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New spaces and Continuity via $\hat{\Omega}$-closed sets

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ABSTRACT: In this paper, we introduce new spaces like ${}_{\hat{\Omega}}T_\delta$ and ${}_{\omega}T_{\hat{\Omega}}$. It turns out that the space ${}_{\hat{\Omega}}T_\delta$ coincide with *semi*-T_1 and in ${}_{\omega}T_{\hat{\Omega}}$-space every closed set is $\hat{\Omega}$-closed set and in *semi*-$T_{\frac{1}{2}}$ every $\hat{\Omega}$-closed set is closed in a topological space. Also we introduce some kinds of generalized continuity such as $\hat{\Omega}$-continuity, $\hat{\Omega}$-irresolute and weakly $\hat{\Omega}$-continuity.

Key Words: $\hat{\Omega}$-closed sets, ${}_{\hat{\Omega}}T_\delta$-spaces, ${}_{\omega}T_{\hat{\Omega}}$-spaces, $\hat{\Omega}$-continuity and $\hat{\Omega}$- irresolute.

Contents

1. Introduction

In the recent years, many new separation axioms were introduced by using generalized closed sets such as [3] T_b, [12] T_d, [2] *semi*-T_1, *semi*-$T_{\frac{1}{2}}$ and [6] $T_{\frac{3}{4}}$. The class of $\hat{\Omega}$-closed sets [8] has been introduced recently by using the δ-closure operator. As the class of $\hat{\Omega}$-closed sets are independent of the class of closed sets, our interest is to find the spaces in which they are interrelated. With these motivation, new separation axioms via $\hat{\Omega}$-closed sets such as ${}_{\hat{\Omega}}T_\delta$ which coincide with *semi*-T_1 and ${}_{\omega}T_{\hat{\Omega}}$ in which every $\hat{\Omega}$-closed set is closed set are introduced and studied. Moreover, we introduce and study various types of the continuity via $\hat{\Omega}$-closed sets. Also we investigate the role of new spaces and it's impacts on all these continuities.

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2. Preliminaries

Throughout this paper (X,τ) (or briefly X) represent a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of (X,τ), we denote the closure of A, the interior of A and the complement of A as $cl(A)$, $int(A)$ and A^c respectively. The family of all open sets in (X,τ) are denoted by $O(X)$ and $O(X,x) = \{U \in X : x \in U \in O(X)\}$ Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. *[7] A subset A of X is called δ-closed in a topological space (X,τ) if $A = \delta cl(A)$, where $\delta cl(A) = \{x \in X : int(cl(U)) \cap A \neq \emptyset,\ U \in O(X,x)\}$.*
The complement of δ-closed set in (X,τ) is called δ-open set in (X,τ). From [7], Lemma 3, $\delta cl(A) = \bigcap\{F \in \delta C(X) : A \subseteq F\}$ and from Corollary 4, $\delta cl(A)$ is a δ-closed for a subset A in a topological space (X,τ).

The following notations are used in this paper. The family of all δ-open (resp.$\hat{\Omega}$-open) sets on X are denoted by $\delta O(X)$ (resp.$\hat{\Omega}O(X)$).

- $\delta O(X,x) = \{U \in X : x \in U \in \delta O(X)\}$
- $\hat{\Omega}O(X,x) = \{U \in X : x \in U \in \hat{\Omega}O(X)\}$

Definition 2.2. *A subset A of a topological space (X,τ) is called*

(i) *semi open [9] if $A \subseteq cl(int(A))$.*

(ii) *pre open [11] if $A \subseteq int(cl(A))$.*

(iii) *regular open [15] if $A = int(cl(A))$.*

(iv) *a δ generalised closed* (briefly δg-closed) set [5] if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X,τ).

(v) $g\alpha$-closed set [8] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α open in (X,τ).

(vi) $\hat{g}$ (or) ω-closed set [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X,τ).

(vii) $\hat{\Omega}$-closed set [8] if $\delta cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X,τ).

The complement of semi open (resp.pre open, regular open δg-closed ω(or $\hat{g}$)-closed) set is called semi closed (resp.pre closed, regular closed, δg-open, ω(or$\hat{g}$)-open,$\hat{\Omega}$-open).

Definition 2.3. *A function $f : (X,\tau) \to (Y,\sigma)$ is called a*

(i) *contra continuous [5] if the inverse image of open set in Y is closed set in X.*

(ii) *perfectly continuous [14] if the inverse image of open set in Y is clopen set in X.*

(iii) *completely continuous [1] if the inverse image of open set in Y is regular open set in X.*

(iv) *super continuous [10] if the inverse image of open set in Y is δ open set in X.*

(v) *semi continuous [9] if the inverse image of open set in Y is semi open set in X.*

(vi) pre continuous [11] if the inverse image of open set in Y is pre open set in X.

(vi) *δg-continuous [6] if the inverse image of open set in Y is δg-open set in X.*

(vii) *ω-continuous [17] if the inverse image of open set in Y is ω-open set in X.*

(viii) *$g\alpha$-continuous [4] if the inverse image of open set in Y is $g\alpha$-open set in X.*

(ix) *δ-closed mapping [13] if the image of every δ-closed set in X is δ-closed in Y.*

Definition 2.4. *A space (X, τ) is said to be $T_{\frac{3}{4}}$ [6] if every δg-open set is δ-open set in X.*

Definition 2.5. *A space (X, τ) is said to be weakly Hausdorff [6] if for every two different point x and y, there exists a regular open set U such that $x \in U$, $y \notin U$.*

3. ${}_{\hat{\Omega}}T_{\delta}$-spaces and ${}_{\omega}T_{\hat{\Omega}}$-spaces

In this section, we introduce two new spaces in which we get the relationship between the class of $\hat{\Omega}$-open sets and that of open sets. These spaces provide a new path to class of $\hat{\Omega}$-open sets to travel along with open sets.

Definition 3.1. *A space (X, τ) is called ${}_{\hat{\Omega}}T_{\delta}$-space if every $\hat{\Omega}$-closed set in X is δ-closed in X.*

Example 3.2. $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. *Here,* $\hat{\Omega}O(X) = \tau = \delta O(X)$.
Therefore, (Y, σ) is ${}_{\hat{\Omega}}T_{\delta}$-space.

Theorem 3.3. *Every $T_{\frac{3}{4}}$-space is ${}_{\hat{\Omega}}T_{\delta}$-space.*

Proof: Suppose that (X, τ) is a $T_{\frac{3}{4}}$-space and A is any $\hat{\Omega}$-closed set in (X, τ). By [8] Theorem 3.5, A is δg-closed set in X. Since X is $T_{\frac{3}{4}}$-space, A is δ-closed set in X.
Thus, X is ${}_{\hat{\Omega}}T_{\delta}$-space. □

Remark 3.4. *Converse is not always possible from the Example 3.2.*

Example 3.5. *By [6] the digital line is $T_{\frac{3}{4}}$. By Theorem 3.3, the digital line is ${}_{\hat{\Omega}}T_{\delta}$-space.*

Let us give some characterizations of ${}_{\hat{\Omega}}T_{\delta}$-space.

