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Contra $\omega\beta-$continuity

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abstract: This paper is dealing with the application of the notion of $\omega\beta-$open sets in topological spaces to present and study a new class of functions called contra $\omega\beta-$continuous functions. This notion is a weak form of contra-continuity. We also discuss the relationships between this new class and other classes of functions and some examples of applications are shown.

Key Words: Contra $\omega\beta-$continuous, Contra $\omega\beta-$closed graphs, Strongly $S-$closed, $\omega\beta-$open set.

Contents

1. Introduction

Recently authors [2], [1] introduced the concepts of $\omega\beta-$open sets and $\omega\beta$-continuity in a topological space and investigated some of their properties. In 1996, Dontchev [7] introduced a new class of functions called contra-continuous functions. He defined a function $f : X \to Y$ to be contra-continuous if the preimage of every open set of Y is closed in X. Two new weaker form of this class of functions are introduced: contra $\omega-$continuity is introduced by Al-Omari and Noorani [4] and contra $\beta-$continuity is investigated by Caldas and Navalagi [6]. In this direction, we will introduce the concept of contra $\omega\beta-$continuous functions which is weaker than contra-continuous, via the notion of $\omega\beta-$open sets. Some characterizations and several basic properties of this class of functions are obtained.

Throughout the present paper, a space (X, τ) mean a topological space on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a space (X, τ). The closure of A and interior of A in (X, τ) are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A of a space (X, τ) is said to be $\beta-$open [11]

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if $A \subseteq Cl(Int(Cl(A)))$. Recall that a subset A of a space (X, τ) is said to be $\omega\beta-$open [2] (resp. $\omega-$open [5]) set if for every $x \in A$ there exists a $\beta-$open (resp. open) set U containing x such that $U - A$ is countable. The complement of an $\omega\beta-$open set is said to be $\omega\beta-$closed [2]. The intersection of all $\omega\beta-$closed sets of X containing A is called the $\omega\beta-$closure of A and is denoted by $\omega\beta Cl(A)$. The union of all $\omega\beta-$open sets of X contained in A is called the $\omega\beta-$interior of A and is denoted by $\omega\beta Int(A)$.

2. Contra $\omega\beta-$continuous functions

We introduce the definition of contra $\omega\beta-$continuous functions in topological spaces and study some of their properties in this section.

Definition 2.1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra $\omega\beta-$continuous if $f^{-1}(V)$ is $\omega\beta-$closed in (X, τ) for each open set V of (Y, σ).*

Observe that if X is a countable set, then every function $f : (X, \tau) \to (Y, \sigma)$ is contra $\omega\beta-$continuous.
It is obvious that every contra-continuous function is contra $\omega\beta-$continuous. However the following example shows that the converse need not be true in general.

Example 2.2. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 2, & x \in \mathbb{Q} \end{cases}$$

Then f is contra $\omega\beta-$continuous but not contra-continuous.

Recall that the kernel of a set A [12], denoted $\ker(A)$, is the intersection of all open supersets of A.

Lemma 2.3. *[10] Let A and B be subsets of a topological space (X, τ), then the following properties hold:*

(i) $x \in Ker(A)$ if and only if $A \cap F \neq \phi$ for any closed set F in (X, τ) containing x.

(ii) $A \subseteq Ker(A)$ and if A is open in (X, τ), then $A = Ker(A)$.

(iii) If $A \subseteq B$, then $Ker(A) \subseteq Ker(B)$.

Proposition 2.4. *[2] Let (X, τ) be a topological space.*

(i) The union of any family of $\omega\beta-$open sets is $\omega\beta-$open.

(ii) The intersection of an $\omega\beta-$open set and an $\omega-$open set is $\omega\beta-$open.

A subfamily m_X of the power set P(X) of a non-empty set X is called a minimal structure on X ([14], [15]) if ϕ, $X \in m_X$. Let (X, τ) be a topological space, then the set of all $\omega\beta-$open sets of X is a minimal structure on X. The notion of contra $m-$continuous functions [13] is defined as follows: for each open set V of (Y, σ), $f^{-1}(V)$ is m_X-closed in (X, τ). By Definition 2.1 and Proposition 2.4 it follows that contra $\omega\beta-$continuous functions are particular cases of contra $m-$continuous functions. Hence, some results from this paper are particular cases of some results from [13].

Theorem 2.5. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*

(i) f is contra $\omega\beta-$continuous.

(ii) $f^{-1}(F)$ is $\omega\beta-$open in (X, τ) for every closed subset F of (Y, σ).

(iii) For each $x \in X$ and each closed set F in (Y, σ) containing $f(x)$, there exists an $\omega\beta-$open set U in (X, τ) containing x such that $f(U) \subseteq F$.

(iv) $f(\omega\beta Cl(A)) \subseteq \ker(f(A))$ for every subset A of (X, τ).

(v) $\omega\beta Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset B of (Y, σ).

