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Note on generalized topological spaces with hereditary classes

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abstract: In this paper, we extend the study of $\Psi_{\mathcal{H}}$ operator introduced and studied in [5] and rectify the errors in the paper. Moreover, characterizations of $\mu-$codense and strongly $\mu-$codense hereditary classes in generalized topological spaces are also given.

Key Words: generalized topology, $\mu-$closed and $\mu-$open sets, hereditary class, $\mu r-$open sets, $\mu-$codense and strongly $\mu-$codense.

Contents

1. Introduction

Let X be a nonempty set. A family μ of subsets of X is called a *generalized topology* (GT) [1] if $\emptyset \in \mu$ and the arbitrary union of members of μ is again in μ. The largest $\mu-$open set contained in a subset A of X is denoted by $i_\mu(A)$ [1] and is called the $\mu-interior$ of A. The smallest $\mu-$closed set containing A is called the $\mu - closure$ of A and is denoted by $c_\mu(A)$ [1]. Throughout the paper, by a space we always mean a generalized space (X, μ). $\sigma(\mu) = \{A \subset X \mid A \subset c_\mu i_\mu(A)\}$ is the family of all $\mu - semiopen$ sets [2]. A subset A of X is said to be $\sigma(\mu) - closed$ if its complement is $\mu-$semiopen. A GT μ is said to be a $quasi - topology$ [4] on X if $M, N \in \mu$ implies $M \cap N \in \mu$.

A *hereditary class* $\mathcal{H}$ of X is a nonempty collection of subset of X such that $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$ [3]. A hereditary class $\mathcal{H}$ of X is an *ideal* [6] if $A \cup B \in \mathcal{H}$ whenever $A \in \mathcal{H}$ and $B \in \mathcal{H}$. With respect to the generalized topology μ of all $\mu-$open sets and a hereditary class $\mathcal{H}$, for each subset A of X, a subset $A^\star(\mathcal{H})$ or simply $A^\star$ of X is defined by $A^\star = \{x \in X \mid M \cap A \notin \mathcal{H}$ for every $M \in \mu$ containing $x\}$ [3]. $\mathcal{H}$ is said to be $\mu - codense$ if $\mu \cap \mathcal{H} = \{\emptyset\}$ [3] and is said to be $strongly\ \mu - codense$ [3] if $M, N \in \mu$ and $M \cap N \in \mathcal{H}$, then $M \cap N = \emptyset$. Every strongly $\mu-$codense hereditary class is $\mu-$codense but the converse is not true [3]. A subset A of X is said to be $\mu - rare$ [3] (resp. $\mu r - open$) if $i_\mu c_\mu(A) = \emptyset$ (resp. $A = i_\mu c_\mu(A)$). If $\mathcal{H}_r$ is the collection of all $\mu-$rare sets in (X, μ), then $\mathcal{H}_r$ is a hereditary class and for this hereditary class, $A^\star \subset c_\mu i_\mu c_\mu(A)$ for every subset A of X [3, Proposition 2.11]. If $c^\star(A) = A \cup A^\star$ for every subset A of X, with respect to μ and a hereditary class $\mathcal{H}$ of subsets of X, then $c^\star \in \Gamma$ and $\mu^\star = \{A \subset X \mid c^\star(X - A) = X - A\}$ is a generalized topology finer than μ [3].

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The following lemmas will be useful in the sequel and we use some of the results without mentioning it, when the context is clear.

Lemma 1.1. [3] *Let (X, μ) be a space with a hereditary class $\mathcal{H}$. If A and B are any two subsets of X, then the following hold.*
(a) *If $A \in \mathcal{H}$, then $A^\star = X - \mathcal{M}_\mu$ where $\mathcal{M}_\mu = \bigcup\{M \mid M \in \mu\}$.*
(b) *If $A \subset A^\star$, then $c_\mu(A) = A^\star = c^\star(A) = c^\star(A^\star)$.*
(c) *$\mathcal{H}$ is $\mu-$codense if and only if $X = X^\star$.*
(d) *$A^\star$ is $\mu-$closed for every subset A of X.*
(e) *If F is $\mu-$closed, then $F^\star \subset F$.*

Lemma 1.2. [7, Theorem 2.4] *If (X, μ) is a quasi-topological space and $\mathcal{H}$ is a hereditary class of subsets of X, then the following statements are equivalent.*
(a) $\mathcal{H}$ is $\mu-$codense.
(b) $\mathcal{H}$ is strongly $\mu-$codense.