Theorem 3.6. *A topological space (X, τ) is ${}_{\hat{\Omega}}T_{\delta}$-space if and only if every singleton set is semi closed in (X, τ).*

Proof: Necessity- Let $x \in X$. By [8] Proposition 4.7, $\{x\}$ is semi closed or $\{x\}^c$ is $\hat{\Omega}$-closed in X. Therefore, if $\{x\}$ is not semi closed then $\{x\}^c$ is $\hat{\Omega}$-closed in X, By hypothesis, $\{x\}^c$ is δ-closed set in (X, τ) and hence $\{x\}$ is δ-open in (X, τ). By [6] Lemma 4.2, $\{x\}$ is regular open and hence $\{x\}$ is semi closed in X.
Sufficiency- Suppose that A is any $\hat{\Omega}$-closed set in (X, τ) and $x \in \delta cl(A) \setminus (A)$. Since A is $\hat{\Omega}$-closed set in (X, τ), [8] by Theorem 4.4, $\delta cl(A) \setminus (A)$ does not contain any non-empty semi closed set in X. Therefore, $\{x\}$ is not a semi closed in (X, τ), a contradiction.
Therefore, $\delta cl(A) = A$ and hence A is δ-closed in (X, τ). □

Theorem 3.7. *In a topological space (X, τ), the following statements are equivalent.*

(i) *(X, τ) is semi-T_1-space.*

(ii) *$\{x\}$ is semi closed in X for every $x \in X$.*

(iii) *X is ${}_{\hat{\Omega}}T_{\delta}$.*

Proof: By [12], (i)$\Leftrightarrow$ (ii) holds and by Theorem 3.6, (iii) $\Leftrightarrow$ (ii) holds. □

Theorem 3.8. *Every ${}_{\hat{\Omega}}T_{\delta}$-space is semi-$T_{\frac{1}{2}}$-space.*

Proof: Since every semi-T_1-space is semi-$T_{\frac{1}{2}}$ and by Theorem 3.6,
every ${}_{\hat{\Omega}}T_{\delta}$-space is semi-$T_{\frac{1}{2}}$-space. □

Remark 3.9. *Converse is not true in general as seen from the following example.*

Example 3.10. *$X = \{a, b, c\}, \sigma = \{\emptyset, \{a\}, X\}$. is semi-$T_{\frac{1}{2}}$ but not ${}_{\hat{\Omega}}T_{\delta}$-space.*

Theorem 3.11. *A topological space (X, τ) is weakly Hausdorff if and only if (X, τ) is ${}_{\hat{\Omega}}T_{\delta}$-space and every singleton set is $\hat{\Omega}$-closed in (X, τ).*

Proof: Necessity- By [6] Lemma 4.17, every $\{x\}$ is δ-closed and hence semi closed in X. By Theorem 3.7, X is ${}_{\hat{\Omega}}T_{\delta}$-space. By [8] Theorem 3.2, $\{x\}$ is $\hat{\Omega}$-closed in (X, τ).
Sufficiency- Let $x \in X$. By hypothesis, $\{x\}$ is δ-closed in X and again by [6] Lemma 4.17, (X, τ) is weakly Hausdorff. □

Definition 3.12. *A space (X, τ) is called ${}_{\omega}T_{\hat{\Omega}}$-space if every ω-closed set in X is $\hat{\Omega}$-closed in X.*

Example 3.13. *Let* $X = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{bcd\}, X\}$. *Then* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$-*space.*

Let us characterize ${}_{\omega}T_{\hat{\Omega}}$-space via closed sets.

Theorem 3.14. *A space* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$-*space if and only if every closed set is* $\hat{\Omega}$-*closed in* (X, τ).

Proof: Necessity- Suppose that A is any closed set in (X, τ). Then by [16], A is ω-closed set in X. By hypothesis, A is $\hat{\Omega}$-closed set in X.
Sufficiency- Suppose that A is ω-closed set in X. Then by [16], $cl(A) \subseteq sker(A)$. Then $sker(cl(A)) \subseteq sker(sker(A)) = sker(A)$. Since $cl(A)$ is closed set in X, by hypothesis, $cl(A)$ is $\hat{\Omega}$-closed set in X. By [8] Theorem 4.10, $\delta cl(cl(A)) \subseteq sker(cl(A)) \subseteq sker(A)$. But $\delta cl(A) \subseteq \delta cl(cl(A))$. Therefore, $\delta cl(A) \subseteq sker(A)$. Again by [8] Theorem 4.10, A is $\hat{\Omega}$-closed in (X, τ). □

Remark 3.15. *From the above discussion, a space* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$-*space if and only if every open set is* $\hat{\Omega}$-*open in* (X, τ).

Let us characterize semi-$T_{\frac{1}{2}}$-space via $\hat{\Omega}$-closed sets.

Theorem 3.16. *Every* $\hat{\Omega}$-*closed set in topological space* (X, τ) *is closed in* X *if and only if every singleton set is either open or semi closed in* (X, τ).

Proof: Necessity- Assume that every $\hat{\Omega}$-closed set in a topological space (X, τ) is closed set in X and suppose that there exists $x \in X$ such that $\{x\}$ is not semi closed subset of X. Then $\{x\}^c$ is not semi open subset of X. Here, X is the only semi open set containing $\{x\}^c$ and $\delta cl(\{x\}^c) \subseteq X$. Therefore, $\{x\}^c$ is $\hat{\Omega}$-closed subset of X and by hypothesis, $\{x\}^c$is closed in X. Thus, $\{x\}$ is open in X.
Sufficiency- Suppose that A is $\hat{\Omega}$-closed set in X and there exists $x \in cl(A) \setminus (A)$.
Case(i)-$\{x\}$ is open in X. Then $A \cap \{x\} \neq \emptyset$, a contradiction to $x \in cl(A) \setminus (A)$.
Case(ii)-$\{x\}$ is semi closed in X. Since $cl(A) \setminus (A) \subseteq \delta cl(A) \setminus (A)$, $\{x\}$ is a semi closed subset of $\delta cl(A) \setminus (A)$, a contradiction to Theorem 4.4 in [8]. In both cases, we arrive at a contradiction. Therefore, $cl(A) = A$ and hence A is closed in X. □

Theorem 3.17. *In a topological space* (X, τ), *the following statements are equivalent.*

(i) (X, τ) *is semi-*$T_{\frac{1}{2}}$-*space.*

(ii) $\{x\}$ *is either open or semi closed in* X *for every* $x \in X$.

(iii) *Every* $\hat{\Omega}$-*closed set in* (X, τ) *is closed set in* (X, τ).

(iv) *Every* ω-*closed set is closed in* X.

Proof: $(i) \Rightarrow (ii)$ By [16] Theorem 4.8, it follows.
$(ii) \Rightarrow (iii)$ By Theorem 3.16, it follows.
$(iii) \Rightarrow (iv)$ Suppose that F is ω-closed set in X. By hypothesis and by Theorem 3.16, every $\{x\}$ is either open or semi closed in X. By [17] Theorem 5.01, F is closed in X.
$(iv) \Rightarrow (i)$ By [17] Theorem 5.01, it holds. □

Remark 3.18. *From the above discussion, a space (X, τ) is semi-$T_{\frac{1}{2}}$ if and only if every $\hat{\Omega}$-open set is open in (X, τ).*

4. $\hat{\Omega}$-continuity

Definition 4.1. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is called $\hat{\Omega}$-continuous if the inverse image of every open set in (Y, σ) is $\hat{\Omega}$-open in (X, τ).*

Example 4.2. $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, X\}$. *If $f\colon (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = a, f(b) = b, f(c) = b$. Then, f is $\hat{\Omega}$-continuous.*

Theorem 4.3. *Let $f\colon (X, \tau) \to (Y, \sigma)$ be any function. Then, the following statements hold.*

(i) *Every super continuous function is $\hat{\Omega}$-continuous function.*

(ii) *Every $\hat{\Omega}$-continuous function is δg-continuous function.*

(iii) *Every $\hat{\Omega}$-continuous function is ω-continuous function.*

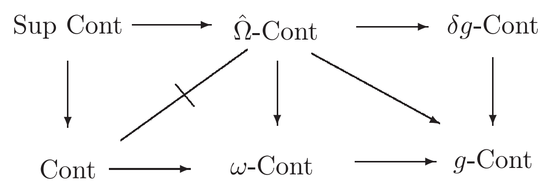
Proof: It follows from the Definition 4.1 and by [8] Theorems 3.2 and 3.5. □

Remark 4.4. *The converse is not always true from the following example.*

Example 4.5. *Example 4.2 shows that f is $\hat{\Omega}$-continuous but not super continuous.* $X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}, \sigma = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, Y\}$.
If $f\colon (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = c, f(c) = f(d) = a$. Then f is g, δg and ω-continuous but not $\hat{\Omega}$-continuous.

