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Properties of b-compact spaces and b-closed spaces

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abstract: In this paper, we give some new characterizations of b-compact sets and b-closed sets by means of nets and filterbases.

Key Words: Topological spaces, b-compact spaces, b-closed spaces.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets.In 1996, Andrijevic [1] introduced a new class of generalized open sets called b-open sets into the field of topology. Andrijevic studied several fundamental and interesting properties of b-open sets. Quite recently, Park has introduced [5] b-closed spaces in topological space. In this paper, to give some characterizations of b-closed spaces. Compactness and properties closely related to compactness play an important role in the applications of General Topology to Real Analysis and Functional Analysis. In the framework of topological spaces several modified forms of compact spaces have been introduced and studied: nearly compact spaces, mildly compact spaces etc. In this paper, we give some new characterizations of b-compact sets and b-closed sets by means of nets and filterbases.

2. Preliminaries

Throughout this paper, spaces always means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of X is said to be b-open [1] $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of b-open is called b-closed [1]. The union of all b-open sets contained in $A \subset X$ is called the b-interior of A, and is

2000 *Mathematics Subject Classification:* 54C60 denoted by $b\,\mathrm{Int}(A)$. The intersection of all b-closed sets containing A is called the b-closure [1] of A and is denoted by $b\,\mathrm{Cl}(A)$. The b-θ-closure [5] of A, denoted by $b\,\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap b\,\mathrm{Cl}(U) \neq \varnothing$ for every b-open set U containing x. A subset A is called b-θ-closed [5] if and only if $A = b\,\mathrm{Cl}_\theta(A)$. The complement of b-θ-closed set is called b-θ-open. A subset S of a topological space (X, τ) is said to be b-regular [5] if it is b-open and b-closed. The family of all b-regular (resp. b-open, b-closed) sets of (X, τ) is denoted by $BR(X)$ (resp. $BO(X)$, $BC(X)$). The family of all b-regular (resp. b-open, b-closed) sets of (X, τ) containing a point $x \in X$ is denoted by $BR(X, x)$ (resp. $BO(X, x)$, $BC(X, x)$).

3. b-compact spaces

Definition 3.1. *Let (X, τ) be a topological space. A class $\{G_i\}$ of b-open subsets of X is said to be b-open cover of X if each point in X belongs to at least one G_i that is $\bigcup_i G_i = X$.*

Definition 3.2. *A subset K of a nonempty set X is said to be b-compact relative to (X, τ) [4] if every cover of K by sets of $BO(X)$ has a finite subcover. We say that (X, τ) is b-compact if X is b-compact.*

We will give several characterizations of the b-compact spaces. The first characterization makes use of the finite intersection condition.

Theorem 3.3. *The following statements are equivalent for any topological space (X, τ):*

(i) X is b-compact.

(ii) Given any family $\mathcal{F}$ of b-open sets, if no finite subfamily of $\mathcal{F}$ covers X, then $\mathcal{F}$ does not cover X.

(iii) Given any family $\mathcal{F}$ of b-closed sets, if $\mathcal{F}$ satisfies the finite intersection condition, then $\cap\{A : A \in \mathcal{F}\} \neq \emptyset$.

(iv) Given any family $\mathcal{F}$ of subsets of X, if $\mathcal{F}$ satisfies the finite intersection condition, then $\cap\{b\,\mathrm{Cl}(A) : A \in \mathcal{F}\} \neq \emptyset$.

Proof: $(i) \Leftrightarrow (ii)$ and $(ii) \Leftrightarrow (iii)$ are obvious. $(iii) \Rightarrow (iv)$: If $\mathcal{F} \subset P(X)$ satisfies the finite intersection condition, then $\cap\{b\,\mathrm{Cl}(A) : A \in \mathcal{F}\}$ is a family of b-closed sets, which obviously satisfies the finite intersection condition. $(iv) \Rightarrow (iii)$ Follows from the fact that $A = b\,\mathrm{Cl}(A)$ for every b-closed set A. □

Definition 3.4. *A point $x \in X$ is said to be b-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in every b-open set containing x. We denote by b-cp$\{x_\alpha\}_{\alpha\in\Delta}$ the set of all b-cluster points of a net $\{x_\alpha\}_{\alpha\in\Delta}$.*

Theorem 3.5. *The set of all b-cluster points of an arbitrary net in X is b-closed.*

Proof: Let $\{x_\alpha\}_{\alpha\in\Delta}$ be a net in X. Set $A = b - cp\{x_\alpha\}_{\alpha\in\Delta}$. Let $x \in X\backslash A$. Then there exists a b-open set U_x containing x and $\alpha_x \in \Delta$ such that $X_\beta \notin U_x$ whenever $\beta \in \Delta$, $\beta \geq \alpha_x$. It turns out that $U_x \subset X\backslash A$, hence $x \in b\,\mathrm{Int}(X\backslash A) = X\backslash b\,\mathrm{Cl}(A)$. This shows that $b\,\mathrm{Cl}(A) \subset A$; hence A is b-closed. □

