Bol. Soc. Paran. Mat. (3s.) **v. 32** 2 (2014): 221–227.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i2.20896

A Fixed Point Theorem in a Generalized Fuzzy Metric Space

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ABSTRACT: We prove a fixed point theorem for uniformly locally contractive fuzzy mapping in a generalized fuzzy metric space.

Key Words: Generalized fuzzy metic space; T-orbitally complete; ε-chainable; locally contractive; (ε, λ)-uniformly locally contractive.

Contents

1. Introduction

The concept of fuzzy set was introduced by L.A. Zadeh in1965. It is the origin of new theory of uncertainty. After the introduction of fuzzy sets, the scope for studies in different branches of science and technology where mathematics has been applied has increased widely. The notion of fuzzy set theory has been applied to introduce the notion of fuzzy real numbers which helps in constructing the sequence of fuzzy real numbers. Recently lots of work has been done on applying fuzzyness by Tripathy and Baruah [9], Tripathy and Borhohain ([10], [11]) Tripathy and Dutta ([12], [13]), Tripathy and Debnath [14], Tripathy and Ray [15], Tripathy and Sarma [16] and others.

The Banach fixed point theorem stats that each self -mapping T of a complete metric space (X, d) such that $d(Tx, Ty) < kd(x, y)(x \neq y, 0 < k < 1)$ has a unique fixed point. The assumption $k < 1$ is non superfluous .With $k = 1$ the mapping of this sort need not have a fixed point. However, if X is compact, then T has a unique fixed point. A lot of generalization of this theorem have been done, mostly by relaxing the contraction condition and sometimes by withdrawing the requirement of completeness or even both.

Recently a very interesting generalization of the concept of metric space was done by Branciari [1], on replacing the triangular inequality of a metric space by a more general inequality. Some works have already been done in this direction ([3], [6]). In this paper, we take uniformly locally contractive fuzzy mappings and show that they can have unique fixed point under some general condition in a generalized

2000 *Mathematics Subject Classification:* 47H10, 54H25

fuzzy metric space.

2. Definitions and preliminaries

LetR^+ denote the set of all non-negative real numbers and N denote the set of all positive integers.

Definition 2.1. *A fuzzy set A on X is a function with domain X and values in [0,1] i.e.* $A : X \to [0,1]$.

Definition 2.2. *A binary operation *: [0,1]*$\times[0,1]\to$ *[0,1] is called a continuous t-norm if ([0,1], *) is an abelian topological monoid with unit 1 such that* $a_1 * b_1 \leq a_2 * b_2$ *whenever* $a_1 \leq a_2, b_1 \leq b_2$ *for all* $a_1, a_2, b_1, b_2 \in$*[0,1].*

Definition 2.3. *Let* X *be an arbitrary set,* be a continuous t-norm and* M *be a fuzzy set in* $X^2 \times [0,\infty)$ *such that for all* $x, y \in X$ *and for all distinct points* $z, w \in X$ *each of them different from* x *and* y *and* $t_1, t_2, t_3, t > 0$*; one has*

(1)$M(x, y, 0) = 0$;
(2)$M(x, y, t) = 1$ *if and only if* $x = y$;
(3)$M(x, y, t) = M(y, x, t)$;
(4)$M(x, y, t_1) * M(y, z, t_2) * M(z, w, t_3) \leq M(x, w, t_1 + t_2 + t_3)$
(5)$M(x, y, \bullet)$*:[0,∞)→ [0,1]is left continuous.*

Then we say that $(X, M, *)$ *is a generalized fuzzy metric space (or shortly g.f.m.s). Any fuzzy metric space is a g.f.m.s but the converse is not true.*

Definition 2.4. *Let*$(X, M, *)$ *be a g.f.m.s. Then*

(a) A sequence (x_n) *in* X *is said to converge to* x *in* X *if for each* $\varepsilon \in$*(0,1) and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x, t) >$ *1-ε for all* $n \geq n_0$.

(b) A sequence $(x_n) \in X$ *is said to be Cauchy if for each* $\varepsilon \in$ *(0,1) and* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) >$ *1-ε, for all* $n, m \geq n_0$.

(c) A g.f.m.s in which every Cauchy sequence is convergent is said to be complete.