Remark 4.6. *The notion of $\hat{\Omega}$-continuous and continuous are independent from the examples 4.2 and 4.5.*

Remark 4.7. *The pictorial representation of the above discussions and the existing results are given below. The reversible implication is never possible.*



The following Theorem gives the conditions under which reversible implication of Theorem 4.3 holds.

Theorem 4.8. *In a topological space (X, τ), the following holds.*

(i) *If (X, τ) is ${}_{\hat{\Omega}}T_{\delta}$-space, then every $\hat{\Omega}$-continuous function is super continuous function.*

(ii) *If (X, τ) is $T_{\frac{3}{4}}$-space, then every δg-continuous function is $\hat{\Omega}$-continuous function.*

(iii) *If (X, τ) is ${}_{\omega}T_{\hat{\Omega}}$-space, then every ω-continuous function is $\hat{\Omega}$-continuous function .*

Proof:

(i) Since in a space ${}_{\hat{\Omega}}T_{\delta}$ every $\hat{\Omega}$-open set is δ-open, it follows.

(ii) In a space $T_{\frac{3}{4}}$, every δg-open set is δ-open. Therefore, by [8] Theorem 3.2, it follows.

(iii) In a space ${}_{\omega}T_{\hat{\Omega}}$, every ω-open set is $\hat{\Omega}$-open. Therefore, it follows.

□

Theorem 4.9. *If a space (X, τ) is ${}_{\omega}T_{\hat{\Omega}}$-space, then every continuous function is $\hat{\Omega}$-continuous function.*

Proof: It follows from the Definition 4.1 and Theorem 3.14. □

Theorem 4.10. *If a space (X, τ) is semi-$T_{\frac{1}{2}}$-space, then every $\hat{\Omega}$-continuous function is continuous function.*

Proof: It follows from the Definition 4.1 and Theorem 3.17. □

Let us decompose perfectly continuous functions via $\hat{\Omega}$-continuous mappings as follows.

Theorem 4.11. *If $f: (X, \tau) \to (Y, \sigma)$ is a function, then the followings are equivalent.*

(i) *f is perfectly continuous.*

(ii) *f is completely continuous and contra continuous.*

(iii) *f is super continuous and contra continuous.*

(iv) *f is $\hat{\Omega}$-continuous and contra continuous.*

(v) *f is ω-continuous and contra continuous.*

(vi) *f is $g\alpha$-continuous and contra continuous.*

(vii) *f is pre-continuous and contra continuous.*

Proof:

(i) Since a clopen set is regular closed and open, it holds.

(ii) Since a regular closed set is δ-closed, it holds.

(iii) Since a δ-closed set is $\hat{\Omega}$-closed, it holds.

(iv) Since a $\hat{\Omega}$-closed set is ω-closed, it holds.

(v) Since a ω-closed set is $g\alpha$-closed, it holds.

(vi) Since a $g\alpha$-closed set is pre-closed, it holds.

(vii) If A is pre-closed and open set in (X, τ), then $cl(int(A)) \subseteq A$, and $A = intA$. Thus $cl(A) \subseteq A$ and hence A is closed. Thus A is clopen in X. Therefore, every pre-continuous and contra continuous function is perfectly continuous.

□

Remark 4.12. *The composition of $\hat{\Omega}$-continuous functions is not always $\hat{\Omega}$- continuous function as seen from the following example.*

Example 4.13. *Let $X = Y = Z = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}, \eta = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Z\}$. If $f: (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = b, f(c) = c$. Then f is $\hat{\Omega}$-continuous function. Define $g: (Y, \sigma) \to (Z, \eta)$ as $g(a) = b, g(b) = c, g(c) = a$. Then g is $\hat{\Omega}$-continuous function. But $g \circ f: (X, \tau) \to (Z, \eta)$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a $\hat{\Omega}$-continuous function.*

Theorem 4.14. *The following statements are true.*

(i) *If $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous and $g: (Y, \sigma) \to (Z, \eta)$ is continuous, then*
$g \circ f: (X, \tau) \to (Z, \eta)$ is $\hat{\Omega}$-continuous.

(ii) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *and* $g\colon (Y,\sigma)\to(Z,\eta)$ *are continuous, then* $g\circ f\colon (X,\tau)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous provided* (X,τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space.*

(iii) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *and* $g\colon (Y,\sigma)\to(Z,\eta)$ *are* $\hat{\Omega}$*-continuous, then* $g\circ f\colon (X,\tau)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous whenever* (Y,σ) *is semi-*$T_{\frac{1}{2}}$*-space.*

(iv) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *is continuous and* $g\colon (Y,\sigma)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous, then*
$g\circ f\colon (X,\tau)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous provided* (X,τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space and* (Y,σ) *is semi-*$T_{\frac{1}{2}}$*-space.*

(v) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *is continuous and* $g\colon (Y,\sigma)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous then,*

$g\circ f\colon (X,\tau)\to(Z,\eta)$ *is continuous whenever* (Y,σ) *is semi-*$T_{\frac{1}{2}}$*-space*

(vi) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *is* $\hat{\Omega}$*-continuous and* $g\colon (Y,\sigma)\to(Z,\eta)$ *is continuous, then*
$g\circ f\colon (X,\tau)\to(Z,\eta)$ *is continuous whenever* (X,τ) *is semi-*$T_{\frac{1}{2}}$*-space.*

Proof: They follow straight forward from their definitions. □

Theorem 4.15. $f\colon (X,\tau)\to(Y,\sigma)$ *is a surjective function such that* $f(U)$ *is* $\hat{\Omega}$*-open set in* Y *for any* $\hat{\Omega}$*-open set in* U *in* X *and* $g\colon (Y,\sigma)\to(Z,\eta)$ *is any function. If* $g\circ f\colon (X,\tau)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous, then* g *is* $\hat{\Omega}$*-continuous function.*

Proof: Suppose that $y\in Y$ and W is any open set in Z containing $g(y)$. Since f is surjective, there exists $x\in X$ such that $f(x)=y$. Then $(g\circ f)(x)=g(f(x))=g(y)\in W$. Since $g\circ f$ is $\hat{\Omega}$-continuous, there exists an $\hat{\Omega}$-open set U in X containing x such that $(g\circ f)(U)\subseteq W$. If we take $V=f(U)$, then by our assumption, V is a $\hat{\Omega}$-open set in Y containing $f(x)$ and $(g\circ f)(U)=g(V)\subseteq W$. Thus, g is $\hat{\Omega}$-continuous function. □

Let us characterize $\hat{\Omega}$-continuous function.