Theorem 3.6. *A topological space X is b-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has at least one b-cluster point.*

Proof: Let X be a b-compact space. Assume that there exist some net $\{x_\alpha\}_{\alpha\in\Delta}$ in X such that b-cp$\{x_\alpha\}_{\alpha\in\Delta}$ is empty. Then for every $x \in X$, there exist $U(x) \in BO(X,x)$ and $\alpha(x) \in \Delta$, such that $x_\beta \notin U(x)$ whenever $\beta \geq \alpha(x)$, $\beta \in \Delta$. Then the family $\{U(x) : x \in X\}$ is a cover of X by b-open sets and has a finite subcover, say, $\{U_k : k = 1, 2, ...n\}$ where $U_k = U(x_k)$ for k = 1,2,....n, $\{x_k : k = 1, 2, ...n\}$. Let us take $\alpha \in \Delta$ such that $\alpha \geq \alpha(x_k)$ for all $k \in \{1, 2, ...n\}$. For every $\beta \in \Delta$ such that $\beta \geq \alpha$ we have, $x_\beta \notin U_k$, k = 1, 2,....n, hence $x_\beta \notin X$, which is a contradiction. Conversely, if X is not b-compact, there exists $\{U_i : i \in I\}$ a cover of X by b-open sets, which has no finite subcover. Let $P(I)$ be the family of all finite subsets of I. Clearly, $(P(I), \subseteq)$ is a directed set. For each $J \in \mathcal{I}$. we may choose $x_j \in X\backslash \cup \{U_i : i \in J\}$. Let us consider the net $\{x_j\}_{j\in P(I)}$. By hypothesis, the set b-$cp\{x_j\}_{j\in P(I)}$ is nonempty. Let $x \in b$-$cp\{x_j\}_{j\in P(I)}$ and let $i_0 \in I$ such that $x \in U_{i0}$. By the definition of b-cluster point, for each $J \in P(I)$ there exist $J^* \in P(I)$ such that $J \subset J^*$ and $x^*_j \in U_{i0}$. For $J = \{i_0\}$, there exists $J^* \in P(I)$ such that $i_0 \in J^*$ and $x^*_j \in U_{i0}$. But $x^*_j \in X\backslash \cup \{U_i : i \in J^*\} \subset X\backslash U_{i_0}$. The contradiction we obtained shows that X is b-compact. □

In the following, we will give a characterization of b-compact spaces by means of filterbases.

Let us recall that a nonempty family $\mathcal{F}$ of subsets of X is said to be a filterbase on X if $\emptyset \notin \mathcal{F}$ and each intersection of two members of $\mathcal{F}$ contains a third member of $\mathcal{F}$. Notice that each chain in the family of all filterbase on X (ordered by inclusion) has an upper bound, for example, the union of all members of the chain. Then, by Zorn's Lemma, the family of all filterbases on X has at least one maximal element. Similarly, the family of all filterbases on X containing a given filterbase $\mathcal{F}$ has at least one maximal element.

Definition 3.7. *A filterbase $\mathcal{F}$ on a topological space X is said to be:*

(i) b-converge to a point $x \in X$ if for each b-open set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset U$.

(ii) b-accumulate at $x \in X$ if $U \cap B \neq \emptyset$ for every b-open set U containing x and every $B \in \mathcal{F}$.

Remark 3.8. *A filterbase $\mathcal{F}$ b-accumulate at x if and only if $x \in \cap\{b\,\mathrm{Cl}(B) : B \in \mathcal{F}\}$. Clearly, if a filterbase $\mathcal{F}$ b-converges to $x \in X$, then $\mathcal{F}$ b-accumulates at x.*

Lemma 3.9. *If a maximal filterbase $\mathcal{F}$ b-accumulate at $x \in X$, then $\mathcal{F}$ b-converges to x.*

Proof: Let $\mathcal{F}$ be a maximal filterbase which b-accumulate at $x \in X$. If $\mathcal{F}$ does not b-converges to x, then there exists a b-open set U_0 containing x such that $U_0 \cap B \neq \emptyset$ and $(X \backslash U_0) \cap B \neq \emptyset$ for every $B \in \mathcal{F}$. Then $\mathcal{F} \cup \{U_0 \cap B : B \in \mathcal{F}\}$ is a filterbase which strictly contains $\mathcal{F}$, which is a contradiction. □

Theorem 3.10. *For a topological space X, the following statements are equivalent:*

(i) X is b-compact;

(ii) Every maximal filterbase b-converges to some point of X;

(iii) Every filterbase b-accumulates at some point of X.