Proof: The proof follows by Theorem 3.2 and Corollarly 3.2 of [13]. □

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is contra $\omega-$continuous [4] if $f^{-1}(V)$ is $\omega-$closed in (X, τ) for each open set V of (Y, σ). Since every $\omega-$open set is $\omega\beta-$open, then every contra $\omega-$continuous function is contra $\omega\beta-$continuous, but the converse is not true as the following example shows.

Example 2.6. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_u$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 2, & x \in \mathbb{Q} \end{cases}$$

Then f is contra $\omega\beta-$continuous but not contra $\omega-$continuous.

A function f is called contra $\beta-$continuous [6] if $f^{-1}(V)$ is $\beta-$closed in (X, τ) for each open set V of (Y, σ). Since every $\beta-$open set is $\omega\beta-$open, then every contra $\beta-$continuous function is contra $\omega\beta-$continuous but the converse is not true as shown by the following example.

Example 2.7. *Let $X = \{1, 2, 3\}$ with the topologies $\tau = \{\phi, X, \{2\}, \{3\}, \{2, 3\}\}$ and $\sigma = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 3, & x = 1 \\ 1, & x = 2 \end{cases}$$

Then f is contra $\omega\beta-$continuous but not contra $\beta-$continuous.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is called $\omega\beta-$continuous [1] if for every $x \in X$ and each open set V in (Y, σ) containing $f(x)$ there exists an $\omega\beta-$open set U containing x such that $f(U) \subseteq V$.

The following two examples show that the concept of $\omega\beta-$continuity and contra $\omega\beta-$continuity are independent of each other.

Example 2.8. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1, & x \in \mathbb{R} - \mathbb{Q} \\ 2, & x \in \mathbb{Q} \end{cases}$$

Then f is $\omega\beta-$continuous but not contra $\omega\beta-$continuous.

Example 2.9. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $Y = \{1, 2\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 2, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

Then f is contra $\omega\beta-$continuous but not $\omega\beta-$continuous.

Proposition 2.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be contra $\omega\beta-$continuous. If one of the following conditions holds, then f is $\omega\beta-$continuous.*

(i) (Y, σ) is regular.

(ii) $\omega\beta Int(f^{-1}(Cl(V))) \subseteq f^{-1}(V)$ for each open set V in (Y, σ).

Proof: (i) Let $x \in X$ and V be an open set of (Y, σ) containing $f(x)$. Since (Y, σ) is regular, there exists an open set W in (Y, σ) containing $f(x)$ such that $Cl(W) \subseteq V$. Since f is contra $\omega\beta-$continuous, so by Theorem 2.5, there exists an $\omega\beta-$open set U in (X, τ) containing x such that $f(U) \subseteq Cl(W)$; hence $f(U) \subseteq V$. Therefore f is $\omega\beta-$continuous.
(ii) Let V be any open set of (Y, σ). Since f is contra $\omega\beta-$continuous and $Cl(V)$ is closed, by Theorem 2.5 $f^{-1}(Cl(V))$ is $\omega\beta-$open in (X, τ) and by (ii), $f^{-1}(Cl(V)) \subseteq \omega\beta Int(f^{-1}(Cl(V))) \subseteq f^{-1}(V)$. So, we obtain $f^{-1}(V) = \omega\beta Int(f^{-1}(Cl(V)))$ and consequently $f^{-1}(V)$ is $\omega\beta-$open in (X, τ). So f is an $\omega\beta-$continuous function. □

Definition 2.11. *A space (X, τ) is called an $\omega\beta-$space (resp. locally $\omega\beta-$ indiscrete) if every $\omega\beta-$open set is open (resp. closed) in (X, τ).*

Proposition 2.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a contra $\omega\beta-$continuous function.*

(i) If (X, τ) is an $\omega\beta-$space, then f is contra-continuous, contra $\omega-$ continuous and contra $\omega\beta-$continuous.

(ii) *If (X,τ) is locally $\omega\beta-$indiscrete, f is continuous.*

(iii) *If (X,τ) is an $\omega\beta-$space and f is a closed surjection, then (Y,σ) is locally indiscrete.*

Proof: (i) and (ii) directly follows from the definitions.
(iii) Let V be open in (Y,σ). Since f is contra $\omega\beta-$continuous, $f^{-1}(V)$ is $\omega\beta-$closed in (X,τ) and hence closed. Since f is closed and surjective, $f(f^{-1}(V)) = V$ is closed in (Y,σ) and so (Y,σ) is locally indiscrete. □

Recall that a function $f:(X,\tau)\to(Y,\sigma)$ is slightly $\omega\beta-$continuous [3] if $f^{-1}(V)$ is $\omega\beta-$open in (X,τ) for each clopen sets V of (Y,σ). Every contra $\omega\beta-$continuous is slightly $\omega\beta-$continuous but the converse is not true as we can see in Example 2.8.

