l^2) + 5a_2\frac{T^{2q-1}}{2q1}\tilde{N}(1+M^2l^2)(1+\|x\|^2_{L^2})\\ &+ 5a_2L_\sigma\tilde{L}\frac{T^{2q-1}}{2q1}(1+M^2l^2)(1+T\|x\|^2_{L^2}).\end{aligned}$$

It follows from from the above inequality and the condition **ii**. that there exists $\beta > 0$ such that

$$\mathbf{E}\|\mathcal{P}(x)(t)\|^2 \leq \beta(1 + \|x\|_{L^2}^2).$$

Therefore $\mathcal{P}$ maps C into itself.

Secondly, we claim that $\mathcal{P}$ is a contraction mapping on C. For $x, y \in C$,

$$\begin{aligned}
& \mathbf{E}\|\mathcal{P}(x)(t) - \mathcal{P}(y)(t)\|^2 \\
\leq \; & 3\mathbf{E}\left\| \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)(f(s,x(s)) - f(s,y(s))ds \right\|^2 \\
+ \; & 3\mathbf{E}\left\| \int_0^t (t-s)^{q-1} \left(\int_0^\tau (\sigma(\theta,x(\theta)) - \sigma(\theta,y(\theta)))dw(\theta) \right) E_{q,q}(A(t-s)^q)ds \right\|^2 \\
+ \; & 3\mathbf{E}\left\| \int_0^t \varphi(t,s)\varphi^\star(T,s)\psi^{-1}[\Upsilon(\phi(0),x_1;x) - \Upsilon(\phi(0),x_1;y)] \right\|^2.
\end{aligned}$$

Using Lemma 2.5, in [30], condition **ii**., and the above notations we get

$$\begin{aligned}
& \mathbf{E}\|\mathcal{P}(x)(t) - \mathcal{P}(y)(t)\|^2 \\
\leq \; & 3a_2 \frac{T^{2q-1}}{2q-1}(1 + M^2 l^2 T) \int_0^t \mathbf{E}\|f(s,x(s)) - f(s,y(s))\|^2 ds \\
+ \; & 3a_2 \frac{T^{2q-1}}{2q-1} L_\sigma (1 + M^2 l^2 T) \int_0^t \left(\int_0^\tau \mathbf{E}\|\sigma(\theta,x(\theta)) - \sigma(\theta,y(\theta))\|^2 d\theta \right) ds \\
\leq \; & 3a_2 \frac{T^{2q-1}}{2q-1}(1 + M^2 l^2 T)(N + LL_\sigma T) \int_0^t \mathbf{E}\|x(s) - y(s)\|^2 ds.
\end{aligned}$$

It results that

$$\sup_{t\in[0,T]} \mathbf{E}\|\mathcal{P}(x)(t) - \mathcal{P}(y)(t)\|^2 \leq 3a_2 \frac{T^{2q-1}}{2q-1}(1 + M^2 l^2 T)(N + LL_\sigma T) \sup_{t\in[0,T]} \mathbf{E}\|x(t) - y(t)\|^2.$$

Therefore we conclude that if $3a_2 \frac{T^{2q-1}}{2q-1}(1 + M^2 l^2 T)(N + LL_\sigma T) < 1$, then $\mathcal{P}$ is a contraction mapping on C, implies that the mapping $\mathcal{P}$ has a unique fixed point $x(\cdot) \in C$. Hence we have

$$\begin{aligned}
x(t) \; = \; & E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s,x(s))ds \\
+ \; & \int_0^t (t-s)^{q-1} \left(\int_0^\tau \sigma(\theta,x(\theta))dw(\theta) \right) E_{q,q}(A(t-s)^q)ds \\
+ \; & \int_{-h}^0 dB_\tau \left[\int_\tau^0 (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) B(s-\tau,\tau) u_0(s)ds \right] \\
+ \; & \int_0^t \left[\int_{-h}^0 (t-(s-\tau))^{q-1} E_{q,q}(A(t-(s-\tau))^q) d_\tau B_t(s-\tau,\tau) \right] u(s)ds
\end{aligned}$$

Thus $x(t)$ is the solution of the system (8), and it is easy to verify that $x(T) = x_1$. Further the control function u(t) steers the system (8) from initial complete state $\phi(0)$ to x_1 on J. Hence the system (8) is globally relatively controllable on J. □

