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Multiplicativity of left centralizers forcing additivity

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abstract: A multiplicative left centralizer for an associative ring R is a map satisfying $T(xy) = T(x)y$ for all x, y in R. T is not assumed to be additive. In this paper we deal with the additivity of the multiplicative left centralizers in a ring which contains an idempotent element. Specially, we study additivity for multiplicative left centralizers in prime and semiprime rings which contain an idempotent element.

Key Words: Prime rings, Semiprime Rings, Centralizers, Pierce decomposition.

Contents

1. Introduction

Let R be an associative ring. The study of the question of when any mapping defined on R is additive has become an active research area in ring theory and operator theory. Of course the starting point of this line of research is the famous paper of Rickart [8] which is often used and cited in the operator algebra settings. Rickart proved that any one-to-one multiplicative mapping of a Boolean ring B onto an arbitrary ring S is necessarily additive (Theorem 1 in [8]). Moreover, if R is an arbitrary ring containing some non-zero minimal right ideals then, under certain conditions on R, any one-to-one multiplicative mapping of R onto an arbitrary ring S is necessarily additive (Theorem 2 in [8]).
In [7] Martindale asked the following question: When is a multiplicative isomorphism defined on R additive? His elegant conclusion assures additivity under the condition that R possess non-trivial idempotent elements. In [3], the second author introduced the definition of multiplicative derivation of R to be a mapping $d : R \longrightarrow R$ such that $d(xy) = d(x)y + xd(y)$ and proved that if R contains non-trivial idempotent elements then any multiplicative derivation is additive.
In [6], Lu and Xie established a condition on R, in the case where R may not contain any non-zero idempotents, that assures that a multiplicative isomorphism is additive, which generalizes Martindale's result. As an application, they showed that under a mild assumption every multiplicative isomorphism from the radical of a nest algebra onto an arbitrary ring is additive.
Our research has been motivated by the cited results. We will follow the above

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mentioned line of investigation and focus our attention on the study of left centralizers in prime and semiprime rings. We recall that an additive mapping $T : R \to R$ is called a left centralizer if $T(xy) = T(x)y$, for all $x, y \in R$. In ring theory it is more common to write that $T : R_R \to R_R$ is a homomorphism of the right R-module R into itself. Let Q be the Martindale right ring of quotient of R, C the extended centroid of R (we refer to Chapter 2 in [2] for definitions and properties of Q and C). For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where $q \in Q$. Moreover in case R has the identity element, then T is a left centralizer if and only if T is the form $T(x) = qx$ for all $x \in R$, where $q \in R$. Notice that the definition of right centralizer is self-explanatory.
An additive mapping $T : R \to R$ is called a two-sided centralizer if T is both a left and right centralizer. If R is a semiprime ring and $T : R \to R$ is a two-sided centralizer, then $T(x) = \lambda x$ for all $x \in R$, where $q \in C$, the extended centroid of R (see Theorem 2.3.2 in [1]).
Centralizers have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting. Recently, several authors have studied such additive mappings on rings satisfying some identical relations. When treating such relations one usually concludes that the form of the map involved can be described, unless the ring is very special (see for example [5], [9], [10], [11], [12]).
In this note, we introduce the notion of the multiplicative left centralizer of a ring R. It is a mapping $T : R \to R$ (not necessarily additive) such that $T(xy) = T(x)y$, for all $x, y \in R$. Here we ask the question when a multiplicative left centralizer on a prime or semiprime ring R is additive, in other words when a multiplicative left centralizer is a left centralizer. Our aim is to establish a sufficient condition which forces additivity.
We also would like to point out the relationship between left centralizers and derivations. In [4], Hvala defined the notion of a generalized derivation as follows: An additive mapping $g : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that

$$g\,(xy) = g(x)y + xd(y) \text{ for all } x, y \in R.$$

He called the maps of the form $x \mapsto ax + xb$ where a, b are fixed elements in R “inner generalized derivations". Then it seems natural to remark that the concept of generalized derivation covers both the concepts of derivation and left centralizer.

In light of this, we call multiplicative generalized derivation any mapping $g : R \to R$ such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in R$, related with a derivation d. In parallel to the works of Martindale [7] and Daif [3], we ask the following question for a multiplicative generalized derivation: When is a multiplicative generalized derivation additive, that is when a multiplicative generalized derivation is a generalized derivation? Under some conditions, we give an answer also for this question, as a consequence of the result we obtain for left centralizers.

2. The Results.

When the ring R has an identity element, it is easy to prove that any multiplicative generalized derivation and also any multiplicative left centralizer is additive as follows. The same conclusion holds in the case R is a commutative prime (or semiprime) ring without an identity element. We omit the proof of these facts for brevity.
Now our aim is to study the case of noncommutative prime and semiprime rings which need not have an identity element but which contain an idempotent e $(e \neq 0, 1)$.