Theorem 4.16. *If* $f\colon (X,\tau)\to(Y,\sigma)$ *is any function, then the following statements are equivalent.*

(i) f *is* $\hat{\Omega}$*-continuous function.*

(ii) *For each* $x\in X$ *and each* $V\in O(Y,f(x))$*, there exists* $U\in\hat{\Omega}O(X,x)$ *such that* $f(U)\subseteq V$*.*

(iii) *The inverse image of every closed set in* (Y,σ) *is* $\hat{\Omega}$*-closed set in* (X,τ)*.*

(iv) $f(\hat{\Omega}cl(A))\subseteq cl(f(A))$ *for each subset* A *of* (X,τ)*.*

(v) $\hat{\Omega}cl(f^{-1}(B))\subseteq f^{-1}(cl(B))$ *for each subset* B *of* (Y,σ)*.*

Proof: $(i) \Rightarrow (ii)$ Suppose that $x \in X$ and V is any open set in (Y, σ) containing $f(x)$. Then $x \in f^{-1}(V)$. Define $U = f^{-1}(V)$. By hypothesis, U is $\hat{\Omega}$-open in (X, τ) containing x. Also $f(U) \subseteq V$.
$(ii) \Rightarrow (i)$ Suppose that V is any open set in (Y, σ) and $x \in f^{-1}(V)$ be arbitrary. By hypothesis, there exists $U_x \in \hat{\Omega}O(X, x)$ such that $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ implies that $f^{-1}(V) = \bigcup\{U_x : x \in f^{-1}(V)\}$. By [8] Theorem 4.16, $f^{-1}(V)$ is $\hat{\Omega}$-open in (X, τ).
$(i) \Rightarrow (iii)$ Let Fbe any closed set in (Y, σ). Then $Y \setminus F$ is open in (Y, σ). By hypothesis, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is $\hat{\Omega}$-open in (X, τ). Therefore, $f^{-1}(F)$ is $\hat{\Omega}$-closed in (X, τ).
$(iii) \Rightarrow (i)$ Let Vbe any open set in (Y, σ). Then $Y \setminus V$ is closed in (Y, σ). By hypothesis, $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is $\hat{\Omega}$-closed in (X, τ). Therefore, $f^{-1}(V)$ is $\hat{\Omega}$-open in (X, τ).
$(iii) \Rightarrow (iv)$ Let A be any subset of X. Then $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. By hypothesis, $f^{-1}(cl(f(A)))$ is a $\hat{\Omega}$-closed in (X, τ) containing A. By the definition of $\hat{\Omega}$-closure, $\hat{\Omega}cl(A) \subseteq f^{-1}(cl(f(A)))$. Therefore, $f(\hat{\Omega}cl(A)) \subseteq cl(f(A))$.
$(iv) \Rightarrow (iii)$ Suppose that F is any closed set in (Y, σ). By hypothesis,
$f(\hat{\Omega}cl(f^{-1}(F))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) \subseteq F$. Then $\hat{\Omega}cl(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus $f^{-1}(F)$ is $\hat{\Omega}$-closed in (X, τ).
$(iv) \Rightarrow (v)$ Let B be any subset of (Y, σ). By hypothesis, $f(\hat{\Omega}cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$. Then $\hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.
$(v) \Rightarrow (iv)$ Let A be any subset of (X, τ). If we define $B = f(A)$, then $A \subseteq f^{-1}(f(A)) = f^{-1}(B)$. Then $\hat{\Omega}cl(A) \subseteq \hat{\Omega}cl(f^{-1}(B))$. By hypothesis, $\hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$. Then $\hat{\Omega}cl(A) \subseteq f^{-1}(cl(B)) = f^{-1}(cl(f(A)))$. Therefore, $f(\hat{\Omega}cl(A)) \subseteq cl(f(A))$. □

Proposition 4.17. *If $f \colon (X, \tau) \to (Y, \sigma)$ is a $\hat{\Omega}$-continuous function and A is both open and pre-closed subset of (X, τ), then the restriction $f|A$ is $\hat{\Omega}$-continuous function.*

Proof: Let F be any open subset of (Y, σ). By hypothesis, $f^{-1}(F)$ is $\hat{\Omega}$-open subset of (X, τ). By [8] Theorem 6.10, $f^{-1}(F) \cap A$ is $\hat{\Omega}$-open in $(A, \tau|A)$. Therefore, $(f|A)^{-1}(F)$ is $\hat{\Omega}$-open in $(A, \tau|A)$. Thus, $f|A$ is $\hat{\Omega}$-continuous function. □

Theorem 4.18. *Let $\{A_\alpha, \alpha \in \Lambda\}$ be a cover of X by open and preclosed subsets of (X, τ). If $f|A_\alpha : (A_\alpha, \tau|A_\alpha) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous function for each $\alpha \in \Lambda$, then $f \colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous function.*

Proof: Let V be any open subset of (Y, σ). Since $f|A_\alpha$ is $\hat{\Omega}$-continuous function, $(f|A_\alpha)^{-1}(V)$ is $\hat{\Omega}$-open subset of $(A_\alpha, \tau|A_\alpha)$. By [8] Theorem 6.9, for each $\alpha \in \Lambda$, $(f|A_\alpha)^{-1}(V)$ is $\hat{\Omega}$-open subset of (X, τ). Since $f^{-1}(V) = \bigcup\{(f|A_\alpha)^{-1}(V) : \alpha \in \Lambda\}$ and since $\hat{\Omega}O(X)$ is closed under arbitrary union, $f^{-1}(V)$ is $\hat{\Omega}$-open subset of (X, τ). Thus, f is $\hat{\Omega}$-continuous function. □

Let us prove a significant theorem as follows.

Theorem 4.19. "The Pasting lemma" *for $\hat{\Omega}$-continuous function.*
Let A and B be any two both open and pre-closed subsets of (X, τ) such that $X = A \cup B$. Let $f: (A, \tau|A) \to (Y, \sigma)$ and $g: (B, \tau|B) \to (Y, \sigma)$ be $\hat{\Omega}$-continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Then the combination $(f \bigtriangledown g)(x) = f(x)$ for every $x \in A$ and $(f \bigtriangledown g)(y) = g(y)$ for every $y \in B$ is $\hat{\Omega}$-continuous function.

Proof: Let F be any closed set in (Y, σ). Then $(f \bigtriangledown g)^{-1}(F) = ((f \bigtriangledown g)^{-1}(F) \cap A) \cup ((f \bigtriangledown g)^{-1}(F) \cap B) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$, where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. Since f and g are $\hat{\Omega}$-continuous, C and D are $\hat{\Omega}$-closed sets in $(A, \tau|_A)$ and $(B, \tau|B)$ respectively. By [8], Theorem 6.6, C and D are $\hat{\Omega}$-closed sets in (X, τ). By [8], Theorem 4.12, $(f \bigtriangledown g)^{-1}(F)$ is $\hat{\Omega}$-closed sets in (X, τ). Thus, the combination $(f \bigtriangledown g)$ is $\hat{\Omega}$-continuous function. □

Theorem 4.20. *If $f: (X, \tau) \to (Y, \sigma)$ is semi-continuous and δ-closed mapping and if A is $\hat{\Omega}$-closed set in (X, τ), then $f(A)$ is $\hat{\Omega}$-closed in (Y, σ).*

Proof: Suppose that $f(A) \subseteq U$, where U is any semi-open subset of Y. Then $A \subseteq f^{-1}(U)$. Since f is semi-continuous, $f^{-1}(U)$ is semi open set in X. Since A is $\hat{\Omega}$-closed set in $X, \delta cl(A) \subseteq f^{-1}(U)$ and hence $f(\delta cl(A)) \subseteq U$. By [7] Corollary 4, $\delta cl(A)$ is δ-closed set and since f is a δ-closed mapping, $\delta cl(f(A)) \subseteq \delta cl(f(\delta cl(A))) \subseteq f(\delta cl(A)) \subseteq U$. Thus, $f(A)$ is $\hat{\Omega}$-closed in (Y, σ). □