Lemma 1.3. [7, Theorem 2.5] *If (X, μ) is a space and $\mathcal{H}$ is a hereditary class of subsets of X, then the following statements are equivalent.*
(a) *$\mathcal{H}$ is strongly $\mu-$codense.*
(b) *$M \subset M^\star$ for every $M \in \mu$.*
(c) *$S \subset S^\star$ for every $S \in \sigma(\mu)$.*
(d) *$c_\mu(M) = M^\star$ for every $M \in \mu$.*
(e) *$c_\mu(S) = S^\star$ for every $S \in \sigma(\mu)$.*

2. Operator $\Psi_{\mathcal{H}}$

If $\mathcal{H}$ is a hereditary class on a space (X, μ), an operator $\Psi_{\mathcal{H}} : \wp(X) \to \wp(X)$ is defined as follows: for every $A \in \wp(X)$, $\Psi_{\mathcal{H}}(A) = \{x \in X \mid$ there exists a $M \in \mu$ such that $x \in M$ and $M - A \in \mathcal{H}\}$. The following Theorem 2.1 gives a characterization of the function $\Psi_{\mathcal{H}}$ which is $\gamma^\star_\mu$ in [5]. Throughout the paper, we use the notation $\Psi_{\mathcal{H}}$.

Theorem 2.1. *Let (X, μ) be a space with a hereditary class $\mathcal{H}$. Then $\Psi_{\mathcal{H}}(A) = X - (X - A)^\star$.*

Proof: Suppose $x \in X - (X - A)^\star$. Then $x \notin (X - A)^\star$ and so there exists $M \in \mu$ containing x such that $M \cap (X - A) \in \mathcal{H}$ which implies that $M - A \in \mathcal{H}$. Therefore, $X - (X - A)^\star \subset \{x \in X \mid$ there exists $M \in \mu(x)$ such that $M - A \in \mathcal{H}\}$. Conversely, assume that $y \in \Psi_{\mathcal{H}}(A)$. Then there exists $M \in \mu$ containing x such that $M - A \in \mathcal{H}$. Since $M - A \in \mathcal{H}, M \cap (X - A) \in \mathcal{H}$ which implies that $y \notin (X - A)^\star$. Therefore, $y \in X - (X - A)^\star$. Thus, $\Psi_{\mathcal{H}}(A) = X - (X - A)^\star$. □

The following Theorem 2.3 gives the properties of the operator $\Psi_{\mathcal{H}}$, where (a) confirms that the range of $\Psi_{\mathcal{H}}$ is a subfamily of μ and (e) is a generalization of Theorem 3.3 of [5]. In Example 3.5 of [5], it is established that the other direction of Theorem 2.3(f) is not true, but $\mathcal{H}$ *stated in the above example is not a hereditary class and μ is not even a generalized topology.* The following Example 2.2 shows that the inequality will not be an equality in Theorem 2.3(f).

Example 2.2. *Consider the generalized topological space* (X, μ) *with a hereditary class* $\mathcal{H}$ *where* $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{b\}, \{c\}\}$. *If* $A = \{a, d\}$, *then* $\Psi_{\mathcal{H}}(A) = X - \{b, c\}^\star = X - \{c, d\} = \{a, b\}$ *and* $\Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A)) = \Psi_{\mathcal{H}}(\{a, b\}) = X - \{c, d\}^\star = X - \{d\} = \{a, b, c\}$ *and so* $\Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A)) \neq \Psi_{\mathcal{H}}(A)$.

Theorem 2.3. *Let* (X, μ) *be a space with a hereditary class* $\mathcal{H}$ *and* A, $B \subset X$. *Then the following hold.*
(a) $\Psi_{\mathcal{H}}(A)$ *is* $\mu-$*open* [5, Theorem 3.1(ii)].
(b) $A^\star = X - \Psi_{\mathcal{H}}(X - A)$ [5, Theorem 3.1(iii)].
(c) *If* $A \subset B$, *then* $\Psi_{\mathcal{H}}(A) \subset \Psi_{\mathcal{H}}(B)$ [5, Theorem 3.1(i)].
(d) $\Psi_{\mathcal{H}}(A \cap B) \subset \Psi_{\mathcal{H}}(A) \cap \Psi_{\mathcal{H}}(B)$.
(e) *If* $U \in \mu^\star$, *then* $U \subset \Psi_{\mathcal{H}}(U)$.
(f) $\Psi_{\mathcal{H}}(A) \subset \Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A))$ [5, Theorem 3.4(i)].
(g) $\Psi_{\mathcal{H}}(A) = \Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A))$ *if and only if* $(X - A)^\star = ((X - A)^\star)^\star$.
(h) $A \cap \Psi_{\mathcal{H}}(A) = i^\star_\mu(A)$.
(i) *If* $H \in \mathcal{H}$, *then* $(A \cup H)^\star = A^\star$ *and hence* $\Psi_{\mathcal{H}}(A - H) = \Psi_{\mathcal{H}}(A)$.
(j) $\Psi_{\mathcal{H}}(\emptyset) = \mathcal{M}_\mu - X^\star$.