Proposition 2.13. *Let (Y,σ) be locally indiscrete. A function $f:(X,\tau)\to(Y,\sigma)$ is contra $\omega\beta-$continuous if and only if f is slightly $\omega\beta-$continuous.*

Definition 2.14. *A function $f:(X,\tau)\to(Y,\sigma)$ is said to be weakly $\omega\beta-$ continuous if for each $x\in X$ and each open set V in (Y,σ) containing $f(x)$ there exists an $\omega\beta-$open set U in (X,τ) containing x such that $f(U)\subseteq Cl(V)$.*

Proposition 2.15. *If a function $f(X,\tau)\to(Y,\sigma)$ is contra $\omega\beta-$continuous, then f is weakly $\omega\beta-$continuous.*

Proof: The proof follows by Theorem 4.1 of [13]. □

Definition 2.16. *A filter base Δ is said to $\omega\beta-$converge (resp. $c-$converge [9]) to a point $x\in X$ if for any $\omega\beta-$open (resp. closed) set U in (X,τ) containing x, there exists $G\in\Delta$ such that $G\subseteq U$.*

Theorem 2.17. *Let a function $f:(X,\tau)\to(Y,\sigma)$ be contra $\omega\beta-$continuous. Then for each point $x\in X$ and each filter base Δ in (X,τ) $\omega\beta-$converging to x, the filter base $f(\Delta)$ is $c-$convergent to $f(x)$.*

Proof: Let $x\in X$ and Δ be any filter base in X $\omega\beta-$converging to x. Since f is contra $\omega\beta-$continuous, then by Theorem 2.5 for any closed set V in (Y,σ) containing $f(x)$, there exists an $\omega\beta-$open set U in (X,τ) containing x such that $f(U)\subseteq V$. Since Δ $\omega\beta-$converges to x, there exists $G\in\Delta$ such that $G\subseteq U$. This means that $f(G)\subseteq V$ and therefore the filter base $f(\Delta)$ is $c-$convergent to $f(x)$. □

Recall that Aljarrah and Noorani [1] introduced the notion of the $\omega\beta-$frontier of A, denoted by $\omega\beta F_r(A)$, as $\omega\beta F_r(A) = \omega\beta Cl(A) - \omega\beta Int(A)$, equivalently $\omega\beta F_r(A) = \omega\beta Cl(A)\cap\omega\beta Cl(X-A)$.

Theorem 2.18. *The set of all points $x\in X$ at which $f:(X,\tau)\to(Y,\sigma)$ is not contra $\omega\beta-$continuous is identical with the union of all the $\omega\beta-$frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof: The proof follows by Theorem 6.6 of [13] □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

Proposition 2.19. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $\omega\beta-$continuous, then f is contra $\omega\beta-$continuous.*

Proof: Let U be an open set in (Y, σ), then $X \times U$ is an open set in $(X \times Y, \tau \times \sigma)$. Since g is contra $\omega\beta-$continuous, $g^{-1}(X \times U) = f^{-1}(U)$ is $\omega\beta-$closed in (X, τ). This shows that f is contra $\omega\beta-$continuous. □

A subset A of a topological space (X, τ) is said to be $\omega\beta-$dense in X if $\omega\beta Cl(A) = X$.

Theorem 2.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be contra $\omega\beta-$continuous and $g : (X, \tau) \to (Y, \sigma)$ be contra $\omega-$continuous. If (Y, σ) is Urysohn, then the following properties hold:*

(i) The set $E = \{x \in X : f(x) = g(x)\}$ is $\omega\beta-$closed in (X, τ).

(ii) $f = g$ on (X, τ) whenever $f = g$ on an $\omega\beta-$dense set $A \subseteq X$.

Proof: (i) Let $x \in X - E$. Then $f(x) \neq g(x)$. By assumption on the space (Y, σ), there exist open sets V and W in (Y, σ) such that $f(x) \in V$, $g(x) \in W$ and $Cl(V) \cap Cl(W) = \phi$. Since f is contra $\omega\beta-$continuous, $f^{-1}(Cl(V))$ is an $\omega\beta-$open set in (X, τ) containing x. Since g is contra $\omega-$continuous, $g^{-1}(Cl(W))$ is an $\omega-$open set in (X, τ) containing x. Let $U = f^{-1}(Cl(V))$ and $G = g^{-1}(Cl(W))$ and set $A = U \cap G$. Then by Proposition 2.4, A is an $\omega\beta-$open set in (X, τ) containing x. Now, $f(A) \cap g(A) = f(U \cap G) \cap g(U \cap G) \subseteq f(U) \cap g(G) \subseteq Cl(V) \cap Cl(W) = \phi$. This implies that $A \cap E = \phi$, where A is $\omega\beta-$open in (X, τ). Hence $x \notin \omega\beta Cl(E)$. So E is $\omega\beta C(X, \tau)$.
(ii) Let $E = \{x \in X : f(x) = g(x)\}$. Since f is contra $\omega\beta-$continuous, g is contra $\omega-$continuous and (Y, σ) is Urysohn, by the previous part, E is $\omega\beta-$closed in (X, τ). By assumption, we have $f = g$ on A, where A is $\omega\beta-$dense in (X, τ). Since $A \subseteq E$, A is $\omega\beta-$dense and $E \in \omega\beta-$closed in (X, τ), so $X = \omega\beta Cl(A) \subseteq \omega\beta Cl(E) = E$. Hence $f = g$ on (X, τ). □

Definition 2.21. *The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra $\omega\beta-$closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $\omega\beta-$open set U in (X, τ) containing x and a closed set V in (Y, σ) containing y such that $(U \times V) \cap G(f) = \phi$.*

This definition is a particular case of Definition 5.1 of [13].