5. $\hat{\Omega}$-irresolute mappings

Definition 5.1. *A function $f: (X, \tau) \to (Y, \sigma)$ is said to be $\hat{\Omega}$-irresolute if $f^{-1}(V)$ is $\hat{\Omega}$-open set in (X, τ) for every $\hat{\Omega}$-open set V in (Y, σ).*

Example 5.2. *Let $X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Define $f: (X, \tau) \to (Y, \sigma)$ as $f(a) = b, f(b) = a, f(c) = c, f(d) = d$. Here f is $\hat{\Omega}$-irresolute function.*

Let us characterizes $\hat{\Omega}$-irresolute function in the following theorems

Theorem 5.3. *A function $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute if and only $f^{-1}(F)$ is $\hat{\Omega}$-closed set in (X, τ) for every $\hat{\Omega}$-closed set F in (Y, σ).*

Proof: Necessity- Suppose that f is $\hat{\Omega}$-irresolute and F is any $\hat{\Omega}$-closed set in (Y, σ). Then $(Y \setminus F)$ is $\hat{\Omega}$-open set in (Y, σ) and by hypothesis, $f^{-1}(Y \setminus F)$ is $\hat{\Omega}$-open set in (X, τ). Then $(X \setminus f^{-1}(F))$ is $\hat{\Omega}$-open set in (X, τ). Thus, $f^{-1}(F)$ is $\hat{\Omega}$-closed set in (X, τ).
Sufficiency- Suppose V is any $\hat{\Omega}$-open set in (Y, σ). Then $(Y \setminus V)$ is $\hat{\Omega}$-closed set in (Y, σ). By hypothesis, $f^{-1}(Y \setminus V)$ is $\hat{\Omega}$-closed set in (X, τ) and hence $(X \setminus f^{-1}(V))$ is $\hat{\Omega}$-closed set in (X, τ). Thus, $f^{-1}(V)$ is $\hat{\Omega}$-open set in (X, τ). □

Theorem 5.4. *A function $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute if and only if $f(\hat{\Omega}cl(A)) \subseteq \hat{\Omega}cl(f(A))$ for every subset A of X.*

Proof: Necessity- Suppose that f is $\hat{\Omega}$-irresolute and A is any subset of X. Let $x \in \hat{\Omega}cl(A)$ and $f(x) \notin \hat{\Omega}cl(f(A))$. By [8] Theorem 5.11, there exists a $\hat{\Omega}$-open set V in Y containing $f(x)$ such that $V \cap f(A) = \emptyset$. Then $f^{-1}(V) \cap A = \emptyset$. Since f is $\hat{\Omega}$-continuous, $f^{-1}(V)$ is $\hat{\Omega}$-open set in X containing x. Since $x \in \hat{\Omega}cl(A)$ and by [8] Theorem 5.11, $f^{-1}(V) \cap A \neq \emptyset$, a contradiction. Therefore, $f(\hat{\Omega}cl(A)) \subseteq \hat{\Omega}cl(f(A))$.
Sufficiency- Let F be any $\hat{\Omega}$-closed set in Y. If $x \in \hat{\Omega}cl(f^{-1}(F))$, then $f(x) \in f(\hat{\Omega}cl(f^{-1}(F)))$. By hypothesis, $f(x) \in \hat{\Omega}cl(f(f^{-1}(F))) \subseteq \hat{\Omega}cl(F) = F$. Then $x \in f^{-1}(F)$. By [8] Theorem 5.3 (v), $f^{-1}(F)$ is $\hat{\Omega}$-closed set in X and hence $\hat{\Omega}$-continuous. □

Theorem 5.5. *A function $f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute iff $\hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(\hat{\Omega}cl(B))$ for each subset B of (Y, σ).*

Proof: Necessity- For any set B in Y by Theorem 5.4, $f(\hat{\Omega}cl(f^{-1}(B))) \subseteq \hat{\Omega}cl(f(f^{-1}(B))) \subseteq \hat{\Omega}cl(B)$. Therefore, $\hat{\Omega}cl(f^{-1}(B)) \subseteq f^{-1}(\hat{\Omega}cl(B))$.
Sufficiency- It suffices to show that inverse image of a closed subset of Y is $\hat{\Omega}$-closed subset of X. Suppose that F is any closed set in X. Then by [8] Theorem 5.3, $F = \hat{\Omega}cl(F)$. By hypothesis, $\hat{\Omega}cl(f^{-1}(F)) \subseteq f^{-1}(\hat{\Omega}cl(F)) = f^{-1}(F)$. Again by [8] Theorem 5.3, $f^{-1}(F)$ is $\hat{\Omega}$-closed in X. □

Theorem 5.6. *$f: (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous function if and only if for every $x \in X$ and for every $V \in \hat{\Omega}O(Y, f(x))$, there exists $U \in \hat{\Omega}O(X, x)$ such that $f(U) \subseteq V$.*

Proof: Necessity- Suppose that $x \in X$ and V is any $\hat{\Omega}$-open set in (Y, σ) containing $f(x)$. Then $x \in f^{-1}(V)$. Define $U = f^{-1}(V)$. By hypothesis, U is $\hat{\Omega}$-open in (X, τ) containing x. Also $f(U) \subseteq V$.
Sufficiency- Suppose that V is any $\hat{\Omega}$-open set in (Y, σ) and $x \in f^{-1}(V)$. By hypothesis, there exists $U_x \in \hat{\Omega}O(X, x)$ such that $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ implies that $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$. By [8] Theorem 4.16, $f^{-1}(V)$ is $\hat{\Omega}$-open in (X, τ).

□

Remark 5.7. *From the following examples, it is shown that the notion of $\hat{\Omega}$-irresolute and $\hat{\Omega}$-continuous functions are independent. Moreover, the concept of $\hat{\Omega}$-irresolute and continuous functions are independent.*

Example 5.8. $X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}.Y\}$. *If $f: (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = a, f(c) = c, f(d) = d$. Then f is $\hat{\Omega}$-irresolute but not continuous.*

Example 5.9. $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}, \sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. *If $f: (X, \tau) \to (Y, \sigma)$ is defined by $f(a) = b, f(b) = c, f(c) = b$. Then f is $\hat{\Omega}$-continuous and continuous but not $\hat{\Omega}$-irresolute.*

Example 5.10. $X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{c\}, \{c, d\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}.Y\}$. *If* $f: (X, \tau) \to (Y, \sigma)$ *is defined by* $f(a) = c, f(b) = d, f(c) = a, f(d) = b$. *Then* f *is* $\hat{\Omega}$*-irresolute but not* $\hat{\Omega}$*-continuous.*

Theorem 5.11. *Let* (Y, σ) *be a* ${}_{\omega}T_{\hat{\Omega}}$*-space. If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-irresolute then it is* $\hat{\Omega}$*-continuous.Moreover, if* (X, τ) *is semi-*$T_{\frac{1}{2}}$*-space, then* f *is continuous.*

Proof: Suppose that V is open in Y. Since Y is ${}_{\omega}T_{\hat{\Omega}}$, V is $\hat{\Omega}$-open in Y. Since f is $\hat{\Omega}$-irresolute, $f^{-1}(V)$ is $\hat{\Omega}$-open in X. Therefore, f is $\hat{\Omega}$-continuous. If X is $semiT_{\frac{1}{2}}$, then $f^{-1}(V)$ is open in X and hence f is continuous. □

Theorem 5.12. *Let* (Y, σ) *be a semi-*$T_{\frac{1}{2}}$*-space. If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-continuous then it is* $\hat{\Omega}$*-irresolute.*

Proof: It follows from their definitions. □

Theorem 5.13. *Suppose* (Y, σ) *is a semi-*$T_{\frac{1}{2}}$*-space and* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space. If* $f: (X, \tau) \to (Y, \sigma)$ *is continuous then it is* $\hat{\Omega}$*-irresolute.*

Proof: It follows from their definitions. □

Composition theorems.