4. Example

In this section, we apply the results obtained in the previous section for the following stochastic fractional dynamical systems with distributed delays in control which involves sequential Caputo derivative

$$\begin{aligned} {}^cD^q x(t) &= Ax(t) + \int_{-1}^{0} d_\tau B(t,\tau)u(t+\tau) + f(t,x(t)) + \sigma(t,x(t))\frac{dw(t)}{dt}; \\ & \quad 0 < q < 1, t \in [0,T] \\ x(0) &= x_0, \end{aligned} \tag{4.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B(t,\tau) = \begin{pmatrix} e^\tau \cos t & e^\tau \sin t \\ -e^\tau \sin t & e^\tau \cos t \end{pmatrix}, \quad u(t+\tau) = \begin{pmatrix} u_1(t+\tau) \\ u_2(t+\tau) \end{pmatrix}$$

$$f(t,x(t)) = \begin{pmatrix} x_1(t)\cos x_2(t) + 3x_2(t) \\ x_2(t)\sin x_1(t) + 2x_1(t) \end{pmatrix},$$

$$\sigma(t,x(t)) = \begin{pmatrix} (2t^2+1)x_1(t)e^{-t} & 0 \\ 0 & x_2(t)e^{-t} \end{pmatrix}.$$

Let us introduce the variables $x_1(t) = x(t)$ and $x_2(t) = {}^cD^{\frac{q}{2}} x_1(t)$. Then ${}^cD^{\frac{q}{2}} x_1(t) = {}^cD^{\frac{q}{2}} x(t) = x_2$.

The Mittag-Leffler matrix of the given system is given by

$$E_q(At^q) = \begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma(1+2jq)} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma(1+(2j+1)q)} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j t^{(2j+1)q}}{\Gamma(1+(2j+1)q)} & \sum_{j=0}^{\infty} \frac{(-1)^j t^{2jq}}{\Gamma(1+2jq)} \end{pmatrix}.$$

Further

$$E_{q,q}(A(T-(s-\tau))^q) =$$

$$\begin{pmatrix} \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(2j+1)q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(j+1)2q]} \\ -\sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q}}{\Gamma[(j+1)2q]} & \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{2jq}}{\Gamma[(2j+1)q]} \end{pmatrix},$$

and

$$(T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) = \begin{pmatrix} \cos_q(t) & \sin_q(t) \\ -\sin_q(t) & \cos_q(t) \end{pmatrix},$$

where $\cos_q(t)$ and $\sin_q(t)$ are given by

$$\cos_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(2j+1)q-1}}{\Gamma[(2j+1)q]},$$
$$\sin_q(t) = \sum_{j=0}^{\infty} \frac{(-1)^j (T-(s-\tau))^{(j+1)2q-1}}{\Gamma[(j+1)2q]}.$$

$$\varphi(T,s) = \int_{-1}^{0} (T-(s-\tau))^{q-1} E_{q,q}(A(T-(s-\tau))^q) d_\tau B_T(s-\tau,\tau) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\beta(s) & \alpha(s) \end{pmatrix},$$

$$\alpha(s) = \int_{-1}^{0} e^\tau [\cos_q(T-(s-\tau))\cos(s-\tau) - \sin_q(T-(s-\tau))\sin(s-\tau)] d\tau$$
$$\beta(s) = \int_{-1}^{0} e^\tau [\sin_q(T-(s-\tau))\cos(s-\tau) - \cos_q(T-(s-\tau))\sin(s-\tau)] d\tau.$$

By simple matrix calculation one can see that the controllability matrix

$$\psi_0^T = \int_0^T \varphi(T,s)\varphi^\star(T,s) ds = \int_0^T [\alpha^2(s)+\beta^2(s)] \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ds$$

is positive definite for any $T > 0$. Further the functions $f(t, x(t))$ and $\sigma(t, x(t))$ satisfies the hypothesis mentioned in Theorem 3.4., and so the fractional system (12) is globally relatively controllable on [0,T].

5. Conclusion

This paper has investigated the global relative controllability of linear and nonlinear stochastic fractional dynamical systems with distributed delays in control. With Lipschitz and linear growth conditions, some sufficient conditions have been presented for global relative controllability of stochastic nonlinear systems in finite dimensional space. As applications, an example have been also discussed.

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