



## Existence of solutions for a fourth order problem at resonance

El. M. Hssini, M. Massar, M. Talbi and N. Tsouli

ABSTRACT: In this work, we are interested at the existence of nontrivial solutions of two fourth order problems governed by the weighted p-biharmonic operator. The first is the following

$$\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda_1 m(x)|u|^{p-2}u + f(x, u) - h \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega,$$

where  $\lambda_1$  is the first eigenvalue for the eigenvalue problem  $\Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u$  in  $\Omega$ ,  $u = \Delta u = 0$  on  $\partial\Omega$ . In the seconde problem, we replace  $\lambda_1$  by  $\lambda$  such that  $\lambda_1 < \lambda < \bar{\lambda}$ , where  $\bar{\lambda}$  is given bellow.

Key Words: p-biharmonic, weight, resonance, saddle point theorem.

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### 1. Introduction and main results

In the present paper, we are concerned with the existence of weak solutions of the following problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda_1 m(x)|u|^{p-2}u + f(x, u) - h & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $p > 1$ ,  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  ( $N \geq 1$ ) with smooth boundary  $\partial\Omega$ ,  $\rho \in C(\bar{\Omega})$ , with  $\inf_{\bar{\Omega}} \rho(x) > 0$ ,  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a bounded Carathéodory function,  $h \in L^{p'}(\Omega)$ , ( $p' = \frac{p}{p-1}$ ),  $m \in C(\bar{\Omega})$  is nonnegative weight function and  $\lambda_1$  design the first eigenvalue for the eigenvalue problem

$$\begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, see for example [1,3,6,7,12]. In [7], Liu and Squassina study the following p-biharmic problem

$$\begin{cases} \Delta(\Delta u|^{p-2}\Delta u) = g(x, u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Under some conditions on  $g(x, u)$  at resonance, the authors established the existence of at least one nontrivial solution.

According to the work of Talbi and Tsouli [10], the eigenvalue problem (1.2) has a nondecreasing and unbounded sequence of eigenvalues, and the first eigenvalue  $\lambda_1$  is given by

$$\lambda_1 = \inf_{u \in X} \left\{ \int_{\Omega} \rho |\Delta u|^p dx : \int_{\Omega} m(x) |u|^p dx = 1 \right\},$$

where  $X := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$  is the reflexive Banach space endowed with the norm

$$\|u\| = \left( \int_{\Omega} \rho |\Delta u|^p dx \right)^{1/p}.$$

Since  $m \in C(\bar{\Omega})$  and  $m \geq 0$ ,  $\lambda_1$  is positive, simple and isolated. Therefore

$$\int_{\Omega} \rho |\Delta u|^p dx \geq \lambda_1 \int_{\Omega} m(x) |u|^p dx \quad \text{for all } u \in X. \tag{1.3}$$

Moreover, there exists a unique positive eigenfunction  $\varphi_1$  associated to  $\lambda_1$ , which can be chosen normalized. Let

$$\lambda_2 := \inf \{ \lambda : \lambda \text{ is a eigenvalue of (1.2), with } \lambda > \lambda_1 \}.$$

The fact that  $\lambda_1$  is isolated implies that  $\lambda_1 < \lambda_2$ . It can also be shown (see Lemma 2.1) that there exists  $\bar{\lambda} \in (\lambda_1, \lambda_2]$  such that

$$\int_{\Omega} \rho |\Delta u|^p dx \geq \bar{\lambda} \int_{\Omega} m(x) |u|^p dx, \tag{1.4}$$

for all  $u \in X$  with  $\int_{\Omega} m(x) \varphi_1^{p-1} u dx = 0$ .

In addition, we study the existence of solutions for the following boundary value problem

$$\begin{cases} \Delta(\rho |\Delta u|^{p-2} \Delta u) = \lambda m(x) |u|^{p-2} u + f(x, u) - h & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

We assume that the function  $f$  satisfy the following hypotheses:

(H) For almost every  $x \in \Omega$ , there exist

$$\lim_{s \rightarrow -\infty} f(x, s) = l(x), \quad \lim_{s \rightarrow +\infty} f(x, s) = k(x). \tag{1.6}$$

Let us recall the minimum principle and the saddle point theorem (see [9]).

**Theorem 1.1.** *Let  $X$  be a Banach space and  $\Phi \in C^1(X, \mathbb{R})$ . Assume that*

(i)  $\Phi$  satisfies the Palais-Smale condition,

(ii)  $\Phi$  is bounded from below  $c = \inf_X \Phi$ .