Proof: $(i) \Rightarrow (ii)$: Let $\mathcal{F}_0$ be a maximal filterbase on X. Suppose that $\mathcal{F}_0$ does not b-converge to any point of X. Then, by Lemma 3.9, $\mathcal{F}_0$ does not b-accumulate at any point of X. For each $x \in X$, there exists a b-open set U_x containing x and $B_x \in \mathcal{F}_0$ such that $U_x \cap B_x = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by b-open sets. By (i), there exists a finite subset $\{x_1, x_2, ...x_n\}$ of X such that $X = \cup\{U_{x_k} : k = 1, 2, ...n\}$. Since $\mathcal{F}_0$ is a filterbase, there exists $B_0 \in \mathcal{F}_0$ such that $B_0 \subset \cap\{B_{x_k} : k = 1, 2, ...n\} = X \backslash \cup \{U_{x_k} : k = 1, 2, ...n\}$, hence $B_0 = \emptyset$. This is a contradiction. $(ii) \Rightarrow (iii)$: Let $\mathcal{F}$ be a filterbase on X. There exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subset \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ b-converges to some point $x_0 \in X$. Let $B \in \mathcal{F}$. For every $U \in BO(X, x_0)$, there exists $B_U \in \mathcal{F}_0$ such that $B_U \subset U$, hence $U \cap B \neq \emptyset$, since it contains the member $B_U \cap B$ of $\mathcal{F}_0$. This shows that $\mathcal{F}$ b-accumulates at x_0. $(iii) \Rightarrow (i)$: Let $\{V_i : i \in I\} = \emptyset$ be any family of b-closed sets such that $\cap\{V_i : i \in I\} = \emptyset$. We shall prove that there exists a finite subset I_0 of I such that $\cap\{V_i : i \in I\}$. By Theorem 3.3, this implies (i). Let $P(I)$ be the family of finite subsets of I. Assume that $\cap\{V_i : i \in J\} = \emptyset$ for every $J \in P(I)$ —-$(\star)$. Then the family $\mathcal{F} = \{\cap\{V_i : i \in J\} : J \in P(I)\}$ is a filterbase on X. By (iii), $\mathcal{F}$ b-accumulates to some point $x_0 \in X$. Since $\{X \backslash V_i : i \in I\}$ is a cover of X, there exists $i_0 \in I$ such that $x_0 \in X \backslash V_{i0}$. Then $X \backslash V_{i0}$ is a b-open set containing x_0, $V_{i0} \in \mathcal{F}$ and $(X \backslash V_{i0}) \cap V_{i0} = \emptyset$. This is a contradiction with the fact that $\mathcal{F}$ b-accumulates at x_0 shows that $(\star)$ is false. □

Definition 3.11. *A point x in a topological space X is said to be a b-complete accumulation point of a subset A of X if $n(S \cap A) = n(S)$ for each $A \in BO(X, x)$, where $n(S)$ denotes the cardinality of S.*

Example 3.12. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the point b is a b-complete accumulation point of a subset $\{a, b\}$ but a is not a b-complete accumulation point of it.*

Definition 3.13. *In a topological space (X, τ), a point x is said to be a b-adherent point of a filterbase $\mathcal{F}$ on X if it lies in the b-closure of all sets of $\mathcal{F}$.*

Example 3.14. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{F} = \{\{a\}, \{a, b\}, \{a, c\}, X\}$. Clearly, the point a is a b-adherent point of $\mathcal{F}$.*

Theorem 3.15. *A topological space* (X, τ) *is* b*-compact if and only if each infinite subset of* X *has a* b*-complete accumulation point.*

Proof: Let the topological space (X, τ) be b-compact and A an infinite subset of X. Let K be the set of all points x in X which are not b-complete accumulation points of S. Now it is obvious that for each point x in K, we are able to find $U(x) \in BO(X, x)$ such that $n(A \cap U(x)) \neq n(S)$. If K is the whole space X, then $\mathcal{F} = \{U(x)x \in X\}$ is a b-cover of X. By hypothesis, X is b-compact. So, there exists a finite subcover $\mathcal{G} = \{U(x_i) : i = 1, 2, ...n\}$, such that $A \subset \cup\{U(x_i) \cap A : i = 1, 2, ...n\}$. Then $n(S) = max\{n(U(x_i) \cap A) : i = 1, 2, ...n\}$ which does not agree with what we assumed. This implies that A has a b-complete accumulation point. Now assume that X is not b-compact and that every infinite subset A of X has a b-complete accumulation point in X. It follows that there exists a b-cover $\mathcal{S}$ with no finite subcover. Set $\alpha = min\{n(\Psi)$: $\Psi \subset \mathcal{S}$, where Ψ is a b-cover of X$\}$. Fix $\Psi = \mathcal{S}$ for which $n(\Psi) = \alpha$ and $\cup\{U : U \in \Psi\} = X$. Then, by hypothesis $\alpha \geq n(N)$, where N denotes the set of all natural numbers. By well-ordering of Ψ by some minimal well-ordering "$\sim$", suppose that U is any member of Ψ. By minimal well-ordering "$\sim$" we have $n(\{V : V \in \Psi, V \sim U\}) < n(\{V : V \in \Psi\})$. Since Ψ cannot have any subcover with cardinality less than α, then for each $U \in \Psi$ we have $X \neq \cup\{V; V \in \Psi, V \sim U\})$. For each $U \in \Psi$, choose a point $x(U) \in X \backslash \cup \{V \cup \{x(V)\}; V \in \Psi, V \sim U\})$. We are always able to do this if not one can choose a cover of smaller cardinality from Ψ. If $H = \{x(U) : U \in \Psi\}$, then to finish the proof we will show that H has no b-complete accumulation point in X. Suppose $z \in X$. Since Ψ is a b-cover of X, z is a point of some set, say W in Ψ. By the fact that $U \sim W$, we have $x(U) \in W$. It follows that $T = \{U : U \in \Psi$ and $x(U) \in W\} \subset \{V; V \in \Psi, V \sim W\}$. But $n(T) < \alpha$. Therefore, $n(H \cap W) < \alpha$. But $n(H) = \alpha \geq n(N)$. Since for two distinct points U and W in Ψ, we have $x(U) \neq x(W)$. This means that H has no b-complete accumulation point in X, which contradicts our assumption. Therefore X is b-compact. □