Proof: (d) The proof follows from (c).
(e) If $U \in \mu^\star$, then $X - U$ is $\mu^\star-$closed. Therefore, $(X - U)^\star \subset X - U$ which implies that $X - (X - U) \subset X - (X - U)^\star$ and so $U \subset \Psi_{\mathcal{H}}(U)$.
(g) Suppose that $(X - A)^\star = ((X - A)^\star)^\star$. Then $\Psi_{\mathcal{H}}(A) = X - (X - A)^\star$ implies that $\Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A)) = X - (X - \Psi_{\mathcal{H}}(A))^\star = X - (X - (X - (X - A)^\star))^\star = X - ((X - A)^\star)^\star = X - (X - A)^\star = \Psi_{\mathcal{H}}(A)$. Hence $\Psi_{\mathcal{H}}(A) = \Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A))$. Conversely, $\Psi_{\mathcal{H}}(A) = \Psi_{\mathcal{H}}(\Psi_{\mathcal{H}}(A))$ implies that $X - (X - A)^\star = X - (X - \Psi_{\mathcal{H}}(A))^\star = X - (X - (X - (X - A)^\star))^\star = X - ((X - A)^\star)^\star$. Therefore, $(X - A)^\star = ((X - A)^\star)^\star$.
(h) Let $x \in A \cap \Psi_{\mathcal{H}}(A)$. Then $x \in A$ and $x \in \Psi_{\mathcal{H}}(A)$. Since $x \in \Psi_{\mathcal{H}}(A)$, there exists $M_x \in \mu$ containing x such that $M_x - A \in \mathcal{H}$. Therefore, $x \in M_x - (M_x - A) \subset A$. Since β is a basis for $\mu^\star$ and $M_x - (M_x - A) \in \beta$, $x \in i^\star_\mu(A)$, where $i^\star_\mu$ is the interior operator in $(X, \mu^\star)$. Conversely, assume that $x \in i^\star_\mu(A)$. Then there exists a $\mu-$open set M_x containing x and $H \in \mathcal{H}$ such that $x \in M_x - H \subset A$. Now $M_x - H \subset A$ implies that $M_x - A \subset H$ which in turn implies that $M_x - A \in \mathcal{H}$ and so $x \in \Psi_{\mathcal{H}}(A)$. Therefore, $x \in A \cap \Psi_{\mathcal{H}}(A)$. Hence $A \cap \Psi_{\mathcal{H}}(A) = i^\star_\mu(A)$.
(i) Suppose that $H \in \mathcal{H}$. Then by Lemma 1.1(a), $(A \cup H)^\star = A^\star \cup H^\star = A^\star \cup (X - \mathcal{M}_\mu) = A^\star$, since $X - \mathcal{M}_\mu$ is the smallest $\mu-$closed set contained in every $\mu-$closed set. Again, $\Psi_{\mathcal{H}}(A - H) = X - (X - (A - H))^\star = X - ((X - A) \cup H)^\star = X - (X - A)^\star = \Psi_{\mathcal{H}}(A)$.
(j) By Theorem 2.1, $\Psi_{\mathcal{H}}(\emptyset) = X - X^\star = (\mathcal{M}_\mu \cup (X - \mathcal{M}_\mu)) - X^\star = (\mathcal{M}_\mu - X^\star) \cup ((X - \mathcal{M}_\mu) - X^\star) = \mathcal{M}_\mu - X^\star$, since $X^\star$ is $\mu-$closed by Lemma 1.1(d) and $X - \mathcal{M}_\mu$ is the smallest $\mu-$closed set contained in every $\mu-$closed set. □

Theorem 2.4 shows that $\Psi_{\mathcal{H}}$ preserves finite intersection under some additional conditions. The proof also follows from Theorem 3.4 of [8] using the fact that $(A \cup B)^\star = A^\star \cup B^\star$.

Theorem 2.4. *Let (X,μ) be a quasi-topological space and $\mathcal{H}$ be an ideal on X. If $A,\ B\subset X$, then $\Psi_{\mathcal{H}}(A\cap B)=\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$.*

Proof: Let $x\in\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$. Then there exist $\mu-$open sets U and V containing x such that $U-A\in\mathcal{H}$ and $V-B\in\mathcal{H}$. If $G=U\cap V$, then G is a $\mu-$open set containing x such that $G-A\in\mathcal{H}$ and $G-B\in\mathcal{H}$. Now $G-(A\cap B)=(G-A)\cup(G-B)\in\mathcal{H}$ and so $x\in\Psi_{\mathcal{H}}(A\cap B)$. Hence $\Psi_{\mathcal{H}}(A\cap B)=\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$.□

Example 2.5 below shows that the conditions *quasi-topology* on X and *ideal* on $\mathcal{H}$ cannot be dropped in Theorem 2.4.