*Then there exists  $u_0 \in X$  such that  $\Phi(u_0) = c$ .*

**Theorem 1.2.** *Let  $X$  be a Banach space. Let  $\Phi : X \rightarrow \mathbb{R}$  be a  $C^1$  functional that satisfies the Palais-Smale condition, and suppose that  $X = V \oplus W$ , with  $V$  a finite dimensional subspace of  $X$ . If there exists  $R > 0$  such that*

$$\max_{v \in V, \|v\|=R} \Phi(v) < \inf_{w \in W} \Phi(w),$$

then  $\Phi$  has at least a critical point on  $X$ .

Now, we are ready to state our main results.

**Theorem 1.3.** *Assume that (1.6) holds. Suppose that  $h \in L^{p'}(\Omega)$  is such that either*

$$\int_{\Omega} k(x)\varphi_1 dx < \int_{\Omega} h(x)\varphi_1 dx < \int_{\Omega} l(x)\varphi_1 dx \tag{1.7}$$

or

$$\int_{\Omega} l(x)\varphi_1 dx < \int_{\Omega} h(x)\varphi_1 dx < \int_{\Omega} k(x)\varphi_1 dx. \tag{1.8}$$

Then problem (1.1) has at least a weak solution .

**Theorem 1.4.** *Assume that (1.6) holds. If  $h \in L^{p'}(\Omega)$  satisfy (1.7) or (1.8), then problem (1.5) with  $\lambda_1 < \lambda < \bar{\lambda}$ , has at least one solution.*

### 2. Preliminaries and proofs of Theorems

We consider the following energy functional  $\Phi : X \rightarrow \mathbb{R}$  defined by

$$\Phi(u) = \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x)|u|^p dx - \int_{\Omega} F(x, u) dx + \int_{\Omega} h u dx,$$

where

$$F(x, t) = \int_0^t f(x, s) ds \text{ for almost every } x \in \Omega, \forall t \in \mathbb{R}.$$

It is well known that  $\Phi \in \mathcal{C}^1(X, \mathbb{R})$ , with derivative at point  $u \in X$  is given by

$$\langle \Phi'(u), v \rangle = \int_{\Omega} \rho |\Delta u|^{p-2} \Delta u \Delta v dx - \lambda_1 \int_{\Omega} m(x)|u|^{p-2} u v dx - \int_{\Omega} f(x, u) v dx + \int_{\Omega} h v dx,$$

for every  $v \in X$ .

Let denote  $V = \langle \varphi_1 \rangle$  the linear spans of  $\varphi_1$  and

$$W = \left\{ u \in X : \int_{\Omega} m(x)\varphi_1^{p-1} u dx = 0 \right\}. \tag{2.1}$$

Then we can decompose  $X$  as a direct sum of  $V$  and  $W$ . In fact, let  $u \in X$ , writing

$$u = \alpha \varphi_1 + w,$$

where  $w \in X$ , and  $\alpha = \lambda_1 \int_{\Omega} m(x)\varphi_1^{p-1} u dx$ .

Since

$$\int_{\Omega} \rho |\Delta \varphi_1|^p dx = 1,$$

$$\int_{\Omega} m(x)\varphi_1^{p-1}w dx = 0.$$

Therefore  $w \in W$ , hence

$$X = V \oplus W.$$

We begin by establishing the existence of  $\bar{\lambda}$  for which (1.4) holds.

**Lemma 2.1.** *There exists  $\bar{\lambda} \in (\lambda_1, \lambda_2]$  such that*

$$\int_{\Omega} \rho|\Delta u|^p dx \geq \bar{\lambda} \int_{\Omega} m(x)|u|^p dx, \quad (2.2)$$

for all  $u \in W$ .