For the next lemmas we call this idempotent element e_1 and formally set $e_2 = 1-e_1$. Then for $R_{ij} = e_i R e_j$ $(i, j = 1, 2)$ we may write R in its Peirce decomposition $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. Moreover we will denote x_{ij} any element of $\S R_{ij}$.
We begin with the following useful result:

Lemma 2.1. *For any $x_{ij} \in R_{ij}$, we have*

$$[T(x_{11} + x_{12} + x_{21} + x_{22}) - T(x_{11} + x_{21}) - T(x_{12} + x_{22})]R = (0).$$

Proof: Let $x_{1*} \in R_{1i}$ and $* = 1, 2$. We get

$$\begin{aligned}
&[T(x_{11} + x_{21}) + T(x_{12} + x_{22})]x_{1*} = T(x_{11} + x_{21})x_{1*} + T(x_{12} + x_{22})x_{1*} \\
&= T[(x_{11} + x_{21})x_{1*}] + T[(x_{12} + x_{22})x_{1*}] = [T(x_{11} + x_{21})x_{1*}] \\
&= T[(x_{12} + x_{22})x_{1*}] = T(x_{11} + x_{12} + x_{21} + x_{22})x_{1*}
\end{aligned}$$

So we have,

$$[T(x_{11} + x_{12} + x_{21} + x_{22}) - T(x_{11} + x_{21}) - T(x_{12} + x_{22})]x_{1*} = 0 \tag{2.1}$$

Also, similarly, for any x_{2*} where $* = 1, 2$ we get

$$[T(x_{11} + x_{12} + x_{21} + x_{22}) - T(x_{11} + x_{21}) - T(x_{12} + x_{22})]x_{2*} = 0 \tag{2.2}$$

Using (2.1) and (2.2) we get,

$$[T(x_{11} + x_{12} + x_{21} + x_{22}) - T(x_{11} + x_{21}) - T(x_{12} + x_{22})]R = (0) \tag{2.3}$$

□

Lemma 2.2. *For any $x_{ij} \in R_{ij}$, $*$ means any value of 1, 2 can be used, we have:*

(I) $T(x_{*1}x_{12}x_{2*} + y_{*1}y_{12}y_{2*}) = T(x_{*1}x_{12}x_{2*}) + T(y_{*1}y_{12}y_{2*})$.

(II) $T(x_{*2}x_{21}x_{1*} + y_{*2}y_{21}y_{1*}) = T(x_{*2}x_{21}x_{1*}) + T(y_{*2}y_{21}y_{1*})$.

Proof: The proof makes use of a clever factoring procedure:

$$\begin{aligned}
&T(x_{*1}x_{12}x_{2*} + y_{*1}y_{12}y_{2*}) = T[(x_{*1}x_{12} + y_{*1})(x_{2*} + y_{12}y_{2*})] \\
&= T(x_{*1}x_{12} + y_{*1})(x_{2*} + y_{12}y_{2*}) \\
&= T(x_{*1}x_{12} + y_{*1})x_{2*} + T(x_{*1}x_{12} + y_{*1})y_{12}y_{2*} \\
&= T(x_{*1}x_{12}x_{2*} + y_{*1}x_{2*}) + T(x_{*1}x_{12}y_{12}y_{2*} + y_{*1}y_{12}y_{2*}) \\
&= T(x_{*1}x_{12}x_{2*}) + T(y_{*1}y_{12}y_{2*}).
\end{aligned}$$

By a similar method, we can get the proof of (II). □

Lemma 2.3. *For any $x_{ij} \in R_{ij}$, we have*

(1) $[T(x_{11}+x_{21}) - T(x_{11}) - T(x_{21})]R_{12}R = (0)$.

(2) $[T(x_{12}+x_{22}) - T(x_{12}) - T(x_{22})]R_{21}R = (0)$.

(3) $[T(x_{21}+y_{21}) - T(x_{21}) - T(y_{21})]R_{12}R = (0)$.

(4) $[T(x_{22}+y_{22}) - T(x_{22}) - T(y_{22})]R_{21}R = (0)$.

(5) $T(x_{11}+y_{11}) = T(x_{11}) + T(y_{11})$.

(6) $T(x_{12}+y_{12}) = T(x_{12}) + T(y_{12})$.