Example 2.5. (a) *Consider the space (X,μ) where $X=\{a,b,c\}$, $\mu=\{\emptyset,\{a,b\},\{a,c\},X\}$ and $\mathcal{H}=\{\emptyset,\{a\}\}$. Clearly, μ is not a quasi-topology on X. If $A=\{b\}$ and $B=\{a,c\}$, then $\Psi_{\mathcal{H}}(A)=\{a,b\}$ and $\Psi_{\mathcal{H}}(B)=\{a,c\}$ which implies $\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)=\{a\}$. But $\Psi_{\mathcal{H}}(A\cap B)=\Psi_{\mathcal{H}}(\emptyset)=\emptyset\neq\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$.*
(b) *Consider the space (X,μ) with a hereditary class $\mathcal{H}$ where $X=\{a,b,c,d\}$, $\mu=\{\emptyset,\{a\},\{d\},\{a,c\},\{a,d\},\{a,c,d\},X\}$ and $\mathcal{H}=\{\emptyset,\{a\},\{c\}\}$. Here $\mathcal{H}$ is not an ideal. If $A=\{b,c,d\}$ and $B=\{a,b\}$, then $\Psi_{\mathcal{H}}(A)=X$ and $\Psi_{\mathcal{H}}(B)=\{a,c\}$ and so $\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)=\{a,c\}$. Also, $\Psi_{\mathcal{H}}(A\cap B)=\{a\}\neq\Psi_{\mathcal{H}}(A)\cap\Psi_{\mathcal{H}}(B)$.*

Theorem 2.6. [8, Theorem 3.3] *Let (X,μ) be a space with a hereditary class $\mathcal{H}$. If $\sigma=\{A\subset X\mid A\subset\Psi_{\mathcal{H}}(A)\}$, then σ is a generalized topology on X and $\sigma=\mu^\star$.*

Proof: Let $A\in\sigma$. Then $A\subset\Psi_{\mathcal{H}}(A)=X-(X-A)^\star$ which implies that $(X-A)^\star\subset X-A$. Therefore, $X-A$ is $\mu^\star-$closed and so A is $\mu^\star-$open. Therefore, $\sigma\subset\mu^\star$. Conversely, $A\in\mu^\star$ and $x\in A$. Then there exists $M\in\mu$ and $H\in\mathcal{H}$ such that $x\in M-H\subset A$. Now $M-H\subset A$ implies that $M-A\subset H$ which in turn implies that $M-A\in\mathcal{H}$ and so $x\in\Psi_{\mathcal{H}}(A)$. Therefore, $\mu^\star\subset\sigma$. Hence $\sigma=\mu^\star$. Since $\mu^\star$ is a generalized topology [3], it follows that σ is a generalized topology. □

Corollary 2.7. *Let (X,μ) be a space with a hereditary class $\mathcal{H}$. Then the following hold.*
(a) $\mathcal{M}_\mu=\mathcal{M}_{\mu^\star}$ [8, Corollary 3.1].
(b) *If $B\subset X-\mathcal{M}_\mu$, then $i^\star_\mu(B)=\emptyset$.*

Theorem 2.8. *Let (X,μ) be a space with a hereditary class $\mathcal{H}$ and $A\subset X$. Then the following properties hold.*
(a) $\Psi_{\mathcal{H}}(A)=\bigcup\{U\in\mu\mid U-A\in\mathcal{H}\}$ [5, Theorem 3.2].
(b) $\Psi_{\mathcal{H}}(A)=\bigcup\{U\in\mu\mid (U-A)\cup(A-U)\in\mathcal{H}\}$, *if A is $\mu-$open.*

Proof: (a) follows immediately from the definition of $\Psi_{\mathcal{H}}$.
(b) Denote $\bigcup\{U\in\mu\mid(U-A)\cup(A-U)\in\mathcal{H}\}$ by $\mathcal{A}$. Since $\mathcal{H}$ is hereditary, $\Psi_{\mathcal{H}}(A)\supset\mathcal{A}$ for every $A\subset X$. Assume $A\in\mu$ and $x\in\Psi_{\mathcal{H}}(A)$. Then there exists $M\in\mu$ such that $x\in M$ and $M-A\in\mathcal{H}$. If $U=M\cup A$, then $U\in\mu$ and $x\in U$. Now $(U-A)\cup(A-U)=(M-A)\cup\emptyset=M-A$ implies $(U-A)\cup(A-U)\in\mathcal{H}$ and so $x\in\mathcal{A}$. Therefore, $\Psi_{\mathcal{H}}(A)\subset\mathcal{A}$. Hence $\Psi_{\mathcal{H}}(A)=\mathcal{A}$. □

The following Theorem 2.9 characterizes $\mu-$codense hereditary classes in quasi-topological spaces.