**Proof:** Let

$$\lambda = \inf \left\{ \int_{\Omega} \rho|\Delta u|^p dx : u \in W, \int_{\Omega} m(x)|u|^p dx = 1 \right\}.$$

This value is attained in  $W$ . To see why this is so, let  $(u_n)$  be a sequence in  $W$ , satisfying  $\int_{\Omega} m(x)|u_n|^p dx = 1$  for all  $n$ , and  $\int_{\Omega} \rho|\Delta u_n|^p dx \rightarrow \lambda$ . It follows that  $(u_n)$  is bounded in  $X$  and therefore, up to subsequence, we may assume that

$$u_n \rightharpoonup u \text{ weakly in } X \quad \text{and} \quad u_n \rightarrow u \text{ strongly in } L^p(\Omega).$$

From the strong convergence of the sequence in  $L^p(\Omega)$  we obtain

$$\int_{\Omega} m(x)|u|^p dx = \lim_{n \rightarrow \infty} \int_{\Omega} m(x)|u_n|^p dx = 1$$

and

$$\int_{\Omega} m(x)\varphi_1^{p-1}u dx = \lim_{n \rightarrow \infty} \int_{\Omega} m(x)\varphi_1^{p-1}u_n dx = 0,$$

so that  $u \in W$ . By the weakly lower semicontinuity of the norm  $\|\cdot\|$ , we get

$$\lambda \leq \int_{\Omega} \rho|\Delta u|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \rho|\Delta u_n|^p dx = \lambda,$$

and hence  $\lambda$  is attained at  $u$ .

Now we claim that  $\lambda > \lambda_1$ . It follows from (1.3) that  $\lambda \geq \lambda_1$ . If  $\lambda = \lambda_1$ , by simplicity of  $\lambda_1$  there is  $\alpha \in \mathbb{R}$  such that  $u = \alpha\varphi_1$ . Since  $u \in W$ ,

$$\alpha \int_{\Omega} m(x)\varphi_1^p dx = 0,$$

which implies  $\alpha = 0$ . This contradicts the fact that  $\int_{\Omega} m(x)|u|^p dx = 1$ . So, choose  $\bar{\lambda} = \min\{\lambda, \lambda_2\}$ . It is clear that  $\bar{\lambda}$  satisfies (2.2) and the proof of lemma is complete.  $\square$

**Lemma 2.2.** *Assume that (1.6) and (1.7) or (1.8) are verified. Then the functional  $\Phi$  satisfies the Palais-Smale condition on  $X$ .*

**Proof:** Let  $(u_n)$  be a sequence in  $X$ , and  $c$  a real number such that:

$$|\Phi(u_n)| \leq c \text{ for all } n, \tag{2.3}$$

$$\Phi'(u_n) \rightarrow 0. \tag{2.4}$$

We claim that  $(u_n)$  is bounded in  $X$ . Indeed, suppose by contradiction that

$$\|u_n\| \rightarrow +\infty, \text{ as } n \rightarrow +\infty.$$

Put  $v_n = u_n/\|u_n\|$ , thus  $(v_n)$  is bounded, for a subsequence still denoted  $(v_n)$ , we can assume that  $v_n \rightharpoonup v$  weakly in  $X$ , by Sobolev injection theorem we have  $v_n \rightarrow v$  strongly in  $L^p(\Omega)$ , and  $v_n \rightarrow v$  a.e. in  $\Omega$ . Dividing (2.3) by  $\|u_n\|^p$ , we get

$$\lim_{n \rightarrow +\infty} \left( \frac{1}{p} \int_{\Omega} \rho |\Delta v_n|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x) |v_n|^p dx - \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^p} dx + \int_{\Omega} h \frac{u_n}{\|u_n\|^p} dx \right) = 0. \tag{2.5}$$

By the hypotheses on  $f, h$  and  $(u_n)$ , we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \frac{F(x, u_n)}{\|u_n\|^p} dx = \lim_{n \rightarrow +\infty} \int_{\Omega} h \frac{u_n}{\|u_n\|^p} dx = 0,$$

while

$$\lim_{n \rightarrow +\infty} \int_{\Omega} m(x) |v_n|^p dx = \int_{\Omega} m(x) |v|^p dx$$

then, from (2.5) we deduce that

$$1 = \lim_{n \rightarrow +\infty} \int_{\Omega} \rho |\Delta v_n|^p dx = \lambda_1 \int_{\Omega} m(x) |v|^p dx.$$

Then  $v \neq 0$ . According to the definition of  $\lambda_1$  and the weak lower semi continuity of norm, one has

$$\lambda_1 \int_{\Omega} m(x) |v|^p dx \leq \int_{\Omega} \rho |\Delta v|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} \rho |\Delta v_n|^p dx = \lambda_1 \int_{\Omega} m(x) |v|^p dx.$$

This implies that

$$v_n \rightarrow v \text{ strongly in } X \quad \text{and} \quad \int_{\Omega} \rho |\Delta v|^p dx = \lambda_1 \int_{\Omega} m |v|^p dx.$$

By the definition of  $\varphi_1$ , we deduce that  $v = \pm \varphi_1$ .