Theorem 5.14. *The following statements are true.*

(i) *If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-irresolute and* $g: (Y, \sigma) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute, then*
$g \circ f: (X, \tau) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute.*

(ii) *If* $f: (X, \tau) \to (Y, \sigma)$ *is continuous and* $g: (Y, \sigma) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute function, then* $g \circ f: (X, \tau) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute whenever* (Y, σ) *is semi-*$T_{\frac{1}{2}}$*-space and* (X, τ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space.*

(iii) *If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-continuous and* $g: (Y, \sigma) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute, then*
$g \circ f: (X, \tau) \to (Z, \eta)$ *is* $\hat{\Omega}$*-irresolute whenever* (Y, σ) *is semi-*$T_{\frac{1}{2}}$*-space.*

(iv) *If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-irresolute and* $g: (Y, \sigma) \to (Z, \eta)$ *is continuous function, then* $g \circ f: (X, \tau) \to (Z, \eta)$ *is* $\hat{\Omega}$*-continuous whenever* (Y, σ) *is* ${}_{\omega}T_{\hat{\Omega}}$*-space.*

(v) *If* $f: (X, \tau) \to (Y, \sigma)$ *is* $\hat{\Omega}$*-irresolute and* $g: (Y, \sigma) \to (Z, \eta)$ *is* $\hat{\Omega}$*-continuous, then*
$g \circ f: (X, \tau) \to (Z, \eta)$ *is* $\hat{\Omega}$*-continuous.*

(vi) *If* $f\colon (X,\tau)\to(Y,\sigma)$ *is* $\hat{\Omega}$*-irresolute and* $g\colon (Y,\sigma)\to(Z,\eta)$ *is* $\hat{\Omega}$*-continuous, then*
$g\circ f\colon (X,\tau)\to(Z,\eta)$ *is continuous whenever* (X,τ) *is semi-*$T_{\frac{1}{2}}$*-space..*

Proof: They follow straight forward from their definitions. □

Let us prove an application of $\hat{\Omega}$-irresolute functions as follows.

Theorem 5.15. "The Pasting lemma" *for* $\hat{\Omega}$*-irresolute function.*
Let A *and* B *be any two both open and pre-closed sets in* (X,τ) *such that* $X = A\cup B$*.Let* $f\colon (A,\tau\mid_A)\to(Y,\sigma)$ *and* $g\colon (B,\tau\mid_B)\to(Y,\sigma)$ *be* $\hat{\Omega}$*-irresolute functions such that* $f(x)=g(x)$ *for every* $x\in A\cap B$*.Then the combination* $(f\bigtriangledown g)(x)=f(x)$ *for every* $x\in A$ *and* $(f\bigtriangledown g)(y)=g(y)$ *for every* $y\in B$ *is* $\hat{\Omega}$*-irresolute function.*

Proof: It is similar to that of 4.18. □

Theorem 5.16. *Let* (X,τ) *be a Hausdorff space and if* $f\colon (X,\tau)\to(Y,\sigma)$ *is* $\hat{\Omega}$*-continuous, then the set* $A=\{x\in X : f(x)=x\}$ *is* $\hat{\Omega}$*-closed set in* (X,τ)*.*

Proof: Suppose that $x\in\hat{\Omega}cl(A)\setminus A$. Then $x\neq f(x)$. Since X is Hausdorff,there exists two disjoint open sets U and V in X such that $x\in U$ and $f(x)\in V$. Since f is $\hat{\Omega}$-continuous and by [8] Theorem 4.12, $f^{-1}(V)\cap U$ is $\hat{\Omega}$-open set containing x. Since $x\in\hat{\Omega}cl(A)$, by [8] Theorem 5.11, $f^{-1}(V)\cap U\cap A\neq\emptyset$. Therefore, $U\cap V\neq\emptyset$ a contradiction. Thus A is $\hat{\Omega}$-closed set in (X,τ). □

Corollary 5.17. *Let* (X,τ) *be a Hausdorff and* ${}_{\omega}T_{\hat{\Omega}}$*-space and if* $f\colon (X,\tau)\to(Y,\sigma)$ *is* $\hat{\Omega}$*-irresolute, then the set* $A=\{x\in X : f(x)=x\}$ *is* $\hat{\Omega}$*-closed set in* (X,τ)*.*

Theorem 5.18. *If* $f\colon (X,\tau)\to(Y,\sigma)$ *and* $g\colon (X,\tau)\to(Y,\sigma)$ *are* $\hat{\Omega}$*-irresolute from* X *into a Hausdorff space* Y*, then the set* $A=\{x\in X : f(x)=g(x)\}$ *is* $\hat{\Omega}$*-closed set in* (X,τ)*.*

Proof: Let $x\in(X\setminus A)$.Then $f(x)\neq g(x)$. Since Y is Hausdorff space, there exists two disjoint open sets U and V in Y such that $f(x)\in U$ and $g(x)\in V$. Since f is $\hat{\Omega}$-continuous and by [8] Theorem 4.12, $f^{-1}(U)\cap g^{-1}(V)$ is $\hat{\Omega}$-open set containing x. Moreover, $f^{-1}(U)\cap g^{-1}(V)\cap A=\emptyset$. Therefore, $x\in f^{-1}(U)\cap g^{-1}(V)\subseteq(X\setminus A)$. This implies that $(X\setminus A)$ is $\hat{\Omega}$-open in (X,τ) and hence A is $\hat{\Omega}$-closed set in (X,τ). □

Corollary 5.19. *Let* (Y,σ) *be a* $\hat{\Omega}$*-*T_2*-space and if* $f\colon (X,\tau)\to(Y,\sigma)$ *and* $g\colon (X,\tau)\to(Y,\sigma)$ *are* $\hat{\Omega}$*-irresolute, then the set* $A=\{x\in X : f(x)=g(x)\}$ *is* $\hat{\Omega}$*-closed set in* (X,τ)*.*

Theorem 5.20. *Let* (X,τ) *be a semi-*$T_{\frac{1}{2}}$*-space. If* $f\colon (X,\tau)\to(Y,\sigma)$ *is* $\hat{\Omega}$*-continuous injective and* Y *is Hausdorff, then* X *is Hausdorff.*

Proof: Suppose that $x, y \in X$ such that $x \neq y$. Since f is injective, $f(x) \neq f(y)$ and since Y is Hausdorff space, there exists two disjoint open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is $\hat{\Omega}$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint $\hat{\Omega}$-open sets in X containing x and y respectively. Since X is *semi-*$T_{\frac{1}{2}}$-space, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in X containing x and y respectively. Thus, (X, τ) is Hausdorff. □

Definition 5.21. *A space is said to be $\hat{\Omega}$-T_2, if for every distinct pair of points $x, y \in X$ there exists disjoint $\hat{\Omega}$-open sets in X such that $x \in U, y \in V$.*

Corollary 5.22. *Let (Y, σ) be Hausdorff space. If $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous injective, then X is $\hat{\Omega}$-T_2.*

Corollary 5.23. *Let (X, τ) be semi-$T_{\frac{1}{2}}$-space and (Y, σ) be $\hat{\Omega}$-T_2-space. If $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute injective, then X is Hausdorff.*

Corollary 5.24. *If $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute injective and Y is $\hat{\Omega}$-T_2, then X is $\hat{\Omega}$-T_2.*

6. weakly $\hat{\Omega}$-continuous

Definition 6.1. *A function $f\colon (X, \tau) \to (Y, \sigma)$ is weakly $\hat{\Omega}$-continuous at $x \in X$ if for each $G \in O(Y, f(x))$, there exists $U \in \hat{\Omega}O(X, x)$ such that $f(U) \subseteq cl(G)$. If f is weakly $\hat{\Omega}$ continuous at each $x \in X$, then f is weakly $\hat{\Omega}$-continuous on X.*

Example 6.2. $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, Y\}$. *Define $f\colon (X, \tau) \to (Y, \sigma)$ as an identity function. Then f is weakly $\hat{\Omega}$-continuous.*

Theorem 6.3. *Every $\hat{\Omega}$-continuous function is weakly $\hat{\Omega}$-continuous function.*

Proof: Let $x \in X$ and $V \in O(Y, f(x))$. By hypothesis,their exists $U \in \hat{\Omega}O(X, x)$ such that $f(U) \subseteq V \subseteq cl(V)$. Thus, f is weakly $\hat{\Omega}$-continuous function. □

Remark 6.4. *The converse is not possible in general, as seen from the example 6.2.*

The following theorem gives the conditions under which the reversible implication of Theorem 6.3 is true.