Theorem 3.16. *For a topological space* (X, τ)*, the following statements are equivalent:*

(i) X *is* b*-compact;*

(ii) Every net in X *with a well-ordered directed set as its domain* b*-accumulates to some point of* X.

Proof: $(i) \Rightarrow (ii)$: Suppose that X is b-compact and $A = \{x_\alpha : \alpha \in \Delta\}$ a net with a well-ordered directed set Δ as domain. Assume that A has no b-adherent point in X. Then for each $x \in X$, there exists $V(x) \in BO(X, x)$ and an $\alpha(x) \in \Delta$ such that $V(x) \cap \{x_\alpha : \alpha \geq \alpha(x)\} = \emptyset$. This implies that $\{x_\alpha : \alpha \geq \alpha(x)\}$ is a subset of $X \backslash V(x)$. Then the collection $\mathcal{F} = \{V(x) : x \in X\}$ is a b-cover of X. Since X is b-compact, $\mathcal{F}$ has a finite subfamily $\{V_{xi} : i = 1, 2, ...n\}$ such that $X = \bigcup\limits_{i=1}^{n}\{V(x_i) : i = 1, 2, ...n\}$. Suppose that the corresponding elements of Δ are $\{\alpha(x_i)\}$, where

$i = 1, 2, ...n$. Since Δ is well-ordered and $\{\alpha(x_i) : i = 1, 2, ...n\}$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_i)\}$. Then for $\beta \geq \{\alpha(x_i)\}$, we have $\{x_\delta : \delta \geq \beta\} \subset \bigcap_{i=1}^{n} \{X \backslash V(x_i)\} = X \backslash \bigcup_{i=1}^{n} V(x_i) = \emptyset$, which is impossible. This shows that A has atleast one b-adherent point in X. $(ii) \Rightarrow (i)$: Now it is enough to prove that each infinite subset has a b-complete accumulation point by utilizing Theorem 3.15. Suppose that S is an infinite subset of X. According to Zorn's Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well-ordered index set. It follows that S has a b-adherent point z. Therefore, z is a b-complete accumulation point of S. This shows that X is b-compact. □

Theorem 3.17. *A topological space X is b-compact if and only if each family of b-closed subsets of X with the finite intersection property has a nonempty intersection.*

Proof: Straightforward. □

Theorem 3.18. *A topological space X is b-compact if and only if each filterbase in X has at least one b-adherent point.*

Proof: Suppose that X is b-compact and $\mathcal{F} = \{F_\alpha : \alpha \in \Delta\}$ a filterbase in it. Since all finite intersections of $\mathcal{F}_\alpha$'s are nonempty, it follows that all finite intersection of $b\,\mathrm{Cl}(F_\alpha)$'s are also nonempty. Now it follows from Theorem 3.16 that $\bigcap_{\alpha \in \Delta} b\,\mathrm{Cl}(F_\alpha) \neq \emptyset$. This implies that $\mathcal{F}$ has at least one b-adherent point. Now suppose $\mathcal{F}$ is a family of b-closed sets. Let each finite intersection be nonempty. The sets F_α with their finite intersection establish a filterbase $\mathcal{F}$. Therefore, $\mathcal{F}$ b-accumulates to some point $z \in X$. It follows that $z \in \bigcap_{\alpha \in \Delta} F_\alpha$. Now we have by Theorem 3.16 X is b-compact. □

Theorem 3.19. *A topological space X is b-compact if and only if each filterbase on X with at least one b-adherent point is b-convergent.*