On the other hand, from (2.3) we have

$$-cp \leq \int_{\Omega} \rho |\Delta u_n|^p dx - \lambda_1 \int_{\Omega} m(x) |u_n|^p dx - p \int_{\Omega} F(x, u_n) dx + p \int_{\Omega} h u_n dx \leq cp \tag{2.6}$$

In view of (2.4), for all  $\varepsilon > 0$  and  $n$  large enough, we have

$$-\varepsilon\|u_n\| \leq -\int_{\Omega} \rho|\Delta u_n|^p dx + \lambda_1 \int_{\Omega} m|u_n|^p dx + \int_{\Omega} f(x, u_n)u_n dx - \int_{\Omega} hu_n dx \leq \varepsilon\|u_n\| \quad (2.7)$$

Let

$$g(x, s) = \begin{cases} \frac{F(x, s)}{s} & \text{if } s \neq 0 \\ f(x, 0) & \text{if } s = 0. \end{cases} \quad (2.8)$$

Suppose that  $v_n \rightarrow -\varphi_1$  (for example), then  $u_n(x) \rightarrow -\infty$  for a.e.  $x \in \Omega$ , it follows from (1.6) that

$$\begin{cases} f(x, u_n) \rightarrow l(x) & \text{a.e } x \in \Omega \\ g(x, u_n) \rightarrow l(x) & \text{a.e } x \in \Omega, \end{cases}$$

Moreover, the Lebesgue's theorem imply

$$\lim_{n \rightarrow +\infty} \int_{\Omega} (f(x, u_n)v_n - pg(x, u_n)v_n) dx = (p-1) \int_{\Omega} l(x)\varphi_1 dx. \quad (2.9)$$

Combining (2.6) and (2.7), we get

$$-cp - \varepsilon\|u_n\| \leq \int_{\Omega} f(x, u_n)u_n dx - p \int_{\Omega} F(x, u_n) dx + (p-1) \int_{\Omega} hu_n dx \leq cp + \varepsilon\|u_n\|.$$

Dividing by  $\|u_n\|$  the last inequalities, we obtain

$$\frac{-cp}{\|u_n\|} - \varepsilon \leq \int_{\Omega} f(x, u_n)v_n dx - p \int_{\Omega} g(x, u_n)v_n dx + (p-1) \int_{\Omega} hv_n dx \leq \frac{cp}{\|u_n\|} + \varepsilon,$$

and passing to the limits, we deduce from (2.9) that

$$\int_{\Omega} l(x)\varphi_1 dx = \int_{\Omega} h(x)\varphi_1 dx,$$

which contradicts both (1.7) and (1.8). Thus  $(u_n)$  is bounded in  $X$ , for a subsequence denoted also  $(u_n)$ , there exists  $u \in X$  such that  $u_n \rightharpoonup u$  weakly in  $X$ , and strongly in  $L^p(\Omega)$ . From

$$\lim_{n \rightarrow +\infty} \langle \Phi'(u_n), (u_n - u) \rangle = 0,$$

that is

$$\begin{aligned} \langle \Phi'(u_n), (u_n - u) \rangle &= \int_{\Omega} \rho|\Delta u_n|^{p-2} \Delta u_n \Delta(u_n - u) dx \\ &\quad - \lambda_1 \int_{\Omega} m(x)|u_n|^{p-2} u_n (u_n - u) dx \\ &\quad - \int_{\Omega} f(x, u_n)(u_n - u) dx + \int_{\Omega} h(u_n - u) dx \\ &= o_n(1). \end{aligned}$$

Using the hypotheses on  $m$ ,  $h$  and  $f$ , we see that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\Omega} m(x)|u_n|^{p-2}u_n(u_n - u)dx &= 0, \quad \lim_{n \rightarrow +\infty} \int_{\Omega} f(x, u_n)(u_n - u)dx = 0 \\ \lim_{n \rightarrow +\infty} \int_{\Omega} h(u_n - u)dx &= 0. \end{aligned}$$

Consequently,

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \rho|\Delta u_n|^{p-2}\Delta u_n\Delta(u_n - u)dx = 0.$$

In the same way, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} \rho|\Delta u|^{p-2}\Delta u\Delta(u_n - u)dx = 0.$$