Theorem 6.5. *If (Y, σ) is regular space, then every weakly $\hat{\Omega}$-continuous function $f\colon (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-continuous function.*

Proof: Let $x \in X$ and V be any open set in Y containing $f(x)$. Since Y is regular,there exists an open set W in Y such that $f(x) \in W \subseteq cl(W) \subseteq V$. Since f is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set in U in X containing x such that $f(U) \subseteq cl(W) \subseteq V$. Therefore, f is $\hat{\Omega}$-continuous function. □

The following theorem states the characterization of weakly $\hat{\Omega}$-continuous function.

Theorem 6.6. *The following statements are equivalent.*

(i) $f\colon (X,\tau) \to (Y,\sigma)$ *is weakly* $\hat{\Omega}$*-continuous function.*

(ii) $\hat{\Omega}cl(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V))$ *for every subset* V *of* Y*.*

(iii) $\hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F)$ *for every regular closed subset* F *of* Y*.*

(iv) $\hat{\Omega}cl(f^{-1}(U) \subseteq f^{-1}(cl(U))$ *for every open subset* U *of* Y*.*

(v) $f^{-1}(U) \subseteq \hat{\Omega}int(f^{-1}(cl(U)))$ *for every open subset* U *of* Y*.*

Proof:
$(i) \Rightarrow (ii)$ Suppose that V is any subset of Y and $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ implies that there exists an open set U in Y containing $f(x)$ such that $U \cap V = \emptyset$. Therefore $U = int(U) \subseteq int(X \setminus V) = (X \setminus cl(V))$. Then $cl(U) \cap int(cl(V)) = \emptyset$ and hence $f^{-1}(int(cl(V))) \subseteq (X \setminus f^{-1}(cl(U)))$. Since f is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set W in X containing x such that $f(W) \subseteq cl(U)$ and then $W \subseteq f^{-1}(cl(U))$. Therefore, $f^{-1}(int(cl(V))) \subseteq (X \setminus W)$ where $(X \setminus W)$ is a $\hat{\Omega}$-closed set in X. By [8] the definition of $\hat{\Omega}$ closure, $\hat{\Omega}cl(f^{-1}(int(cl(V)))) \subseteq (X \setminus W)$. Then $W \cap \hat{\Omega}cl(f^{-1}(int(cl(V)))) = \emptyset$. Since $x \in W, x \notin \hat{\Omega}cl(f^{-1}(int(cl(V))))$.
$(ii) \Rightarrow (iii)$ Suppose that F is any regular closed set in Y. By hypothesis, $\hat{\Omega}cl(f^{-1}(int(cl(int(F))))) \subseteq f^{-1}(cl(int(F)))$. Then, $\hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F)$.
$(iii) \Rightarrow (iv)$ Suppose that U is any open set in Y and then $cl(U)$ is regular closed set in Y. By hypothesis, $\hat{\Omega}cl(f^{-1}(int(cl(U)))) \subseteq f^{-1}(cl(U))$. Since U is open, $\hat{\Omega}cl(f^{-1}(U) \subseteq \hat{\Omega}cl(f^{-1}(int(cl(U))) \subseteq f^{-1}(cl(U))$. Thus, $\hat{\Omega}cl(f^{-1}(U)) \subseteq f^{-1}(cl(U))$.
$(iv) \Rightarrow (v)$ Suppose that U is open in Y and then $(Y \setminus cl(U))$ is open in Y. By hypothesis, $\hat{\Omega}cl(f^{-1}(Y \setminus cl(U)) \subseteq f^{-1}(cl(Y \setminus cl(U)))$. That is, $\hat{\Omega}cl(X \setminus f^{-1}(cl(U))) \subseteq f^{-1}(cl(Y \setminus cl(U)))$. By [8] Theorem 5.15 (i), $(X \setminus \hat{\Omega}int(f^{-1}(cl(U))) \subseteq f^{-1}(cl(Y \setminus cl(U))) \subseteq (X \setminus f^{-1}(U))$. Therefore, $f^{-1}(U)) \subseteq \hat{\Omega}int(f^{-1}(cl(U)))$.
$(v) \Rightarrow (i)$ Suppose that $x \in X$ and U be any open set in Y containing $f(x)$. By hypothesis,
$f^{-1}(U) \subseteq \hat{\Omega}int(f^{-1}(cl(U)))$. By [8] Remark 5.13, $\hat{\Omega}int(f^{-1}(cl(U)))$ is a $\hat{\Omega}$ open set in X such that $\hat{\Omega}int(f^{-1}(cl(U))) \subseteq f^{-1}(cl(U))$. Therefore, $f(\hat{\Omega}int(f^{-1}(cl(U)))) \subseteq cl(U)$. If we take $W = \hat{\Omega}int(f^{-1}(cl(U)))$, then W is $\hat{\Omega}$-open set in X containing x such that $f(W) \subseteq cl(U)$. □

Theorem 6.7. $f\colon (X,\tau) \to (Y,\sigma)$ *is weakly* $\hat{\Omega}$*-continuous function if and only if* $\hat{\Omega}cl(f^{-1}(int(F))) \subseteq f^{-1}(F)$ *for every closed subset* F *of* Y*.*

Proof: **Necessity-** Suppose that F is any closed set in Y and then $(Y \setminus F)$ is open set in Y. By Theorem 6.6, (v), $f^{-1}(Y \setminus F) \subseteq \hat{\Omega}int(f^{-1}(cl(Y \setminus F)))$. Then,
$(X \setminus f^{-1}(F)) \subseteq \hat{\Omega}int(f^{-1}(Y \setminus intF))) = \hat{\Omega}int[(f^{-1}(Y)) \setminus f^{-1}(intF)] = \hat{\Omega}int[(X \setminus f^{-1}(intF)] = (X \setminus \hat{\Omega}cl(f^{-1}(intF)))$. Therefore, $\hat{\Omega}cl(f^{-1}(intF)) \subseteq f^{-1}(F)$.
Sufficiency- Suppose that $x \in X$ and U be any open set in Y containing $f(x)$.