Proof: Suppose that X is b-compact, $x \in X$ and $\mathcal{F}$ is a filterbase on X. The b-adherence of $\mathcal{F}$ is a subset of $\{x\}$. Then the b-adherence of $\mathcal{F}$ is equal to $\{x\}$ by Theorem 3.16. Assume that there exists $V \in BO(X, x)$ such that for all $F \in \mathcal{F}$, $F \cap (X \backslash V) \neq \emptyset$. Then $\Psi = \{F \backslash V : F \in \mathcal{F}\}$ is a filterbase on X. It follows that the b-adherence of Ψ is nonempty. However, $\bigcap_{F \in \mathcal{F}} b\,\mathrm{Cl}(F \backslash V) = (\bigcap_{F \in \mathcal{F}} b\,\mathrm{Cl}(F)) \cap (X \backslash V) = \{x\} \cap (X \backslash V) = \emptyset$, a contradiction. Hence for each $V \in BO(X, x)$, there exists an $F \in \mathcal{F}$ with $F \subset V$. This shows that $\mathcal{F}$ b-converges to x. To prove the converse, it suffices to show that each filterbase in X has at least one b-accumulation point. Assume that $\mathcal{F}$ is a filterbase on X with no b-adherent point. By hypothesis, $\mathcal{F}$ b-converges to some point $z \in X$. Suppose F_α is an arbitrary element of $\mathcal{F}$. Then for each $V \in BO(X, x)$, there exists $F_\beta \in \mathcal{F}$ such that $F_\beta \subset V$. Since $\mathcal{F}$ is a filterbase, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where $F_\alpha \neq \emptyset$. This means that $F\alpha \cap V \neq \emptyset$ for every $V \in BO(X, x)$ and corresponding for each α, z is a point of $b\,\mathrm{Cl}(F_\alpha)$. It follows that $z \in \bigcap\limits_\alpha b\,\mathrm{Cl}(F_\alpha)$. Therefore, z is a b-adherent point of $\mathcal{F}$, a contradiction. This shows that X is b-compact. □

Definition 3.20. *A subset K of a topological space X is said to be b-closed [5] relative to X if for any cover $\{U_i : i \in I\}$ of K by b-open sets, there exists a finite subset I_0 of I such that $K \subset \cup\{b\,\mathrm{Cl}(U_i) : i \in I_0\}$.*
We say that X is b-closed if X is b-closed relative to X. Obviously every set which is b-compact relative to X is also b-closed relative to X.

Definition 3.21. *Let X be a topological space. A point $x \in X$ is said to be b-θ-cluster point of a net $\{x_\alpha\}_{\alpha\in\Delta}$ if $\{x_\alpha\}_{\alpha\in\Delta}$ is frequently in the b-closure of every b-open set containing x.*

Example 3.22. *Let $(\Re, \tau_u)$ be the usual topological space. Then the net $(S_n)_{n\in N} = (n + (-1)^n n)_{n\in N}$ in $\Re$ has 0 as a b-θ-cluster point.*

Theorem 3.23. *A topological space X is b-compact if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X, has at least one b-cluster point.*

Proof: Similar to the proof of Theorem 3.6 □

Definition 3.24. *A filterbase $\mathcal{F}$ on a topological space X is said to:*

(i) b-θ-converge to a point $x \in X$ if for each b-open set U containing x, there exists $B \in \mathcal{F}$ such that $B \subset b\,\mathrm{Cl}(U)$.

(ii) b-θ-accumulate at $x \in X$ if $b\,\mathrm{Cl}(U) \cap B \neq \emptyset$ for each b-open set U containing x and every $B \in \mathcal{F}$.

Remark 3.25. *A filterbase $\mathcal{F}$ is b-θ-convergent to a point $x \in X$ if and only if $\mathcal{F}$ contains the collection $\{b\,\mathrm{Cl}(U) : U \in BO(X, x)\}$.*

Theorem 3.26. *For a topological space X, the following statements are equivalent:*

(i) X is b-closed;

(ii) Every maximal filterbase b-θ-converges to some point of X;

(iii) Every filterbase b-θ-accumulates to some point of X;

(iv) For every family $\{V_i : i \in I\}$ of b-closed sets that $\cap\{V_i : i \in I\} = \emptyset$, there exists a finite subset I_0 of I such that $\cap\{b\,\mathrm{Int}(V_i) : i \in I_0\} = \emptyset$.

Proof: $(i) \Rightarrow (ii)$: Let $\mathcal{F}$ be a maximal filterbase on X. Suppose that $\mathcal{F}$ does not b-θ-converge to any point of X. Then by Theorem 3.23 $\mathcal{F}$ does not b-θ-accumulates at any point of X. For each $x \in X$, there exist a b-open set U_x containing x and $B_x \in \mathcal{F}$ such that $b\,\mathrm{Cl}(U_x) \cap B_x = \emptyset$. The family $\{U_x : x \in X\}$ is a cover of X by