Theorem 2.9. *Let (X, μ) be a quasi-topological space with a hereditary class $\mathcal{H}$. Then the following conditions are equivalent.*
(a) *$\mathcal{H}$ is $\mu-$codense.*
(b) $\Psi_{\mathcal{H}}(\emptyset) = \emptyset$.
(c) *If $A \subset X$ is $\mu-$closed, then $\Psi_{\mathcal{H}}(A) - A = \emptyset$.*
(d) *If $A \subset X$, then $i_\mu c_\mu(A) = \Psi_{\mathcal{H}}(i_\mu c_\mu(A))$.*
(e) *If A is $\mu r-$open, then $A = \Psi_{\mathcal{H}}(A)$.*
(f) *If $U \in \mu$, then $\Psi_{\mathcal{H}}(U) \subset i_\mu c_\mu(U) \subset U^\star$.*

Proof: (a)⇒(b). $\Psi_{\mathcal{H}}(\emptyset) = \cup\{U \in \mu \mid U - \emptyset = U \in \mathcal{H}\} = \emptyset$, since $\mu \cap \mathcal{H} = \{\emptyset\}$.
(b)⇒(c). Suppose $A \subset X$ is $\mu-$closed. If $x \in \Psi_{\mathcal{H}}(A) - A$, then there exists a $U_x \in \mu$ containing x such that $U_x - A \in \mathcal{H}$. But $U_x - A \in \mu$ implies that $U_x - A \in \{U \mid U \in \mathcal{H}\}$ and so $\Psi_{\mathcal{H}}(\emptyset) \neq \emptyset$, a contradiction. Therefore, $\Psi_{\mathcal{H}}(A) - A = \emptyset$.
(c)⇒(d). Since $i_\mu c_\mu(A) \in \mu$ for every subset A of X, by Theorem 2.3(e), $i_\mu c_\mu(A) \subset \Psi_{\mathcal{H}}(i_\mu c_\mu(A))$. By (c), $\Psi_{\mathcal{H}}(c_\mu(A)) \subset c_\mu(A)$ and so $\Psi_{\mathcal{H}}(c_\mu(A)) = i_\mu(\Psi_{\mathcal{H}}(c_\mu(A))) \subset i_\mu c_\mu(A)$. By Theorem 2.3(b), $\Psi_{\mathcal{H}}(i_\mu c_\mu(A)) \subset \Psi_{\mathcal{H}}(c_\mu(A)) \subset i_\mu c_\mu(A)$ and so $\Psi_{\mathcal{H}}(i_\mu c_\mu(A)) = i_\mu c_\mu(A)$.
(d)⇒(e). Let A be a $\mu r-$open subset of X. Then $A = i_\mu c_\mu(A)$ and so $\Psi_{\mathcal{H}}(A) = \Psi_{\mathcal{H}}(i_\mu c_\mu(A)) = i_\mu c_\mu(A) = A$.
(e)⇒(a). Since $\emptyset$ is $\mu r-$open, $\emptyset = \Psi_{\mathcal{H}}(\emptyset) = \cup\{U \in \mu \mid U \in \mathcal{H}\}$, by Theorem 2.8(a). Hence $\mu \cap \mathcal{H} = \{\emptyset\}$.
(c)⇒(f). If $U \in \mu$, then $X - U$ is $\mu-$closed and so $\Psi_{\mathcal{H}}(X - U) = X - U$ which implies that $X - (X - (X - U))^\star = X - U$ so that $X - U^\star = X - U$. Hence $U^\star = U$. Also, $c_\mu(U)$ is $\mu-$closed implies that $\Psi_{\mathcal{H}}(c_\mu(U)) - c_\mu(U) = \emptyset$ which implies that $\Psi_{\mathcal{H}}(c_\mu(U)) \subset c_\mu(U)$. Therefore, $\Psi_{\mathcal{H}}(U) = i_\mu(\Psi_{\mathcal{H}}(U)) \subset i_\mu(\Psi_{\mathcal{H}}(c_\mu(U))) \subset i_\mu c_\mu(U) \subset c_\mu(U) = U^\star$, by Lemma 1.1(b). Hence $\Psi_{\mathcal{H}}(U) \subset i_\mu c_\mu(U) \subset U^\star$.
(f)⇒(a). Suppose $U \in \mu$. Then $U \subset \Psi_{\mathcal{H}}(U) \subset i_\mu c_\mu(U) \subset U^\star$ which implies that $\mathcal{H}$ is strongly $\mu-$codense, by Lemma 1.3 and so $\mathcal{H}$ is $\mu-$codense. □

The following Example 2.10 shows that the condition *quasi-topology* on μ cannot be dropped in Theorem 2.9.