Lemma 2.22. *The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is contra $\omega\beta-$closed if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $\omega\beta-$open set U in (X, τ) containing x and a closed set V in (Y, σ) containing y such that $f(U) \cap V = \phi$.*

Proposition 2.23. *If $f : (X, \tau) \to (Y, \sigma)$ is contra $\omega\beta-$continuous and (Y, σ) is Uryshon, then $G(f)$ is contra $\omega\beta-$closed in $(X \times Y, \tau \times \sigma)$.*

Proof: The proof follows by Theorem 5.1 of [13]. □

Lemma 2.24. *[8] Let $G(f)$ be the graph of a function $f : X \to Y$, for any subset $A \subseteq X$ and $B \subseteq Y$, we have $f(A) \cap B = \phi$ if and only if $(A \times B) \cap G(f) = \phi$.*

Theorem 2.25. *Let $f : (X, \tau) \to (Y, \sigma)$ be $\omega\beta-$continuous. If one of the following conditions holds, then $G(f)$ is contra $\omega\beta-$closed in $(X \times Y, \tau \times \sigma)$.*

(i) (Y, σ) *is* T_1.

(ii) (Y, σ) *is* T_2.

Proof: (i) The proof follows by Theorem 5.2 of [13].
(ii) It follows from (i). □

A space (X, τ) is said to be $\omega\beta - \mathrm{T}_1$ if for each pair of distinct points x and y of X, there exist $\omega\beta-$open sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Theorem 2.26. *Let $f : (X, \tau) \to (Y, \sigma)$ have a contra $\omega\beta-$closed graph. Then the space (X, τ) is $\omega\beta - \mathrm{T}_1$ if f is injective.*

Proof: Let x and y be any two distinct points in (X, τ). Then, we have $(x, f(y)) \in (X \times Y) - G(f)$. So, there exist an $\omega\beta-$open set U in (X, τ) containing x and a closed set F in (Y, σ) containing $f(y)$ such that $f(U) \cap F = \phi$, hence $U \cap f^{-1}(F) = \phi$. Therefore, we have $y \notin U$. This implies that (X, τ) is $\omega\beta - \mathrm{T}_1$. □

The composition of two contra $\omega\beta-$continuous functions need not be contra $\omega\beta-$continuous.

Example 2.27. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$ and let $Y = \{1, 2\}$ with the topologies $\sigma = \{\phi, Y, \{1\}\}$ and $\rho = \{\phi, Y, \{2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the function defined by*

$$f(x) = \begin{cases} 2, & x \in \mathbb{R} - \mathbb{Q} \\ 1, & x \in \mathbb{Q} \end{cases}$$

and $g : (Y, \sigma) \to (Y, \rho)$ be the identity function. Then f, g are contra $\omega\beta-$ continuous, but $g \circ f$ is not contra $\omega\beta-$continuous.

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\omega\beta-$irresolute [1] if the inverse image of each $\omega\beta-$open set in (Y, σ) is $\omega\beta-$open in (X, τ).

Theorem 2.28. *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$ be functions, then the following properties hold:*

(i) $g \circ f$ *is* $\omega\beta-$*continuous, if* f *is contra* $\omega\beta-$ *continuous and* g *is contra- continuous.*

(ii) $g \circ f$ *is contra* $\omega\beta-$*continuous, if* f *is contra* $\omega\beta-$*continuous and* g *is continuous.*

(iii) $g \circ f$ *is contra* $\omega\beta-$ *continuous, if* f *is* $\omega\beta-$*irresolute and* g *is contra* $\omega\beta-$*continuous.*

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be $\omega\beta-$ open [1] if the image of each $\omega\beta-$open set in (X, τ) is $\omega\beta-$open (Y, σ).

