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On sufficient conditions of meromorphic starlike functions

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ABSTRACT: In this paper, we investigate interesting properties and sufficient conditions for meromorphic starlike functions in the punctured unit disc.

Key Words: Analytic functions, Meromorphic functions, Meromorphic starlike functions,Linear operator, Hadamard product(or Convolution).

Contents

1. Introduction

Let $\mathcal{M}$ denote the class of functions $f(z)$ of the form

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k\ z^k, \tag{1.1}$$

which are analytic in the punctured open unit disc $E^*=\{z : z\in\mathbb{C} \text{ and } 0<|z|<1\}$ $=E\backslash\{0\}$.

If $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\ b_kz^k\ , \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f\star g)(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kb_kz^k=(g\star f)(z)\quad (z\in E). \tag{1.3}$$

A function $f\in\mathcal{M}$ is said to be meromorphic starlike of order α $(0\le\alpha<1)$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>-\alpha\quad (z\in E^*).$$

We denote by $\mathcal{MS}(\alpha)$, the class of all such functions. A function $f\in\mathcal{M}$ is said to be meromorphic convex of order α $(0\le\alpha<1)$ if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>-\alpha\quad (z\in E^*).$$

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We denote by $\mathcal{MC}(\alpha)$, the class of all such functions.

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Rushcheweyh derivative [12], the Carlson-Shaffer operator [1], the Dzoik-Srivastava operator [4], the Noor integral operator [11] also see, [3,6,7,10]. Motivated by the work of N. E. Cho and K. I. Noor [2,9], we introduce a family of integral operators defined on the space of meromorphic functions in the class $\mathcal{M}$,see [13]. By using these integral operators, we derive some interesting properties of meromorphic starlike functions classes introduced here.

For a complex parameters $\alpha_1, ...\alpha_q$ and $\beta_1, ...\beta_s$ $(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^- = \{0, -1, -2, ...\};$ $j = 1, ...s)$, we now define the function $\phi(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z)$ by

$$\begin{aligned}\phi(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) &= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}...(\beta_s)_{k+1} \quad (k+1)!} z^k, \\ (q &\leq s+1;\ s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \mathbb{N} = \{1, 2, ...\};\ z \in E),\end{aligned}$$

where $(v)_k$ is the Pochhammer symbol(or shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } v \in \mathbb{C}\backslash\{0\} \\ v(v+1)...(v+k-1) & \text{if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

Now we introduce the following operator

$$I_\mu^p(\alpha_1, ...\alpha_q, \beta_1, ...\beta_s) : \mathcal{M} \longrightarrow \mathcal{M}$$

as follows:

Let $F_{\mu,p}(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\mu+1}{\mu}\right)^p z^k, p \in \mathbb{N}_0, \mu \neq 0$ and let $F_{\mu,p}^{-1}(z)$ be defined such that

$$F_{\mu,p}(z) * F_{\mu,p}^{-1}(z) = \phi(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z).$$

Then

$$I_\mu^p(\alpha_1, ...\alpha_q, \beta_1, ...\beta_s)f(z) = F_{\mu,p}^{-1}(z) * f(z). \tag{1.4}$$

From (1.4) it can be easily seen

$$\begin{aligned}&I_\mu^p(\alpha_1, ...\alpha_q, \beta_1, ...\beta_s)f(z) \\ = \;& \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^p \frac{(\alpha_1)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}...(\beta_s)_{k+1} \quad (k+1)!} z^k.\end{aligned} \tag{1.5}$$

For conveniences, we shall henceforth denote

$$I_\mu^p(\alpha_1, ...\alpha_q, \beta_1, ...\beta_s)f(z) = I_\mu^p(\alpha_1, \beta_1)f(z). \tag{1.6}$$

For the choices of the parameters $p = 0, q = 2,\ s = 1$, the operator $I_\mu^p(\alpha_1, \beta_1)f(z)$ is reduced to an operator by N. E. Cho and K. I. Noor [2] and K. I. Noor [9] and when $p = 0$, $q = 2$, $s = 1, \alpha_1 = \lambda$, $\alpha_2 = 1, \beta_1 = (n+1)$, the operator $I_\mu^p(\alpha_1, \beta_1)f(z)$ is reduced to an operator recently introduced by S. -M. Yuan et. al. in [14].