Therefore

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \int_{\Omega} (\rho|\Delta u_n|^{p-2}\Delta u_n - \rho|\Delta u|^{p-2}\Delta u)\Delta(u_n - u)dx \\ &\geq \lim_{n \rightarrow \infty} (||u_n||^{p-1} - ||u||^{p-1})(||u_n|| - ||u||) \geq 0, \end{aligned}$$

hence  $||u_n|| \rightarrow ||u||$ . By the uniform convexity of  $X$ , it follows that  $u_n \rightarrow u$  strongly in  $X$  and  $\Phi$  satisfies the (PS) condition.  $\square$

**Lemma 2.3.** *Assume that (1.6) and (1.7) are satisfied. Then the functional  $\Phi$  is coercive on  $X$ .*

**Proof:** Suppose by contadiction that  $\Phi$  is not coercive, then there exists a sequence  $(u_n)$  such that  $||u_n|| \rightarrow +\infty$ , and  $|\Phi(u_n)| \leq C$ .

In the proof of lemma 2.2, we have showed that  $v_n = u_n/||u_n|| \rightarrow \pm\varphi_1$ .

Since

$$\begin{aligned} 0 &\leq \int_{\Omega} \rho|\Delta u_n|^p dx - \lambda_1 \int_{\Omega} m|u_n|^p dx, \\ &- \int_{\Omega} F(x, u_n)dx + \int_{\Omega} hu_n dx \leq \Phi(u_n) \leq C. \end{aligned} \tag{2.10}$$

Assume  $v_n \rightarrow -\varphi_1$  (for example). Dividing (2.10) by  $||u_n||$ , we get

$$- \int_{\Omega} \frac{F(x, u_n)}{||u_n||} dx + \int_{\Omega} h \frac{u_n}{||u_n||} dx \leq \frac{C}{||u_n||}.$$

Passing to the limits, we have

$$\int_{\Omega} l(x)\varphi_1 dx \leq \int_{\Omega} h(x)\varphi_1 dx$$

which contradicts (1.7).  $\square$

**Proof:** [Proof of Theorem 1.3]. If (1.7) holds, the coerciveness of the functional  $\Phi$  and the Palais-Smale condition entrain, from theorem 1.1, that  $\Phi$  attains its minimum, so problem (1.1) admits at least a weak solution in  $X$ .

If (1.8) holds, then  $\Phi$  has the geometry of the saddle point theorem 1.2. Indeed, splitting  $X = V \oplus W$ . Let  $u \in W$ , using Höder inequality and the properties of  $F$ , since  $\bar{\lambda} > \lambda_1$

$$\begin{aligned} \Phi(u) &\geq \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx - \frac{\lambda_1}{p} \int_{\Omega} m(x) |u|^p dx - \int_{\Omega} F(x, u) dx + \int_{\Omega} h(x) u dx \\ &\geq \frac{1}{p} \left(1 - \frac{\lambda_1}{\bar{\lambda}}\right) \|u\|^p - C(\|b\|_{\infty} |\Omega|^{\frac{1}{p'}} + \|h\|_{p'}) \|u\|, \end{aligned} \quad (2.11)$$

where  $C$  is the embedding constants of Sobolev,  $\|\cdot\|_{p'}$  and  $\|\cdot\|_{\infty}$  denote the norms in  $L^{p'}(\Omega)$  and  $L^{\infty}(\Omega)$  respectively. Then  $\Phi$  is bounded from below on  $W$ , is a consequence of the assumption that  $p > 1$ , so that

$$\inf_{w \in W} \Phi(w) > -\infty. \quad (2.12)$$

On the other hand, for every  $t \in \mathbb{R}$ , one has

$$\begin{aligned} \Phi(t\varphi_1) &= - \int_{\Omega} F(x, t\varphi_1) dx + t \int_{\Omega} h(x) \varphi_1 dx \\ &= t \left( \int_{\Omega} h(x) \varphi_1 dx - \int_{\Omega} g(x, t\varphi_1) \varphi_1 dx \right) \end{aligned}$$

where  $g$  has been defined by (2.8). From the Lebesgue theorem, it follows that

$$\lim_{t \rightarrow +\infty} \left( \int_{\Omega} h(x) \varphi_1 dx - \int_{\Omega} g(x, t\varphi_1) \varphi_1 dx \right) = \int_{\Omega} (h(x) - k(x)) \varphi_1 dx, \quad (2.13)$$

and the limit is negative by (1.8). Analogously, if  $t$  tends to  $-\infty$ , we have the same result with  $k(x)$  exchanged with  $l(x)$ , so that the limit is positive by (1.8). In both cases we get

$$\lim_{t \rightarrow \pm\infty} \Phi(t\varphi_1) = -\infty \quad (2.14)$$

By (2.12) and (2.14), there exists  $R > 0$  such that

$$\max_{v \in V, \|v\|=R} \Phi(v) < \inf_{w \in W} \Phi(w).$$

Hence,  $\Phi$  satisfies the hypotheses of Theorem 1.2, and there exists a critical point of  $\Phi$ , that is a solution of (1.1).  $\square$