Theorem 2.29. *Let* $f : (X, \tau) \to (Y, \sigma)$ *be a surjective* $\omega\beta-$*irresolute and* $\omega\beta-$*open function and* $g : (Y, \sigma) \to (Z, \rho)$ *be any function. Then* $g \circ f : (X, \tau) \to (Z, \rho)$ *is contra* $\omega\beta-$*continuous if and only if* g *is contra* $\omega\beta-$*continuous.*

Proof: Suppose $g \circ f : (X, \tau) \to (Z, \rho)$ is contra $\omega\beta-$continuous. Let F be a closed set in (Z, ρ). Then $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is $\omega\beta-$open in (X, τ). Since f is $\omega\beta-$open and surjective, $g^{-1}(F)) = f(f^{-1}(g^{-1}(F)))$ is $\omega\beta-$open in (Y, σ) and we obtain that g is contra $\omega\beta-$continuous.
For the converse, suppose g is contra $\omega\beta-$continuous. Let V be a closed set in (Z, ρ). Then $g^{-1}(V)$ is $\omega\beta-$open in (Y, σ). Since f is $\omega\beta-$irresolute, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is $\omega\beta-$open in (X, τ) and so $g \circ f$ is a contra $\omega\beta-$continuous function. □

Proposition 2.30. *[2] Let* (Y, τ_Y) *be a subspace of* (X, τ)*,* $A \subseteq Y$ *and* Y *be a* $\beta-$*open set in* (X, τ)*. Then* A *is* $\omega\beta-$*open in* (X, τ) *if and only if* A *is* $\omega\beta-$*open in* (Y, τ_Y)*.*

Theorem 2.31. *Let* $X = A \cup B$ *be a topological space with a topology* τ *and* Y *be a topological space with a topology* σ*. Let* $f : (A, \tau_A) \to (Y, \sigma)$ *and* $g : (B, \tau_B) \to (Y, \sigma)$ *be contra* $\omega\beta-$*continuous functions such that* $f(x) = g(x)$ *for every* $x \in A \cap B$*. Suppose* A *and* B *are* $\beta-$*open sets in* (X, τ)*. Then the composition* $h : (X, \tau) \to (Y, \sigma)$ *is contra* $\omega\beta-$*continuous.*

Proof: Let V be any closed set in (Y, σ). So $h^{-1}(V) = f^{-1}(V) \cup g^{-1}(V)$. Since f, g are contra $\omega\beta-$continuous, $f^{-1}(V)$ is $\omega\beta-$open in (A, τ_A) and $g^{-1}(V)$ is $\omega\beta-$open in (B, τ_B). Since A and B are $\beta-$open in (X, τ), by Proposition 2.30 $f^{-1}(V)$ and $g^{-1}(V)$ are $\omega\beta-$open in (X, τ). So $h^{-1}(V)$ is $\omega\beta-$open in (X, τ). □

Definition 2.32. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is said to be perfectly* $\omega\beta-$*continuous if the inverse of every open set in* (Y, σ) *is* $\omega\beta-$*clopen in* (X, τ)*.*

Clearly, every perfectly $\omega\beta-$continuous function is both $\omega\beta-$continuous and contra $\omega\beta-$continuous, but the converse need not be true as in Example 2.9 f is contra $\omega\beta-$continuous but not perfectly $\omega\beta-$continuous and in Example 2.8 f is $\omega\beta-$continuous but not perfectly $\omega\beta-$continuous. Although $\omega\beta-$continuity and contra $\omega\beta-$continuity are independent notions, every $\omega\beta-$continuous and contra $\omega\beta-$continuous function is perfectly $\omega\beta-$continuous.

Theorem 2.33. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(i) f is perfectly $\omega\beta-$continuous.

(ii) f is $\omega\beta-$continuous and contra $\omega\beta-$continuous.

For functions defined above, we obtain the following diagram

Contra-continuous $\to$ contra $\omega-$continuous

$\downarrow$ $\downarrow$

Contra $\beta-$continuous $\to$ contra $\omega\beta-$continuous $\leftarrow$ perfect $\omega\beta-$continuous

$\downarrow$ $\downarrow$

Weakly $\omega\beta-$continuous $\leftarrow$ $\omega\beta-$continuous

$\downarrow$

Slightly $\omega\beta-$continuous

3. Application

In this section, we will apply the concepts in order to prove the invariance of certain properties of the domain and the range under the actions of the above functions.
A space (X, τ) is said to be $\omega\beta-$connected [1] provided that X is not the union of two disjoint nonempty $\omega\beta-$open sets.

Proposition 3.1. *Let $f : (X, \tau) \to (Y, \sigma)$ be surjective and contra $\omega\beta-$continuous. If (X, τ) is $\omega\beta-$connected, then (Y, σ) is connected and is not a discrete space.*

Proof: The proof follows by Theorem 6.5 of [13]. □

Proposition 3.2. *If every contra $\omega\beta-$continuous function from a space (X, τ) into any $\mathrm{T}_\circ-$space (Y, σ) is constant, then (X, τ) is $\omega\beta-$connected*

Proof: Suppose that (X, τ) is not $\omega\beta-$connected and every contra $\omega\beta-$continuous function from (X, τ) into any $\mathrm{T}_\circ-$space (Y, σ) is constant. Since (X, τ) is not $\omega\beta-$connected, there exists a proper nonempty $\omega\beta-$clopen subset A of (X, τ). Let $Y = \{a, b\}$ and $\sigma = \{\phi, Y, \{a\}, \{b\}\}$ be a topology for Y. Let $f : (X, \tau) \to (Y, \sigma)$ be a function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then f is not constant and contra $\omega\beta-$continuous such that (Y, σ) is $\mathrm{T}_\circ$. This is a contradiction. Hence (X, τ) must be $\omega\beta-$connected. □

A space (X, τ) is called hyperconnected [18] if the closure of every open set is the entire set X. It is well known that every hyperconnected space is connected but not conversely.