It can be easily verified from the above definition of the operator $I_\mu^p(\alpha_1, \beta_1)f(z)$ that

$$z(I_\mu^{p+1}(\alpha_1, \beta_1)f(z))' = \mu I_\mu^p(\alpha_1, \beta_1)f(z) - (\mu + 1)I_\mu^{p+1}(\alpha_1, \beta_1)f(z), \tag{1.7}$$

and

$$z(I_\mu^p(\alpha_1, \beta_1)f(z))' = \alpha_1 I_\mu^p(\alpha_1 + 1, \beta_1)f(z) - (\alpha_1 + 1)I_\mu^p(\alpha_1, \beta_1)f(z). \tag{1.8}$$

By using the operator $I_\mu^p(\alpha_1, \beta_1)f(z)$, we now studies some properties of meromorphic starlike functions. Also, see the interested work by R. M. El-Ashwa et. al [5]

Definition 1.1. *Let $f \in \mathcal{M}$, $0 \le \alpha < 1$, $z \in E^*$. Then*

$$f \in \mathcal{M}S_\mu^p(\alpha_1, \beta_1), \text{ if and only if } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{MS}(\alpha).$$

Also,

$$f \in \mathcal{M}C_\mu^p(\alpha_1, \beta_1), \text{ if and only if } I_\mu^p(\alpha_1, \beta_1)f \in \mathcal{MC}(\alpha), \ z \in E^*.$$

We note that for $z \in E^*$,

$$f \in \mathcal{M}C_\mu^p(\alpha_1, \beta_1) \Leftrightarrow -zf' \in \mathcal{M}S_\mu^p(\alpha_1, \beta_1).$$

Lemma 1.2. bf(Jack [8]).

Suppose $w(z)$ be a nonconstant analytic functions in E with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in E$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = \zeta w(z_0)$, where $\zeta \ge 1$ is some real number.

2. Main Results

Theorem 2.1. *If $f \in \mathcal{M}$ satisfies*

$$\left| \frac{I_\mu^p(\alpha_1, \beta_1)f(z)}{I_\mu^{p+1}(\alpha_1, \beta_1)f(z)} - 1 \right|^\gamma \left| \frac{I_\mu^{p-1}(\alpha_1, \beta_1)f(z)}{I_\mu^p(\alpha_1, \beta_1)f(z)} - 1 \right|^\beta < \Psi(p, \mu, \alpha_1, \beta_1, \alpha, \beta, \gamma), z \in E^*, \tag{2.1}$$

for some real numbers α, β and γ such that $0 \le \alpha < 1$, $\beta \ge 0$, $\gamma \ge 0$, and $\beta + \gamma > 0$, then

$$f \in \mathcal{M}S_\mu^p(\alpha_1, \beta_1, \alpha) \quad p \in \mathbb{N}_0,$$

where

$$\Psi(p, \mu, \alpha_1, \beta_1, \alpha, \beta, \gamma) = \begin{cases} (1-\alpha)^\gamma \left(1 - \alpha + \frac{1}{2\mu}\right)^\beta, & 0 \le \alpha < \frac{1}{2} \\ (1-\alpha)^{\beta+\gamma} (1 + \frac{1}{\mu})^\beta, & \frac{1}{2} \le \alpha < 1. \end{cases} \tag{2.2}$$

Proof:

Case (i). Let $0 \leq \alpha < \frac{1}{2}$ and by setting

$$\frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} = \frac{1+(1-2\alpha)\,w(z)}{1-w(z)}, \quad z \in E. \tag{2.3}$$

Then w is analytic in E, $w(0)=0$ and $w(z) \neq 1$ in E. A simple computation yields that

$$\frac{z(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))'}{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))} - \frac{z(I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z))'}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} = \frac{(1-2\alpha)\,zw'(z)}{[1+(1-2\alpha)\,w(z)]} + \frac{zw'(z)}{1-w(z)}. \tag{2.4}$$

By making use of the identity (1.7), we deduce that

$$\begin{aligned} &\mu\,\frac{(I_{\mu}^{p-1}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))} - \mu\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} \\ &= \left\{\frac{(1-w(z)\,(1-2\alpha)\,zw'(z) + [1+(1-2\alpha)\,w(z)]\,zw'(z)}{[1+(1-2\alpha)\,w(z)]\,(1-w(z))}\right\}, \end{aligned}$$

$$\begin{aligned} \frac{(I_{\mu}^{p-1}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))} &= \frac{[1+(1-2\alpha)\,w(z)]}{1-w(z)} \\ &\quad + \frac{2\,(1-\alpha)\,zw'(z)}{\mu\,[1+(1-2\alpha)\,w(z)]\,(1-w(z))} \\ \frac{(I_{\mu}^{p-1}(\alpha_1,\beta_1)f(z))}{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))} - 1 &= \frac{2\,(1-\alpha)\,w(z)}{1-w(z)} + \frac{2\,(1-\alpha)\,zw'(z)}{\mu\,[1+(1-2\alpha)\,w(z)]\,(1-w(z))} \end{aligned} \tag{2.5}$$

and

$$\frac{(I_{\mu}^{p}(\alpha_1,\beta_1)f(z))}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} - 1 = \frac{2\,(1-\alpha)\,w(z)}{1-w(z)}.$$