Note 2.5. *A sequence* (x_n) *in a g.f.m.s is converges to* $x \in X$, *if and only if* $\lim_{n\to\infty} M(x_n, x, t) = 1$.

A sequence (x_n) *in a g.f.m.s is a Cauchy sequence if and only if*

$\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ *for each* $t > 0$ *and* $p > 0$.

Definition 2.6. *Let T be a mapping of a g.f.m.s $(X, M, *)$ into itself. $(X, M, *)$ is said to be T-orbitally complete if and only if every Cauchy sequence which is contained in $\{x, Tx, T^2x, T^3x, \cdots\}$ for some $x \in X$ converges in X.*

A T-orbitally complete g.f.m.s may not be complete.

Throughout the paper by X we will mean a generalized fuzzy metric space.

We introduce the following definitions in this article.

Definition 2.7. *Let $(X, M, *)$ be g.f.m.s. A finite sequence $x = x_0, x_1, ..., x_n = y$ is called ε-chain from x to y if there exists a positive number $0 < e < 1$ such that $M(x_{i-1}, x_i, t) > \varepsilon$ for every $t > 0$ and $i = 1, 2, ..., n$. A g.f.m.s $(X, M, *)$ is called ε-chainable if for any $x, y \in X$ there exists ε-chain from x to y.*

Definition 2.8. *Let $(X, M, *)$ be a g.f.m.s. A mapping $T : X \to X$ is called locally contractive if for every $x \in X$ there exist $\varepsilon_x > 0$ and $\lambda_x \in$ (0,1) such that for all$p, q \in y : M(x, y, t) > \varepsilon$, the relation $\lambda_x M(T(p), T(q), t) \geq M(p, q, t)$ holds.*

Definition 2.9. *Let $(X, M, *)$ be a g.f.m.s. A mapping $T : X \to X$ is called (ε, λ) uniformly locally contractive at all points $x \in X$ and ε, λ do not depend on x, i.e. for $\varepsilon > 0$ and$\lambda \in (0, 1)$ $M(x, y, t) > \varepsilon \Rightarrow \lambda M(Tx, Ty, t) > M(x, y, t)$ for all $x, y \in X$.*

Note 2.10. *From the definition it is clear that a uniformly locally contractive fuzzy mapping is continuous.*

3. Main results

In this section we establish the main result of this paper.

Theorem 3.1. *If T is an (ε, λ) uniformly contractive fuzzy mapping defined on a T- orbitally complete ε- chainable g.f.m.s. X satisfying the following condition,*

$$\textit{forall } x, y, z \in X \textit{ and } t > 0, M(x, y, t) > \varepsilon \textit{ and } M(y, z, t) > \varepsilon \Rightarrow M(x, z, t) > \varepsilon. \tag{3.1}$$

Then T has a unique fixed point in X.

Proof: Step I Let $x \in X$.Since X is ε-chainable,we can find finite number of points, $x = x_0, x_1, x_2, ..., x_{n-1}, x_n = T_x$ such that $M(x_{i-1}, x_i, t) > \varepsilon$ for all $i = 1, 2, ..., n$ and $t > 0$.

Without any loss of generality we can assume that the points $x_1, x_2, ..., x_{n-1}$ are distinct (and different from x and T_x if $n > 1$)

We show that

$$M(x, T_x, t) > \varepsilon, \tag{3.2}$$

therefore $M(x, T_x, t) \geq M(x, x_1, \frac{t}{n}) * M(x_1, x_2, \frac{t}{n}) * M(x_2, x_3, \frac{t}{n}) * ... * M(x_{n-1}, T_x, \frac{t}{n})$ $> \varepsilon * \varepsilon * ... * \varepsilon > \varepsilon$.

Since, T is (ε, λ) uniformly locally contractive fuzzy mapping, we have,

$M(x_{i-1}, x_i, t) > \varepsilon \Rightarrow \lambda M(Tx_{i-1}, Tx_i, t) > M(x_{i-1}, x_i, t) > \varepsilon$ i.e.$M(Tx_{i-1}, Tx_i, t) > \frac{\varepsilon}{\lambda} > \varepsilon$ for all $i = 1, 2, ..., n$

and therefore,

$\lambda^2 M(T^2 x_{i-1}, T^2 x_i, t) = \lambda(\lambda M(T(Tx_{i-1}), T(Tx_i), t) > \lambda M(Tx_{i-1}, Tx_i, t) > \lambda\varepsilon$.