b-open sets. By (i), there exists a finite subset $\{x_1, x_2, ...x_n\}$ of X such that $X = \cup\{b\,\mathrm{Cl}(U_{xk}) : k = 1, 2, ...n\}$. Since $\mathcal{F}$ is a filterbase, there exists $F \in \mathcal{F}$ such that $\mathcal{F}_0 \subset \cap\{B_{xk} : k = 1, 2, ...n\}$. It follows that $\mathcal{F}_0 \subset \cap\{X\backslash b\,\mathrm{Cl}(U_{x_k} : k = 1, 2, ...n\}$ $= X\backslash \cup \{b\,\mathrm{Cl}(U_)x_k : k = 1, 2, ...n\} = \emptyset$, hence $F = \emptyset$. This is a contradiction with the definition of filterbase. $(ii) \Rightarrow (iii)$: Let $\mathcal{F}_0$ be a filterbase on X. There exists a maximal filterbase $\mathcal{F}$ such that $\mathcal{F}_0 \subset \mathcal{F}$. By (ii), $\mathcal{F}$ b-θ-converges to some point $x_0 \in X$. Let $F \in \mathcal{F}_0$. For every $U \in BO(X, x_0)$, there exists $B_U \in \mathcal{F}$ such that $B_U \subset b\,\mathrm{Cl}(U)$, hence $b\,\mathrm{Cl}(U) \cap B \supset B_U \cap B$ is nonempty, since it contains a member of $\mathcal{F}$. This shows that $\mathcal{F}_0$ b-θ-accumulates at x_0. $(iii) \Rightarrow (iv)$: Let $\{V_i : i \in I\}$ be any family of b-closed set such that $\cap\{V_i : i \in I\} = \emptyset$. Let $P(I)$ be the family of all finite subsets of I. Assume that $\cap\{b\,\mathrm{Int}(V_i) : i \in J\} = \emptyset$ for every $J \in P(I)$ — (**). Then, the family $\mathcal{F} = \{\cap\{b\,\mathrm{Int}(V_i) : i \in I\} : J \in \mathcal{I}\}$ is a filterbase on X. By (iii), $\mathcal{F}$ b-accumulates at some point $x \in X$. Since $\{X\backslash V_i : i \in I\}$ is a cover of X, there exists $i_0 \in I$ such that $x \in X\backslash V_{i_0}$. Then $X\backslash V_{i_0}$ is a b-open set containing x, $b\,\mathrm{Int}(V_{i_0}) \in \mathcal{F}$ and $b\,\mathrm{Cl}(X\backslash V_{i_0}) \cap b\,\mathrm{Int}(V_{i_0}) = \emptyset$. This contradicts the fact that that $\mathcal{F}$ b-accumulates at x. It follows that (**) is false. $(iv) \Rightarrow (i)$: Let $\{U_i : i \in I\}$ be a cover of X by b-open sets. Denote $V_i = X\backslash U_i$. Then $\{V_i : i \in I\}$ is a family of b-closed sets such that $\cap\{V_i : i \in I\} = \emptyset$. There exists a finite subset $\mathcal{I}_0$ of $\mathcal{I}$ such that $\cap\{b\,\mathrm{Int}(V_i) : i \in I\} = \emptyset$. Then $X = X\backslash \cap \{b\,\mathrm{Int}(V_i) : i \in I_0\} = \cup\{X\backslash b\,\mathrm{Int}(V_i) : i \in I_0\} = \cup\{b\,\mathrm{Cl}(U_i) : i \in I_0\}$. □

Theorem 3.27. *A topological space X is b-closed if and only if each net $\{x_\alpha\}_{\alpha\in\Delta}$ in X has at least one b-θ-cluster point.*

Proof: Similar to the proof of Theorem 3.6. □

Definition 3.28. *If $\mathcal{F}$ is a filterbase on a topological space X, then the section of $\mathcal{F}$, denoted by $sec\mathcal{F}$, is given by $sec\mathcal{F} = \{A \subset X : A \cap G \neq \emptyset$, for all $G \in \mathcal{F}\}$.*

Theorem 3.29. *If a filterbase $\mathcal{F}$ on a topological space X, b-θ-adheres to some point $x \in X$, then $\mathcal{F}$ is b-θ-convergent to x.*

Proof: Let a filterbase $\mathcal{F}$ on X, b-θ-adhere at $x \in X$. Then for each $U \in BO(X, x)$ and each $G \in \mathcal{F}$, $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$ so that $b\,\mathrm{Cl}(U) \in sec\mathcal{F}$, for each $U \in BO(X, x)$, and hence $X\backslash b\,\mathrm{Cl}(U) \notin \mathcal{F}$. Then $b\,\mathrm{Cl}(U) \in \mathcal{F}$ (as $\mathcal{F}$ is a filterbase and $X \in \mathcal{F}$), for each $U \in BO(X, x)$. Hence $\mathcal{F}$ must b-θconverge to x. □

The following example shows that the converse of the Theorem 3.29 is not true in general.

Example 3.30. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{F} = P(X)\backslash\{\emptyset\}$. Then $\mathcal{F}$ is b-θ-convergent to a but not b-θ-adheres to a.*

Remark 3.31. *Let X be a topological space. Then for any $x \in X$, we adopt the following notions:*

(i) $\mathcal{F}(b_\theta, x) = \{A \subset X : x \in b\,\mathrm{Cl}_\theta(A)\}$.

(ii) $sec\mathcal{F}(b_\theta, x) = \{A \subset X : A \cap G \neq \emptyset, forall G \in \mathcal{F}(b_\theta, x)\}$

In the next two theorems, we characterize the b-θ-adherence and b-θ-convergence of filterbase in terms of the above notions.