Thus, we have

$$\begin{aligned} &\left|\frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)} - 1\right|^{\gamma} \left|\frac{I_{\mu}^{p-1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)} - 1\right|^{\beta} \\ &= \left|\frac{2\,(1-\alpha)\,w(z)}{1-w(z)}\right|^{\gamma} \left|\frac{2\,(1-\alpha)\,w(z)}{1-w(z)} + \frac{2\,(1-\alpha)\,zw'(z)}{\mu\,[1+(1-2\alpha)\,w(z)]\,(1-w(z))}\right|^{\beta} \\ &= \left|\frac{2\,(\alpha-1)\,w(z)}{1-w(z)}\right|^{\beta+\gamma} \left|1 + \frac{zw'(z)}{\mu\,[1+(1-2\alpha)\,w(z)]\,(w(z))}\right|^{\beta}. \end{aligned}$$

Suppose that there exists a point $z_0 \in E$ such that $\max\limits_{|z|\leq|z_0|} |w(z)| = |w(z_0)| = 1$. Then by using Lemma 1.2, we have $w(z_0) = e^{i\theta}$, $0 < \theta \leq 2\pi$ and $z_0 w'(z_0) = \zeta w(z_0)$,

$\zeta \geq 1$. Therefore

$$\begin{aligned}\left|\frac{I_\mu^p(\alpha_1,\beta_1)f(z_0)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z_0)}-1\right|^\gamma \left|\frac{I_\mu^{p-1}(\alpha_1,\beta_1)f(z_0)}{I_\mu^{p}(\alpha_1,\beta_1)f(z_0)}-1\right|^\beta &= \left|\frac{2\,(1-\alpha)\,w(z_0)}{1-w(z_0)}\right|^{\beta+\gamma} \\ &\quad \left|1+\frac{\zeta w'(z_0)}{\mu\,[1+(1-2\alpha)\,w(z_0)]\,(w(z_0))}\right|^\beta \\ &= \frac{2^{\beta+\gamma}\,(1-\alpha)^{\beta+\gamma}}{|1-e^{i\theta}|^{\beta+\gamma}}\left|1+\frac{\zeta}{\mu\,[[1+(1-2\alpha)\,e^{i\theta}]\,e^{i\theta}]}\right|^\beta \\ &\geq (1-\alpha)^{\beta+\gamma}\left(1+\frac{\zeta}{[2\mu\,(\alpha-1)]}\right)^\beta \\ &\geq (1-\alpha)^{\beta+\gamma}\left(1+\frac{1}{[2\mu\,(1-\alpha)]}\right)^\beta \\ &= (1-\alpha)^\gamma\left(1-\alpha+\frac{1}{2\mu}\right)^\beta,\end{aligned}$$

which contradicts (2.1) for $0 \leq \alpha < \frac{1}{2}$. Therefore, we must have $|w(z)| < 1$ for all $z \in E$, and hence $f \in \mathcal{MS}_\mu^p(\alpha_1,\beta_1,\alpha)$ $p \in \mathbb{N}_0$.

Case (ii). When $\frac{1}{2} < \alpha < 1$. Let $w(z)$ be defined by

$$\frac{I_\mu^p(\alpha_1,\beta_1)f(z)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)} = \frac{\alpha}{\alpha-(1-\alpha)\,w(z)},\ z \in E,$$

where $w(z) \neq \frac{\alpha}{(1-\alpha)}$ in E. Then $w(z)$ is analytic in E and $w(0) = 0$. Proceeding likewise as in Case (i) and using identity (1.7), we obtain

$$\frac{z(I_\mu^p(\alpha_1,\beta_1)f(z))'}{(I_\mu^p(\alpha_1,\beta_1)f(z))}-\frac{z(I_\mu^{p+1}(\alpha_1,\beta_1)f(z))'}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)} = \frac{(1-\alpha)\,zw'(z)}{[\alpha-(1-\alpha)\,w(z)]}.$$

$$\mu\frac{(I_\mu^{p-1}(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)f(z))}-\mu\frac{(I_\mu^{p}(\alpha_1,\beta_1)f(z))}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)} = \frac{(1-\alpha)\,zw'(z)}{[\alpha-(1-\alpha)\,w(z)]}.$$

implies that

$$\frac{(I_\mu^{p-1}(\alpha_1,\beta_1)f(z))}{(I_\mu^p(\alpha_1,\beta_1)f(z))} = \frac{(1-\alpha)\,w(z)}{\alpha-(1-\alpha)\,w(z)}+\frac{(1-\alpha)\,zw'(z)}{\mu\,[\alpha-(1-\alpha)\,w(z)]}.$$