Remark 3.3. *A contra $\omega\beta-$continuous surjection do not necessarily preserve hyperconnectedness. Let* $X=\{1,2,3\}$, $\tau=\{\phi,X,\{1\}\}$ *and* $\sigma=\{\phi,X,\{2\},\{3\},\{2,3\}\}$. *The identity* $f:(X,\tau)\to(X,\sigma)$ *is contra* $\omega\beta-$*continuous and* (X,τ) *is hyperconnected, but* (X,σ) *is not hyperconnected.*

Definition 3.4. *A space* (X,τ) *is said to be*

(i) $\omega\beta-T_2$ *[2] if for each two distinct points* $x,y\in X$, *there exist* $\omega\beta-$ *open sets* U *and* V *in* (X,τ) *such that* $x\in U$, $y\in V$ *and* $U\cap V=\phi$.

(ii) Weakly Hausdroff [16] if each element of X *is an intersection of regular closed sets.*

(iii) Ultra Hausdorff [17] if every two distinct points of X *can be separated by disjoint clopen sets.*

(iv) Ultra normal [17] (resp. $\omega\beta-$*normal [2]) if each pair of non-empty disjoint closed sets can be separated by disjoint clopen (resp.* $\omega\beta-$*open) sets.*

Theorem 3.5. *Let* $f:(X,\tau)\to(Y,\sigma)$ *be a contra* $\omega\beta-$*continuous injection, then the following properties holds:*

(i) (X,τ) *is* $\omega\beta-\mathrm{T}_1$ *if* (Y,σ) *is weakly Hausdorff.*

(ii) (X,τ) *is* $\omega\beta-\mathrm{T}_2$ *if* (Y,σ) *is a Urysohn space or ultra Hausdorff.*

(iii) (X,τ) *is* $\omega\beta-$*normal if* (Y,σ) *is ultra normal and* f *is closed.*

Proof: (i) Suppose that (Y,σ) is weakly Hausdorff. For any distinct points x and y in (X,τ), there exist regular closed sets A, B in (Y,σ) such that $f(x)\in A$, $f(y)\notin A$, $f(x)\notin B$ and $f(y)\in B$. Since f is contra $\omega\beta-$continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $\omega\beta-$open sets in (X,τ) such that $x\in f^{-1}(A)$, $y\notin f^{-1}(A)$, $x\notin f^{-1}(B)$ and $y\in f^{-1}(B)$. This shows that (X,τ) is $\omega\beta-\mathrm{T}_1$.
(ii) The proof follows by Corollary 6.1 and Theorem 6.2 of [13].
(iii) Let F_1 and F_2 be disjoint closed subsets of (X,τ). Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of (Y,σ). Since (Y,σ) is ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen set V_1 and V_2, respectively. Since f is contra $\omega\beta-$continuous, $F_i\subseteq f^{-1}(V_i)$ and $f^{-1}(V_i)$ is $\omega\beta-$open in (X,τ) for $i=1,2$ and $f^{-1}(V_1)\cap f^{-1}(V_2)=\phi$. Thus (X,τ) is $\omega\beta-$normal. □

4. Covering properties

In this section we study the properties of compact and strongly $S-$closed spaces under the contra $\omega\beta-$continuous functions.

Definition 4.1. *A space* (X,τ) *is said to be*

(i) Strongly $S-$*closed [7] if every closed cover of* X *has a finite subcover.*

(ii) $\omega\beta-$*compact [3] if every* $\omega\beta-$*open cover of* X *has a finite subcover.*

(iii) Mildly $\omega\beta-$compact if every $\omega\beta-$clopen cover of X has a finite subcover.

A subset A of a space X is said to be $\omega\beta-$compact relative to X if for any cover $\{V_\alpha : \alpha \in \Delta\}$ of A by $\omega\beta-$open sets of X, there exists a finite subset $\Delta_\circ$ of Δ such that $A \subseteq \cup\{V_\alpha : \alpha \in \Delta_\circ\}$.

Theorem 4.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be a contra $\omega\beta-$continuous surjection.*

(i) If A is $\omega\beta-$compact relative to (X, τ), then $f(A)$ is strongly $S-$closed in (Y, σ).

(ii) If (X, τ) is strongly $S-$closed, then (Y, σ) is compact.