Theorem 3.32. *The filterbase $\mathcal{F}$ on a topological space X, b-θ-adheres to a point $x \in X$ if and only if $\mathcal{F} \subset \mathcal{F}(b_\theta, x)$.*

Proof: A filterbase $\mathcal{F}$ on a topological space X b-θ-converges at $x \in X$. $\Rightarrow b\,\mathrm{Cl}(U) \cap G \neq \emptyset$ for all $U \in BO(X, x)$ and all $G \in \mathcal{F}$.
$\Rightarrow x \in b\,\mathrm{Cl}_\theta$ for all $G \in \mathcal{F}$.
$\Rightarrow G \in \mathcal{F}(b_\theta, x)$ for all $G \in \mathcal{F}$.
$\Rightarrow \mathcal{F} \subset \mathcal{F}(b_\theta, x)$.
Conversely, let $\mathcal{F} \subset \mathcal{F}(b_\theta, x)$. Then for all $G \in \mathcal{F}$, $x \in b\,\mathrm{Cl}_\theta$, so that for all $U \in BO(X)$ and for all $G \in \mathcal{F}$, $b\,\mathrm{Cl}(U) \cap G \neq \emptyset$. Hence, $\mathcal{F}$ b-θ-adheres to a point x. □

Theorem 3.33. *A filterbase $\mathcal{F}$ on a topological space X is b-θ-convergent to a point x of X if and only if $sec\mathcal{F}(b_\theta, x) \subset \mathcal{F}$.*

Proof: Let $\mathcal{F}$ be a filterbase on X, b-θ-converging to $x \in X$. Then for each $U \in BO(X)$ there exist $G \in \mathcal{F}$ such that $G \subset b\,\mathrm{Cl}(U)$; hence $b\,\mathrm{Cl}(U) \in \mathcal{F}$ for each $U \in BO(X)$ $(\star)$. Now $B \in sec\mathcal{F}(b_\theta, x) \Rightarrow X \backslash B \notin sec\mathcal{F}(b_\theta, x) \Rightarrow x \notin b\,\mathrm{Cl}_\theta(X \backslash B) \Rightarrow$ there exists $U \in BO(X)$ such that $b\,\mathrm{Cl}(U) \cap (X \backslash B) = \emptyset \Rightarrow b\,\mathrm{Cl}(U) \subset B$, where $U \in BO(X) \Rightarrow B \in \mathcal{F}$ (by $(\star)$). Conversely, let if possible, $\mathcal{F}$ not belong to b-θ-converge to x. Then for some $U \in BO(X)$, $b\,\mathrm{Cl}(U) \notin \mathcal{F}$ and hence $b\,\mathrm{Cl}(U) \notin sec\mathcal{F}(b_\theta, x)$. Thus for some $A \in \mathcal{F}(b_\theta, x)$, $A \cap b\,\mathrm{Cl}(U) = \emptyset$ $(\star\star)$. But $A \in \mathcal{F}(b_\theta, x) \Rightarrow x \in b\,\mathrm{Cl}_\theta(A)$ $\Rightarrow b\,\mathrm{Cl}(U) \cap A \neq \emptyset$, contradiction $(\star\star)$. □

We shall now derive some new characterizations of b-closedness in terms of filterbase and the associated concepts.

Theorem 3.34. *A subset A of a topological space X is b-closed relative to X if and only if every filterbase $\mathcal{F}$ on X with $A \in \mathcal{F}$, b-θ-converges to a point in A.*

Proof: Let A be b-closed relative to X and $\mathcal{F}$ a filterbase on X satisfying $A \in \mathcal{F}$ such that $\mathcal{F}$ does not b-θ-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_\alpha \in BO(X, a)$ such that $b\,\mathrm{Cl}(U_\alpha) \notin \mathcal{F}$. Now $\{U_a : a \in A\}$ is a cover of A by b-open sets of X. Then $A \subset \overset{n}{\underset{i=1}{\cup}} b\,\mathrm{Cl}(U_{ai}) = U$ (say) for some positive integer n. Since $\mathcal{F}$ is a filterbase, $U \in \mathcal{F}$ and hence $A \in \mathcal{F}$, which is a contradiction. Conversely, let A be not b-closed relative to X. Then for some cover $U = \{U_\alpha : \alpha \in \Lambda\}$ of A by b-open sets of X, $\mathcal{F} = \{A \backslash U_{\alpha \in \Lambda_0} : \Lambda_0 \subset \Lambda\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultra-filter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a filterbase on X with $A \in \mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each $x \in A$, there exist $\beta \in \Lambda$ such that $x \in U_\beta$, as U is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A \backslash b\,\mathrm{Cl}(U_\beta) \neq \emptyset$, so that $G \nsubseteq b\,\mathrm{Cl}(U_\beta)$ for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot b-θ-converge to any point of A. This contradiction proves the desired result. □

Theorem 3.35. *Let X be any topological space such that every filterbase $\mathcal{F}$ on X, with the property that $\overset{n}{\underset{i=1}{\cap}} b\,\mathrm{Cl}_\theta(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ...G_n\}$ of $\mathcal{F}$ b-θ-adheres in X, then X is a b-closed space.*