Example 2.10. *Consider the space (X, μ) with hereditary class $\mathcal{H}$ as in Example 2.2. Clearly, μ is not a quasi-topology on X and $\mathcal{H}$ is $\mu-$codense. If $A = \{a, d\}$, then A is $\mu-$closed and $\Psi_{\mathcal{H}}(A) - A = \{a, b\} - \{a, d\} = \{b\} \neq \emptyset$.*

A hereditary class $\mathcal{H}$ is said to be $\star - $*strongly* $\mu -$ *codense* [5] if for $M, N \in \mu$, $(M \cap N) \cap A \in \mathcal{H}$ and $(M \cap N) - A \in \mathcal{H}$, then $M \cap N = \emptyset$. Nothing is mentioned about the set A. In the proof of Lemma 3.9(i) of [5], $A = X$, in the proof of Lemma 3.9(ii) of [5], $A = \emptyset$ and in Example 3.10 of [5], A is a nonempty proper subset of X. Hence the set A in the definition of $\star-$strongly $\mu-$codense hereditary class is any subset A of X. Also, in [5], it is proved that every $\star-$strongly $\mu-$codense hereditary class is strongly $\mu-$codense but the converse is not true [5, Example

3.10]. However, the converse holds if $\mathcal{H}$ is an ideal as shown by Theorem 2.12 below. Corollary 2.14 follows from Theorem 2.12 and Theorem 3.12 of [5]. If $\mu = \{\emptyset\}$, the trivial generalized topology, in a space (X, μ), then every hereditary class $\mathcal{H}$ is a $\star-$strongly $\mu - codense$ hereditary class. In this context, we have the following Theorem 2.11.

Theorem 2.11. *Let (X, μ) be a space where $\mu = \{\emptyset\}$. Then the following hold.*
(a) *Every hereditary class is a $\star-$strongly $\mu - codense$ hereditary class.*
(b) *$A^\star = X$ for every subset A of X.*
(c) *$\Psi_{\mathcal{H}}(A) = \emptyset$ for every subset A of X.*

Remark 2.1. If $\mu = \{\emptyset\}$, Theorem 2.11 shows that every hereditary class is $\star-$strongly $\mu-$codense and $\Psi_{\mathcal{H}}(A) = \emptyset$ and so the results established in Corollary 3.11, Theorem 3.12, Corollary 3.13, Theorem 3.14, Corollary 3.15, Theorem 3.17 and Theorem 3.18 of [5] are vacuously true.

Theorem 2.12. *Let (X, μ) be a space with an ideal $\mathcal{H}$. If $\mathcal{H}$ is strongly $\mu-$codense, then $\mathcal{H}$ is $\star-$strongly $\mu-$codense.*

Proof: Let $M, N \in \mu$ and $A \subset X$ with $(M \cap N) - A \in \mathcal{H}$ and $(M \cap N) \cap A \in \mathcal{H}$. Now $M \cap N = ((M \cap N) - A) \cup ((M \cap N) \cap A) \in \mathcal{H}$, since $\mathcal{H}$ is an ideal. Since $\mathcal{H}$ is strongly $\mu-$codense, $M \cap N = \emptyset$. Hence $\mathcal{H}$ is $\star-$strongly $\mu-$codense. □

Corollary 2.13. *Let (X, μ) be a quasi-topological space with an ideal $\mathcal{H}$. If $\mathcal{H}$ is $\mu-$codense, then $\mathcal{H}$ is $\star-$strongly $\mu-$codense.*

Corollary 2.14. *Let (X, μ) be a space with an ideal $\mathcal{H}$. If $\mathcal{H}$ is strongly $\mu-$codense, then $\Psi_{\mathcal{H}}(A) \subset A^\star$ for every subset A of X.*

Proof: Follows from Theorem 2.12 and Theorem 3.12 of [5]. □

Corollary 2.15. *Let (X, μ) be a space with a strongly $\mu-$codense ideal $\mathcal{H}$ and $A \subset X$. If $A \in \mathcal{H}$, then $\Psi_{\mathcal{H}}(A) = \emptyset$.*

Proof: Follows from Corollary 2.14 and Lemma 1.1(a). □

The following Example 2.16 shows that the above Corollary 2.14 is not true for $\mu-$codense ideals.