Then, $(Y \setminus U)$ is closed in Y not containing $f(x)$. By hypothesis, $\hat{\Omega}cl(f^{-1}(int(Y \setminus U))) \subseteq f^{-1}(Y \setminus U)$. Since $x \notin f^{-1}(int(Y \setminus U))$, $x \notin \hat{\Omega}cl(f^{-1}(int(Y \setminus U)))$. By [8] Theorem 5.11, there exists $\hat{\Omega}$-open set V in X such that $V \cap f^{-1}(int(Y \setminus U)) = \emptyset$. Also $f(V) \cap int(Y \setminus U) = \emptyset$ implies that $f(V) \subset cl(U)$. Thus f is weakly $\hat{\Omega}$-continuous function. □

Remark 6.8. *If f and g are weakly $\hat{\Omega}$-continuous functions, then their composition is not always weakly $\hat{\Omega}$-continuous function from the following example. However, composition is possible if we consider g as a continuous function.*

Example 6.9. *$X = Y = Z = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}, \sigma = \{\emptyset, \{a\}, \{bcd\}, Y\}, \eta = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, Z\}$. Define $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ by $f(a) = b, f(b) = c, f(c) = d, f(d) = b, g(a) = a, g(b) = a, g(c) = d, g(d) = b$. Then f and g are weakly $\hat{\Omega}$-continuous functions where as $g \circ f : (X, \tau) \to (Z, \eta)$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ is not a weakly $\hat{\Omega}$-continuous function.*

Theorem 6.10. *If $f : (X, \tau) \to (Y, \sigma)$ is weakly $\hat{\Omega}$-continuous function and g is continuous,then $g \circ f : (X, \tau) \to (Z, \eta)$ is weakly $\hat{\Omega}$-continuous function.*

Proof: Suppose that $x \in X$ and W is any open set in Z containing $(g \circ f)(x)$. Since g is continuous, $g^{-1}(W)$ is open set in Y containing $f(x)$. Since f is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set U in X containing x such that $f(U) \subseteq cl(g^{-1}(W))$. Since g is continuous, $(g \circ f)(U) \subseteq g(f(U)) \subseteq g(cl(g^{-1}(W))) \subseteq cl(gg^{-1}(W)) \subseteq cl(W)$. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is weakly $\hat{\Omega}$-continuous function. □

Theorem 6.11. *$f : (X, \tau) \to (Y, \sigma)$ is a surjective function such that $f(U)$ is $\hat{\Omega}$-open set in Y for any $\hat{\Omega}$-open set in U in X and $g : (Y, \sigma) \to (Z, \eta)$ is any function. If $g \circ f : (X, \tau) \to (Z, \eta)$ is weakly $\hat{\Omega}$-continuous,then g is weakly $\hat{\Omega}$-continuous function.*

Proof: Suppose that $y \in Y$ and W is any open set in Z containing $g(y)$. Since f is surjective, there exists $x \in X$ such that $f(x) = y$. Then $(g \circ f)(x) = g(f(x)) = g(y) \in W$. Since $g \circ f$ is weakly $\hat{\Omega}$-continuous, there exists an $\hat{\Omega}$-open set U in X containing x such that $(g \circ f)(U) \subseteq cl(W)$. If we take $V = f(U)$, then by our assumption, V is a $\hat{\Omega}$-open set in Y containing $f(x)$ and $(g \circ f)(U) = g(V) \subseteq cl(W)$. Thus, g is weakly $\hat{\Omega}$-continuous function. □

Theorem 6.12. *If $f : (X, \tau) \to (Y, \sigma)$ is $\hat{\Omega}$-irresolute function and g is weakly $\hat{\Omega}$-continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is weakly $\hat{\Omega}$-continuous function.*

Proof: Suppose that $x \in X$ and W is any open set in Z containing $(g \circ f)(x)$. Since g is weakly $\hat{\Omega}$-continuous,there exists $\hat{\Omega}$-open set V in Y containing $f(x)$ such that $g(V) \subseteq cl(W)$. Since f is $\hat{\Omega}$-irresolute,there exists $\hat{\Omega}$-open set U in X containing x such that $f(U) \subseteq V$. Then $(g \circ f)(U) \subseteq g(f(U)) \subseteq g(V) \subseteq cl(W)$. Therefore, $g \circ f : (X, \tau) \to (Z, \eta)$ is weakly $\hat{\Omega}$-continuous function. □

Theorem 6.13. *If $f\colon (X,\tau) \to (Y,\sigma)$ is weakly $\hat{\Omega}$-continuous and Y is Hausdorff, then f has $\hat{\Omega}$-closed points inverses.*

Proof: Suppose that $y \in Y$ and $x \in \{x \in X : f(x) \neq y\}$.Since $f(x) \neq y$ in a Hausdorff space Y, there exists two disjoint open sets U and V in Y such that $f(x) \in U$ and $y \in V$.Moreover,$cl(U) \cap V = \emptyset$ implies that $y \notin cl(U)$. Since f is weakly $\hat{\Omega}$-continuous,there exists $\hat{\Omega}$-open set W in X containing x such that $f(W) \subseteq cl(U)$. If $W \nsubseteq \{x \in X : f(x) \neq y\}$,then choose a point $z \in W$ such that $f(z) = y$. Then $y = f(z) \in f(W) \subseteq cl(U)$, a contradiction.Hence $x \in W \subseteq \{x \in X : f(x) \neq y\}$ and hence $\{x \in X : f(x) \neq y\}$ is $\hat{\Omega}$-open in X.Therefore, $f^{-1}(y) = \{x \in X : f(x) = y\}$ is $\hat{\Omega}$-closed in X. □

Definition 6.14. *If $A \subseteq X$, weakly $\hat{\Omega}$-continuous retraction of X onto A is a weakly $\hat{\Omega}$-continuous function $f\colon X \to A$ such that $f(a) = a$ for each $a \in A$.*

Theorem 6.15. *If $A \subseteq X$, and $f\colon X \to A$ is a weakly $\hat{\Omega}$-continuous retraction of a Hausdorff space X onto A, then A is $\hat{\Omega}$-closed in X.*

Proof: Suppose that there exists $x \in \hat{\Omega}cl(A) \setminus A$. Then $x \neq f(x)$. Since X is Hausdorff, there exists two disjoint open sets U and V in X such that $x \in U$ and $f(x) \in V$.Also $U \cap cl(V) = \emptyset$. Let W be an arbitrary $\hat{\Omega}$-open set in X containing x. By [8] Theorem 4.12, $U \cap W$ is a such that $\hat{\Omega}$-open set in X containing x. Since $x \in \hat{\Omega}cl(A)$, by [8] Theorem 5.11, $U \cap W \cap A \neq \emptyset$. If we choose $y \in U \cap W \cap A$, then $y \in A$ implies that $y = f(y) \in U$ and hence $f(y) \notin cl(V)$. That is,there exists $y \in W$ such that $f(y) \notin cl(V)$. Therefore, $f(W) \nsubseteq cl(V)$ for any $\hat{\Omega}$-open set in X containing x, a contradiction. Thus, A is $\hat{\Omega}$-closed in X. □

Definition 6.16. *A space (X,τ) is called Urysohn space if for every pair of distinct points x and y in X, there exists open sets U and V in X such that $x \in U$, $y \in V$ and $cl(U) \cap cl(V) = \emptyset$.*

Theorem 6.17. *If $f\colon (X,\tau) \to (Y,\sigma)$ is injective weakly $\hat{\Omega}$-continuous from X into a Urysohn space Y, then X is $\hat{\Omega}$-T_2-space.*

Proof: Suppose that $x, y \in X$ such that $x \neq y$. Since f is injective, $f(x) \neq f(y)$ and since Y is Urysohn space, there exists two open sets U and V in Y such that $f(x) \in U, f(y) \in V$ and $cl(U) \cap cl(V) = \emptyset$. Since f is weakly $\hat{\Omega}$-continuous, there exists $\hat{\Omega}$-open set U_1 and V_1 in X containing x and y respectively such that $f(U_1) \subseteq cl(U), f(V_1) \subseteq cl(V)$. Moreover, $U_1 \cap V_1 = \emptyset$. Thus, X is $\hat{\Omega}$-T_2-space. □

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