Proof: Let U be any ultrafilter on X. Then $\mathcal{U}$ is a filterbase on X and also for each finite subcollection $\{U_1, U_2, ...U_n\}$ of $\mathcal{U}$, $\overset{n}{\underset{i=1}{\cap}} b\,\mathrm{Cl}_\theta(U_i) \supset \overset{n}{\underset{i=1}{\cap}} U_i \neq \emptyset$ so that $\mathcal{U}$ is a filterbase on X with the given condition. Hence by hypothesis, $\mathcal{U}$ b-θ-adheres. Consequently, the space X is b-closed. □

Definition 3.36. *A filterbase $\mathcal{F}$ on a topological space X is said to be b-θ-conjoint if for every finite subfamily $A_1, A_2, ..., A_n$ of $\mathcal{F}$ $b\,\mathrm{Int}(\overset{n}{\underset{i=1}{\cap}} b\,\mathrm{Cl}_\theta(A_i) \neq \emptyset$.*

Theorem 3.37. *In a b-closed topological space X, every b-θ-conjoint filterbase b-θ-adheres in X.*

Proof: Consider any b-θ-conjoint filterbase $\mathcal{F}$ on a b -closed space X. We first note from Theorem 3.33 that for $A \subset X$, $b\,\mathrm{Cl}_\theta(A)$ is b-closed. Thus $\{b\,\mathrm{Cl}_\theta(A) : A \in \mathcal{F}\}$ is a collection of b-closed sets in X such that $b\,\mathrm{Int}(\overset{n}{\underset{i=1}{\cap}} b\,\mathrm{Cl}_\theta(A_i)) \neq \emptyset$ for any finite subcollection $A_1, A_2....A_n$ of $\mathcal{F}$. Thus by Theorem 3.34, $\overset{n}{\underset{i=1}{\cap}}\{b\,\mathrm{Cl}_\theta(A) : A \in \mathcal{F}\} \neq \emptyset$, that is, there exists $x \in X$ such that $x \in b\,\mathrm{Cl}_\theta(A)$ for all $A \in \mathcal{F}$. Hence $\mathcal{F} \subset \mathcal{F}(b_\theta, x)$ so that by Theorem 3.32, $\mathcal{F}$ b-θ-adheres at $x \in X$. □

The following example shows that the converse of the Theorem 3.37 is not true in general.

Example 3.38. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\mathcal{F} = \{c\}, \{a, c\}, \{b, c\}, X\}$. Clearly, $\mathcal{F}$ is b-θ-adheres at c but not b-θ-conjoint.*

Theorem 3.39. *Every filterbase $\mathcal{F}$ on a topological space X with the property that $\{b\,\mathrm{Cl}_\theta(G) : G \in \mathcal{F}_0\} \neq \emptyset$ for every finite subset $\mathcal{F}_0$ of $\mathcal{F}$, b-θ-adheres in X if and only if for every family $\mathcal{F}$ of subsets of X for which the family $\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property, we have $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\} \neq \emptyset$.*

Proof: Let every filterbase on a topological space X satisfying the given condition, b-θ-adheres in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property. Let U be the collection of all those families of subsets of X for which $\mathcal{F}^* = \{b\,\mathrm{Cl}_\theta(G) : G \in \mathcal{F}\}$ has the finite intersection property and $\mathcal{F} \subset \mathcal{G}$. Then $\mathcal{F} \in \mathcal{U}$ and $\mathcal{U}$ is a partially ordered set under inclusion in which every chain clearly has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a filterbase with the stipulated property. Hence $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\} \supset \cap\{b\,\mathrm{Cl}_\theta(U) : U \in \mathcal{U} \neq \emptyset$. Conversely, if $\mathcal{F}$ is a filterbase on X with the given property, then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\cap\{b\,\mathrm{Cl}_\theta(F) : \mathcal{F} \in \mathcal{F}_0\} \neq \emptyset$. So, by hypothesis, $\cap\{b\,\mathrm{Cl}_\theta(F) : F \in \mathcal{F}\} \neq \emptyset$. Hence $\mathcal{F}$ b-θ-adheres in X. □

Theorem 3.40. *[5] For a topological space X, the following statements are equivalent:*

(i) X is b-regular;

(ii) For each and each open set U of X containing x, there exists a b-open set V such that.

Theorem 3.41. *Let X be a b-regular space. Then a subset K of X is b-compact if and only if K is b-closed relative to X.*

Proof: Let $\{U_i : i \in I\}$ be a cover of K by b-open sets. For each $x \in K$ there exists $i(x) \in I$ such that $x \in U_{i(x)}$ and by the assumption that X is b-regular, there exists, according to Theorem 3.40, a b-open set V_x such that $x \in V_x \subset b\,\mathrm{Cl}(V_x) \subset U_{i(x)}$. The family $\{V_x : x \in X\}$ is a cover of K by b-open sets. Since K is b-closed relative to X, there exists a finite subset $\{x_1, x_2, ...x_n\}$ of K such that $K \subset \cup\{b\,\mathrm{Cl}(V_{xk}) : k = 1, 2, ...n\}$. Which shows that K is b-compact relative to X. The necessity is obvious. □

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