Then, we have

$$\begin{aligned}
&\left|\frac{I_\mu^p(\alpha_1,\beta_1)f(z_0)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z_0)}-1\right|^\gamma \left|\frac{I_\mu^{p-1}(\alpha_1,\beta_1)f(z_0)}{I_\mu^p(\alpha_1,\beta_1)f(z_0)}-1\right|^\beta \\
=\;& \left|\frac{(1-\alpha)\,w(z)}{\alpha-(1-\alpha)\,w(z)}\right|^\gamma \left|\frac{(1-\alpha)\,w(z)}{\alpha-(1-\alpha)\,w(z)}+\frac{(1-\alpha)\,zw'(z)}{\mu\left[\alpha-(1-\alpha)\,w(z)\right]}\right|^\beta \\
=\;& \left|\frac{(1-\alpha)\,w(z)}{\alpha-(1-\alpha)\,w(z)}\right|^{\beta+\gamma} \left|1+\frac{zw'(z)}{\mu w(z)}\right|^\beta \\
=\;& \left|\frac{(1-\alpha)\,w(z)}{\alpha-(1-\alpha)\,w(z)}\right|^{\beta+\gamma} |w(z)|^\gamma \left|w(z)+\frac{zw'(z)}{\mu}\right|^\beta .
\end{aligned}$$

Suppose that there exists a point $z_0 \in E$ such that $\max\limits_{|z|\leq|z_0|} w(z) = |w(z_0)| = 1$, then by applying Lemma 1.2, we have $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = \zeta w(z_0)$, $\zeta \geq 1$. Therefore,

$$\begin{aligned}
\left|\frac{I_\mu^p(\alpha_1,\beta_1)f(z_0)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z_0)}-1\right|^\gamma \left|\frac{I_\mu^{p-1}(\alpha_1,\beta_1)f(z_0)}{I_\mu^p(\alpha_1,\beta_1)f(z_0)}-1\right|^\beta &= \left|\frac{(1-\alpha)\,w(z_0)}{\alpha-(1-\alpha)\,w(z_0)}\right|^{\beta+\gamma} \\
&\quad \left|1+\frac{\zeta w(z_0)}{\mu w(z_0)}\right|^\beta \\
&= \frac{(1-\alpha)^{\beta+\gamma}}{\left|\alpha-(1-\alpha)\,e^{i\theta}\right|^{\beta+\gamma}}\left(1+\frac{\zeta}{\mu}\right)^\beta \\
&\geq (1-\alpha)^{\beta+\gamma}\,(1+\frac{1}{\mu})^\beta
\end{aligned}$$

which contradicts (2.1) for $\frac{p}{2} < \alpha < p$. Therefore, we must have $|w(z)| < 1$ for all $z \in E$, and hence $f \in \mathcal{MS}_\mu^p(\alpha_1,\beta_1,\alpha)$ $\;p \in \mathbb{N}_0$. This completes the proof of our Theorem. □

Corollary 2.2. *Taking $\mu = 1$ in Theorem 3.1, we have*

$$\begin{aligned}
&\left|\frac{I_1^p(\alpha_1,\beta_1)f(z)}{I_1^{p+1}(\alpha_1,\beta_1)f(z)}-1\right|^\gamma \left|\frac{I_1^{p-1}(\alpha_1,\beta_1)f(z)}{I_1^p(\alpha_1,\beta_1)f(z)}-1\right|^\beta \\
&\qquad < \begin{cases} (1-\alpha)^\gamma\left(1-\alpha+\frac{1}{2}\right)^\beta, & 0 \leq \alpha < \frac{1}{2} \\ (1-\alpha)^{\beta+\gamma}\,(2)^\beta, & \frac{1}{2} \leq \alpha < 1. \end{cases}
\end{aligned}$$

for some real numbers α, β and γ such that $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$, and $\beta+\gamma > 0$, then

$$f \in \mathcal{MS}_1^p(\alpha_1,\beta_1,\alpha) \quad p \in \mathbb{N}_0,$$

Corollary 2.3. *If we take* $\beta = 1$, $\gamma = 0$ *and* $f \in \mathcal{M}$ *satisfies*

$$\left| \frac{I_\mu^{p-1}(\alpha_1,\beta_1)f(z)}{I_\mu^{p}(\alpha_1,\beta_1)f(z)} - 1 \right|^\beta < \begin{cases} \left(1-\alpha+\frac{1}{2\mu}\right), & 0 \le \alpha < \frac{1}{2}, \\ (1-\alpha)\left(1+\frac{1}{\mu}\right), & \frac{1}{2} \le \alpha < 1 \end{cases}, \quad z \in E,\ p \in \mathbb{N}_0 \tag{2.7}$$

for some real numbers α *such that* $0 \le \alpha < 1$,
then

$$\operatorname{Re}\left(\frac{I_\mu^{p}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)} \right) > \alpha,\ z \in E^*$$

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