Example 2.16. *Consider the space (X, μ) where $X = \{a, b, c\}$, $\mu = \{\emptyset, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$. Clearly, $\mathcal{H}$ is a $\mu-$codense ideal. If $A = \{a, c\}$, then $A^\star = \{c\}$ and $\Psi_{\mathcal{H}}(A) = \{a, c\}$ which implies that $\Psi_{\mathcal{H}}(A) \not\subseteq A^\star$.*

In [5], before Lemma 3.9, it is stated that in a space (X, μ), every ideal $\mathcal{H}$ is $\star-$strongly $\mu-$codense. But this statement is not true, even if $\mathcal{H}$ is a $\mu-$codense ideal, as shown by the following Example 2.17.

Example 2.17. *Consider the space* (X, μ) *with hereditary class* $\mathcal{H}$ *where* $X = \{a, b, c\}, \mu = \{\emptyset, \{a\}, \{a, c\}, \{b, c\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{c\}\}$*. Clearly,* $\mathcal{H}$ *is a* $\mu-$*codense ideal. If* $M = \{a, c\}$ *and* $N = \{b, c\}$*, then* $M \cap N = \{c\}$*. Also, for every* $A \subset X$*,* $(M \cap N) \cap A \in \mathcal{H}$ *and* $(M \cap N) - A \in \mathcal{H}$*. But* $M \cap N \neq \emptyset$*. Hence* $\mathcal{H}$ *is not* $\star-$*strongly* $\mu-$*codense. Note that an ideal need not be a strongly* $\mu-$*codense hereditary class.*

In the rest of this section, we derive some properties of the $\Psi_{\mathcal{H}}-$operator. The following Theorem 2.18 gives characterizations of $\mu-$codense hereditary classes which is a generalization of Lemma 1.1(c).

Theorem 2.18. *Let* (X, μ) *be a space with a hereditary class* $\mathcal{H}$*. Then the following are equivalent.*
(a) $\mathcal{H}$ *is* $\mu-$*codense.*
(b) $\mathcal{M}_{\mu}^{\star} = X$.
(c) $\Psi_{\mathcal{H}}(X - \mathcal{M}_{\mu}) = \emptyset$.

Proof: (a)$\Leftrightarrow$(b). Suppose $x \in X$ and $x \notin \mathcal{M}_{\mu}^{\star}$. Then there exists $M \in \mu$ such that $x \in M$ and $M \cap \mathcal{M}_{\mu} \in \mathcal{H}$ which implies that $M \in \mathcal{H}$ and so $M = \emptyset$, since $\mathcal{H}$ is $\mu-$codense. Therefore, $x \in \mathcal{M}_{\mu}^{\star}$. Hence $\mathcal{M}_{\mu}^{\star} = X$. Conversely, suppose $M \in \mu \cap \mathcal{H}$. If $M \neq \emptyset$, then there exists $x \in M$ and so $x \in \mathcal{M}_{\mu}^{\star}$ which implies that $M \cap \mathcal{M}_{\mu} = M \notin \mathcal{H}$, a contradiction. Therefore, $\mu \cap \mathcal{H} = \{\emptyset\}$.
(b)$\Leftrightarrow$(c). $\Psi_{\mathcal{H}}(X - \mathcal{M}_{\mu}) = X - (X - (X - \mathcal{M}_{\mu}))^{\star} = X - \mathcal{M}_{\mu}^{\star}$. So (b) and (c) are equivalent. □

The following Example 2.19 shows that the converse of Theorem 2.9(e) is not true and Theorem 2.20 gives a partial converse.

Example 2.19. *Consider the space* (X, μ) *where* $X = \{a, b, c\}$*,* $\mu = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ *and* $\mathcal{H} = \{\emptyset, \{b\}\}$*. Clearly,* $\mathcal{H}$ *is a* $\mu-$*codense ideal. Here* $\mathcal{H}_r = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$ *so that* $\mathcal{H} \subset \mathcal{H}_r$*. If* $A = \{a, b\}$*, then* $\Psi_{\mathcal{H}}(A) = X - \{c\}^{\star} = X - \{c\} = \{a, b\}$ *and so* $\Psi_{\mathcal{H}}(A) = A$*. But* $i_{\mu}c_{\mu}(A) = i_{\mu}(X) = X \neq \{a, b\}$ *and so* A *is not* $\mu r-$*open.*