**Proof:** [Proof of Theorem 1.4]. The result of lemma 2.2 holds true for the Euler functional associated to problem (1.5), that is

$$\Phi_{\lambda}(u) = \frac{1}{p} \int_{\Omega} \rho |\Delta u|^p dx - \frac{\lambda}{p} \int_{\Omega} m(x) |u|^p dx - \int_{\Omega} F(x, u) dx + \int_{\Omega} h u dx \quad (2.15)$$



for every  $u \in X$ . Indeed, Let  $(u_n)$  be a sequence satisfying (2.3) and (2.4), suppose that  $(u_n)$  is unbounded, and define  $v_n = u_n/||u_n||$ , so that, up to subsequence,  $(v_n)$  converges weakly to a function  $v$  in  $X$ . Dividing (2.4) by  $||u_n||^{p-1}$ , and then taking  $\langle \Phi'_\lambda(u_n), v_n - v \rangle = o_n(1)$ , we get

$$\lim_{n \rightarrow +\infty} \int_{\Omega} \rho |\Delta v_n|^{p-2} \Delta v_n \Delta (v_n - v) dx = 0$$

this fact implies (as in proof of lemma 2.2) that  $v_n \rightarrow v$  strongly in  $X$ . since  $\langle \Phi'_\lambda(u_n), \psi/||u_n||^{p-1} \rangle = o_n(1)$ , with  $\psi \in X$ ,

$$\int_{\Omega} \rho |\Delta v|^{p-2} \Delta v \Delta \psi dx = \lambda \int_{\Omega} m |v|^{p-2} v \psi dx,$$

so that  $v$  solve the problem  $\Delta(\rho |\Delta u|^{p-2} \Delta u) = \lambda m(x) |u|^{p-2} u$  with Navier boundary condition on  $\partial\Omega$ . But this equation, being  $\lambda \in (\lambda_1, \bar{\lambda}) \subset (\lambda_1, \lambda_2)$ , has zero as the only solution by definition of  $\bar{\lambda}$ . Thus  $v = 0$ , a contradiction with the strong convergence of  $v_n$  to  $v$ . Hence  $(u_n)$  is bounded. This implies, by same argument in proof of lemma 2.2, that  $(u_n)$  is strongly convergent.

On the other hand, as in the second part of the proof of Theorem 1.3, rewrite everything with  $\lambda$  instead of  $\lambda_1$  and use the fact that  $\lambda < \bar{\lambda}$ , we get the coerciveness of  $\Phi_\lambda$  on  $W$ .

Now, recalling that

$$\int_{\Omega} \rho |\Delta t\varphi_1|^p dx = \lambda_1 \int_{\Omega} m(x) |t\varphi_1|^p dx, \quad \text{for every } t \in \mathbb{R}$$

thus

$$\Phi_\lambda(t\varphi_1) = \frac{\lambda_1 - \lambda}{p} |t|^p \int_{\Omega} m |\varphi_1|^p dx + t \left( \int_{\Omega} h(x) \varphi_1 dx - \int_{\Omega} g(x, t\varphi_1) \varphi_1 dx \right),$$

since  $\lambda > \lambda_1$  and  $p > 1$ , we have, as before

$$\lim_{t \rightarrow \pm\infty} \Phi_\lambda(t\varphi_1) = -\infty.$$

Using again the saddle point theorem, the desired result follows. □

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*EL Miloud Hssini; Mohammed Massar; Najib Tsouli*  
*University Mohamed I, Faculty of Sciences,*  
*Department of Mathematics, Oujda, Morocco.*  
*E-mail address: hssini1975@yahoo.fr; massarmed@hotmail.com; tsouli@hotmail.com*

*and*

*Mohamed Talbi*  
*Centre R gional de M tiers de l' ducation et de Formation (CRMEF),*  
*Oujda, Morocco.*  
*E-mail address: talbi\_md@yahoo.fr*