Proof: (i) Let $\{V_\alpha : \alpha \in \Delta\}$ be any cover of $f(A)$ by closed sets of the subspace $f(A)$. For $\alpha \in \Delta$, there exists a closed set A_α of (Y, σ) such that $V_\alpha = A_\alpha \cap f(A)$. For each $x \in A$, there exists $\alpha_x \in \Delta$ such that $f(x) \in A_{\alpha_x}$. Now by Theorem 2.5, there exists an $\omega\beta-$open set U_x in (X, τ) containing x such that $f(U_x) \subseteq A_{\alpha_x}$. Since the family $\{U_x : x \in A\}$ is a cover of A by $\omega\beta-$open sets of (X, τ), there exists a finite subset $A_\circ$ of A such that $A \subseteq \cup\{U_x : x \in A_\circ\}$. Therefore, we obtain $f(A) \subseteq \cup\{f(U_x) : x \in A_\circ\} \subseteq \cup\{A_{\alpha_x} : x \in A_\circ\}$. Thus $f(A) = \cup\{V_{\alpha_x} : x \in A_\circ\}$ and hence $f(A)$ is strongly $S-$closed.
(ii) Let $\{V_\alpha : \alpha \in \Delta\}$ be any open cover of Y. Since f is contra $\omega\beta-$continuous, $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an $\omega\beta-$closed cover of the strongly $S-$closed space (X, τ). We have $X = \cup\{f^{-1}(V_\alpha) : \alpha \in \Delta_\circ\}$ for some finite $\Delta_\circ$ of Δ. Since f is surjective, $Y = \cup\{V_\alpha : \alpha \in \Delta_\circ\}$. This shows that (Y, σ) is compact. □

A subset A of X is said to be $\omega\beta-$regular closed if $A = \omega\beta Cl(\omega\beta Int(A))$.

Definition 4.3. *An open cover $\{U_\alpha : \alpha \in \Delta\}$ of a topological space (X, τ) is called an $\omega\beta-$regular cover if for every $\alpha \in \Delta$, there exists a non-empty $\omega\beta-$regular closed subset C_α of (X, τ) such that $C_\alpha \subseteq U_\alpha$ and $X = \underset{\alpha\in\Delta}{\cup} \omega\beta Int(C_\alpha)$.*

Definition 4.4. *A topological space (X, τ) is said to be weakly $\omega\beta-$regular-Lindelof if every $\omega\beta-$regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X has a countable subset $\{\alpha_n : n \in \mathbb{N}\} \subseteq \Delta$ such that $X = \omega\beta Cl(\underset{n\in\mathbb{N}}{\cup} U_{\alpha_n})$.*

Proposition 4.5. *The image of a weakly $\omega\beta-$regular-Lindelof space under a contra $\omega\beta-$continuous and continuous function is Lindelof.*

Proof: Let $f : (X, \tau) \to (Y, \sigma)$ be a contra $\omega\beta-$continuous and continuous function from a weakly $\omega\beta-$regular-Lindelof space (X, τ) into (Y, σ). Let $U = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of $f(X)$. For each $x \in X$, let $U_{\alpha_x} \in U$ such that $f(x) \in U_{\alpha_x}$. Since f is contra $\omega\beta-$continuous and continuous, it follows that $f^{-1}(U_{\alpha_x})$ is $\omega\beta-$clopen in (X, τ) and hence $\{f^{-1}(U_{\alpha_x}) : x \in X\}$ is an $\omega\beta-$regular cover of the weakly $\omega\beta-$regular-Lindelof space (X, τ). Thus there exists a countable subfamily $\{x_n : n \in N\}$ such that $X = \omega\beta Cl(\underset{n\in N}{\cup} f^{-1}(U_{\alpha_{x_n}})) = \omega\beta Cl(f^{-1}(\underset{n\in N}{\cup} U_{\alpha_{x_n}}))$.

Since $\underset{n\in N}{\cup} U_{\alpha_{x_n}}$ is open in (Y,σ) and f is contra $\omega\beta-$continuous, $f^{-1}(\underset{n\in \mathbb{N}}{\cup} U_{\alpha_{x_n}})$ is $\omega\beta-$closed in (X,τ). Thus $\omega\beta Cl(\underset{n\in N}{\cup} f^{-1}(U_{\alpha_{x_n}})) = f^{-1}(\underset{n\in N}{\cup} U_{\alpha_{x_n}})$. So $X = f^{-1}(\underset{n\in N}{\cup} U_{\alpha_{x_n}})$ and hence $f(X) = f(f^{-1}(\underset{n\in N}{\cup} U_{\alpha_{x_n}})) \subseteq \underset{n\in N}{\cup} U_{\alpha_{x_n}}$. This implies that $f(X)$ is Lindelof. □

Theorem 4.6. *Let $f : (X,\tau) \to (Y,\sigma)$ be a surjection. If one of the following conditions holds, then (Y,σ) is strongly $S-$closed.*

(i) f is contra $\omega\beta-$continuous and (X,τ) is $\omega\beta-$compact.