Theorem 2.20. *Let* (X, μ) *be a quasi-topological space with a hereditary class* $\mathcal{H}$ *and* $A \subset X$*. If* $\mathcal{H}_r \subset \mathcal{H}$*, then* $A = \Psi_{\mathcal{H}}(A)$ *implies that* A *is* $\mu r-$*open.*

Proof: Suppose that $A = \Psi_{\mathcal{H}}(A)$. Then A is $\mu-$open and so $A = i_{\mu}(A) \subset i_{\mu}c_{\mu}(A)$. Let $x \in i_{\mu}c_{\mu}(A)$. Then there exists $G \in \mu$ containing x such that $G \subset c_{\mu}(A)$ which implies that $G - A \subset c_{\mu}(A) - A$. Now $i_{\mu}c_{\mu}(c_{\mu}(A) - A) \subset i_{\mu}(c_{\mu}(A) - i_{\mu}(A)) = i_{\mu}(c_{\mu}(A) - A) = i_{\mu}c_{\mu}(A) - c_{\mu}(A) = \emptyset$ and so $c_{\mu}(A) - A \in \mathcal{H}_r$ which implies that $c_{\mu}(A) - A \in \mathcal{H}$, since $\mathcal{H}_r \subset \mathcal{H}$. Therefore, $G - A \in \mathcal{H}$ so that $x \in \Psi_{\mathcal{H}}(A)$. Hence $i_{\mu}c_{\mu}(A) \subset \Psi_{\mathcal{H}}(A) = A$. Thus, A is $\mu r-$open. □

Corollary 2.21. *Let* (X, μ) *be a quasi-topological space with a* $\mu-$*codense hereditary class* $\mathcal{H}$ *such that* $\mathcal{H}_r \subset \mathcal{H}$*. Then for every* $A \subset X$*,* $A = \Psi_{\mathcal{H}}(A)$ *if and only if* A *is* $\mu r-$*open.*

Proof: Follows from Theorem 2.9(e) and Theorem 2.20. □

The following Theorem 2.22 gives characterizations of $\mu-$codense hereditary classes in terms of the generalized topology of semiopen sets $\sigma(\mu)$ of the generalized topology μ.

Theorem 2.22. *Let (X, μ) be a space with a hereditary class $\mathcal{H}$. Then the following are equivalent.*
(a) *$\mathcal{H}$ is strongly $\mu-$codense.*
(b) *If A is $\sigma(\mu)-$closed, then $\Psi_{\mathcal{H}}(A) \subset A$.*
(c) *$\Psi_{\mathcal{H}}(c_\mu(A)) = i_\mu c_\mu(A)$ for every $A \subset X$.*
(d) *$\Psi_{\mathcal{H}}(A) = i_\mu(A)$ for every $\mu-$closed set A.*
(e) *$\Psi_{\mathcal{H}}(c_\sigma(A)) \subset i_\sigma c_\sigma(A)$ for every subset A of X.*
(f) *$\Psi_{\mathcal{H}}(A) \subset i_\sigma(A)$ for every $\sigma(\mu)-$closed set A.*

Proof: (a)⇒(b). Suppose A is $\sigma(\mu)-$closed. By Lemma 1.3(c), $X - A \subset (X - A)^\star$. Therefore, $X - (X - A)^\star \subset A$ which implies that $\Psi_{\mathcal{H}}(A) \subset A$.
(b)⇒(a). If $A \in \sigma$, then $X - A$ is $\sigma-$closed. Therefore, by (b), $\Psi_{\mathcal{H}}(X - A) \subset X - A$ and so $A \subset A^\star$. By Lemma 1.3(c), $\mathcal{H}$ is strongly $\mu-$codense.
(a)⇒(c). If $A \subset X$, $\Psi_{\mathcal{H}}(c_\mu(A)) = X - (X - c_\mu(A))^\star = X - c_\mu(X - c_\mu(A))$, by Lemma 1.3(e) and so $\Psi_{\mathcal{H}}(c_\mu(A)) = i_\mu c_\mu(A)$.
The equivalence of (c) and (d) is clear.
(c)⇒(b). If A is $\sigma(\mu)-$closed, by (c), $\Psi_{\mathcal{H}}(c_\mu(A)) = i_\mu c_\mu(A) \subset A$. Since $\Psi_{\mathcal{H}}$ is monotonic, it follows that $\Psi_{\mathcal{H}}(A) \subset A$ which proves (b).
Clearly, (e) and (f) are equivalent.
(b)⇒(f) follows from the fact that $\Psi_{\mathcal{H}}(A) \in \mu$ and (e)⇒(b) is clear. □

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