$\Rightarrow M(T^2 x_{i-1}, T^2 x_i, t) > \varepsilon$.

In a similar way, we have

$\lambda^3 M(T^3 x_{i-1}, T^3 x_i, t) = \lambda^2(\lambda M(T^2 x_{i-1}), T(T^2 x_i), t) > \lambda^2 M(T^2 x_{i-1}, T^2 x_i, t) > \lambda^2 \varepsilon \Rightarrow M(T^3 x_{i-1}, T^3 x_i, t) > \varepsilon$.

. . .

$\lambda^m M(T^m x_{i-1}, T^m x_i, t) = \lambda^{m-1}(\lambda M(T(T^{m-1} x_{i-1}, T(T^{m-1} x_i), t)$
$> \lambda^{m-1} M(T^{m-1} x_{i-1}, T^{m-1} x_i, t) > \lambda^{m-1}\varepsilon \Rightarrow M(T^m x_{i-1}, T^m x_i, t) > \varepsilon$ and $M(T^m x_0, T^m x_2, t) > \varepsilon$(using equation (3.1)).

Now,

$M(T^m x, T^{m+1} x, t) = M(T^m x_0, T^m x_n, t)[since,\ x_0 = x \ and\ x_n = T_x]$
$> M(T^m x_0, T^m x_1, \frac{t}{n})$.

$$* M(T^m x_1, T^m x_2, \frac{t}{n}) * ... * M(T^m x_{n-1}, T^m x_n, \frac{t}{n}) > \varepsilon \ forall\ t > 0 \ and\ m \in N. \tag{3.3}$$

Note that even some of the points $T^m x_0, \ldots\ldots, T^m x_n$ are equal than also the result is obviously true.

Now, for all $t > 0$ and $j < k$ we have

$M(T^jx, T^kx, t)$
$\geq M(T^jx, T^{j+1}x, \frac{t}{k-j}) * M(T^{j+1}x, T^{j+2}x, \frac{t}{k-j}) * \ldots * M(T^{k-1}x, T^kx, \frac{t}{k-j})$
$> \varepsilon$.

$\Rightarrow T^jx$ is a Cauchy sequence in X.

Since X is T-orbitally complete, T^jx is convergent in X Let $\lim_{j\to\infty} T^jx = u$ Again since T is continuous [by Note 2.10].

$T(u) = T(lim_{j\to\infty}T^jx = u) = lim_{j\to\infty}T^{j+1}x = u$.

This show that u is a fixed point of T.

Step II: To show the fixed point is unique, let us assume that n is another fixed point of T i.e. $T_v = v$. Since X is ε-chainable, we can find an ε-chain, $u = x_0, x_1, x_2 \ldots\ldots x_n = v = T_v$. Then Proceeding as in step I, we can show that $M(u, v, t) \geq M(u, x_1, \frac{t}{n}) * M(x_1, x_2, \frac{t}{n}) * \ldots * M(x_{n-1}, v, \frac{t}{n}) \to 1 * 1 * \ldots * 1 = 1$ as $n \to \infty \Rightarrow u = v$.

This completes the proof of the theorem. □

Remark 3.2. *We now give a simple example to show that the condition (3.1) in Theorem 3.1 is strictly weaker than the requirement of a g.f.m.s to be a metric space.*

Example 3.3. *Let $X = a, b, c, d$, $a * b = min(a, b)$ and $M : X^2 \times [0, \infty) \to [0, 1]$ be defined by $M(a, b, t) = 0.2; M(a, c, t) = M(b, c, t) = 0.25; M(a, d, t) = M(b, d, t) = (M(c, d, t) = 0.2$ and $M(x, x, t) = 0$ for all $x \in X$ Further, let $T : X \to X$ be the mapping*

$$Tx = \begin{cases} c, if x \in \{a, b, c\}; \\ a, if x = d. \end{cases}$$

Then it can be easily verified that $(X, M, *)$ is a ε-chainable g.f.m.f. where $\varepsilon = 0.1$ satisfying the condition (1) but it is not a fuzzy metric space,
*$M(a, b, t_1 + t_2) = 0.2 < M(a, c, t_1) * M(c, b, t_2)$ = 0.25 * 0.25 = 0.25 and T has a unique fixed point c.*

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