(ii) f is perfectly $\omega\beta-$continuous and (X,τ) is mildly $\omega\beta-$compact.

Proof: (i) The proof follows by Theorem 6.4 of [13].
(ii) Let $\{V_i : i \in \Delta\}$ be a closed cover of Y. Since f is perfectly $\omega\beta-$continuous, $\{f^{-1}(V_i) : i \in \Delta\}$ is an $\omega\beta-$clopen cover of X. Clearly, there exists a finite $\Delta_\circ \subseteq \Delta$ such that $X = \underset{i\in\Delta_\circ}{\cup} f^{-1}(V_i)$ as (X,τ) is mildly $\omega\beta-$compact. Hence $Y = \underset{i\in\Delta_\circ}{\cup} V_i$. This shows that (Y,σ) is strongly $S-$closed. □

5. Strongly contra $\omega\beta-$closed graphs

Definition 5.1. *The graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$ is said to be strongly contra $\omega\beta-$closed if for each $(x,y) \in (X \times Y) - G(f)$, there exist an $\omega\beta-$open set U in (X,τ) containing x and a regular closed set V in (Y,σ) containing y such that $(U \times V) \cap G(f) = \phi$.*

Lemma 5.2. *For a graph $G(f)$ of a function $f : (X,\tau) \to (Y,\sigma)$, the following properties are equivalent*

(i) $G(f)$ is strongly contra $\omega\beta-$closed.

(ii) For each point $(x,y) \in (X \times Y) - G(f)$, there exist an $\omega\beta-$open set U in (X,τ) containing x and a regular closed set V in (Y,σ) containing y such that $f(U) \cap V = \phi$.

Theorem 5.3. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function such that (Y,σ) is a Uryshon space. Then $G(f)$ is strongly contra $\omega\beta-$closed in $(X \times Y, \tau \times \sigma)$ if one of the following properties hold:*

(i) f is weakly $\omega\beta-$continuous.

(ii) f is contra $\omega\beta-$continuous.

Proof: We will prove (i), since the prove of (ii) comes from (i). Suppose that $(x,y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since (Y,σ) is Uryshon, there exist open sets V and W in (Y,σ) containing y and $f(x)$, respectively, such that $Cl(V) \cap Cl(W) = \phi$. Since f is weakly $\omega\beta-$continuous, by Definition 2.14, there exists an

$\omega\beta-$open set U in (X, τ) containing x such that $f(U) \subseteq Cl(W)$. This shows that $f(U) \cap Cl(V) = f(U) \cap Cl(Int(V)) = \phi$, where $Cl(Int(V))$ is regular closed in (Y, σ) and hence by Lemma 5.2, $G(f)$ is strongly contra $\omega\beta-$closed. □

Theorem 5.4. *If the collection of all $\omega\beta-$open sets form a topological space in (X, τ) and $f : (X, \tau) \to (Y, \sigma)$ has a contra $\omega\beta-$closed graph, then the inverse image of a strongly $S-$closed subspace K of (Y, σ) is $\omega\beta-$closed in (X, τ).*

Proof: Assume that K is a strongly $S-$closed subspace of (Y, σ) and $x \notin f^{-1}(K)$. For each $k \in K$, $(x, k) \notin G(f)$. By Lemma 5.2 there exist an $\omega\beta-$open set U_k in (X, τ) containing x and a closed set V_k in (Y, σ) containing k such that $f(U_k) \cap V_k = \phi$; hence $f(U_k) \cap (V_k \cap K) = \phi$, where $V_k \cap K$ is closed in (K, σ_K). Since K is a strongly $S-$closed subspace of (Y, σ), then there exists a finite subset $K_\circ \subseteq K$ such that $K \subseteq \cup\{V_k : k \in K_\circ\}$. Set $U = \cap\{U_k : k \in K_\circ\}$, then U is an $\omega\beta-$open set in (X, τ) containing x and $f(U) \cap K \subseteq f(U_k) \cap [\cup(V_k : k \in K_\circ)] = \phi$. Therefore $U \cap f^{-1}(K) = \phi$ and hence $x \notin \omega\beta Cl(f^{-1}(K))$. This shows that $f^{-1}(K)$ is $\omega\beta-$closed in (X, τ). □

Theorem 5.5. *Let (Y, σ) be a strongly $S-$closed space. If the collection of all $\omega\beta-$open sets form a topological space in (X, τ) and a function $f : (X, \tau) \to (Y, \sigma)$ has the contra $\omega\beta-$closed graph, then f is contra $\omega\beta-$continuous.*

Proof: Suppose that (Y, σ) is strongly $S-$closed and $G(f)$ is contra $\omega\beta-$closed. By Theorem 3.5 of [7] we can see that the open set in (Y, σ) is strongly $S-$closed and by Theorem 5.4 $f^{-1}(U)$ is $\omega\beta-$closed in (X, τ) for every open U in (Y, σ). Therefore, f is contra $\omega\beta-$continuous. □

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