Proof: Here we make use of the results (I) and (II) of Lemma 2.2:
(1) Using (I), we get

$$\begin{aligned} &T(x_{11}+x_{21})y_{12}y_{2*} = T(x_{11}y_{12}y_{2*} + x_{21}y_{12}y_{2*}) \\ &= T(x_{11}y_{12}y_{2*}) + T(x_{21}y_{12}y_{2*}) \\ &= T(x_{11})y_{12}y_{2*} + T(x_{21})y_{12}y_{2*}. \end{aligned}$$

This means,

$$[T(x_{11}+x_{21}) - T(x_{11}) - T(x_{21})]R_{12}R_{2*} = (0).$$

But also since $R_{12}R_{1*} = (0)$ we have,

$$[T(x_{11}+x_{21}) - T(x_{11}) - T(x_{21})]R_{12}R_{1*} = (0).$$

So we arrive at

$$[T(x_{11}+x_{21}) - T(x_{11}) - T(x_{21})]R_{12}R = (0).$$

(2) Using (II) and the same argument as (1) we get the proof.
(3) Using (I) and the same argument as (1) we get the proof.
(4) Using (II) and the same argument as (1) we get the proof.
(5) Let x_{11} and y_{11} be two elements in the subring R_{11}, then $T(x_{11}+y_{11}) = T[e_1(x_{11}+y_{11})] = T(e_1)(x_{11}+y_{11}) = T(e_1)x_{11}+T(e_1)y_{11} = T(e_1x_{11})+T(e_1y_{11}) = T(x_{11}) + T(y_{11})$.
(6) Use the same argument as (5). □

Theorem 2.4. *Let R be a prime ring with an idempotent element $e_1 \neq 0, 1$. Let $T : R \to R$ be a multiplicative left centralizer, i.e., $T(xy) = T(x)y$ for all $x, y \in R$. Then T is additive.*

Proof: The set $I = R_{12}R_{21} + R_{12} + R_{21} + R_{21}R_{12}$ is an ideal of R. By (1) of Lemma 2.3 and the fact that $R_{*1}R_{2*} = (0)$ we get,

$$[T(x_{11} + x_{21}) - T(x_{11}) - T(x_{21})]IR = (0). \tag{2.4}$$

Similarly, using Lemma 2.3, we get

$$[T(x_{12} + x_{22}) - T(x_{12}) - T(x_{22})]IR = (0), \tag{2.5}$$

$$[T(x_{21} + y_{21}) - T(x_{21}) - T(y_{21})]IR = (0), \tag{2.6}$$

$$[T(x_{22} + y_{22}) - T(x_{22}) - T(y_{22})]IR = (0), \tag{2.7}$$

$$[T(x_{11} + y_{11}) - T(x_{11}) - T(y_{11})]IR = (0), \tag{2.8}$$

and

$$[T(x_{12} + y_{12}) - T(x_{12}) - T(y_{12})]IR = (0). \tag{2.9}$$

By the previous equations (2.4)-(2.9) and Lemma 2.1, for all a, b in R, we get

$$[T(a + b) - T(a) - T(b)]IR = (0). \tag{2.10}$$

Since R is not zero, by the primeness of R we have

$$[T(a + b) - T(a) - T(b)]I = (0). \tag{2.11}$$

This means that $[T(a+b) - T(a) - T(b)]$ is a left annihilator of I. In a prime ring either it must be zero or else I must be zero. If $[T(a + b) - T(a) - T(b)] = (0)$ this means that T is additive. Now if $I = (0)$ this means $R = R_{11} + R_{22}$. But R_{11} and R_{22} are orthogonal ideals in R this gives $R_{22} = 0$ since $R_{11} \neq (0)$ because it contains e_1. In this case we have $R = R_{11}$. Since, by (5) of Lemma 2.3, T is additive in R_{11}, we have that T is additive in R. □

Corollary 2.5. *Let R be a prime ring with an idempotent element $e \neq 0, 1$. Let $G : R \to R$ be a multiplicative generalized derivation, i.e., $G(xy) = G(x)y + xD(y)$, for all $x, y \in R$, related with a derivation D. Then G is additive.*

Proof: Since $G - D$ is a left centralizer, then $G - D$ is additive by the previous theorem. Since D is additive we get that G is additive. □

Now we study the additivity for a semiprime ring containing an idempotent element $e_1 \neq 0, 1$.

Proposition 2.6. *If R is a semiprime ring and J be an ideal generated by $\{T(a + b) - T(a) - T(b) : \forall a, b \in R\}$. Then $J \subseteq R_{22}$.*

Proof: Assume an ideal $I = R_{12}R_{21} + R_{12} + R_{21} + R_{21}R_{12}$ and J be an ideal generated by $\{T(a+b) - T(a) - T(b)\}$. Then, by Lemmas 2.1 and 2.3 we have $JIR = (0)$, and by the semiprimeness of R we get $(JI)^2 = 0$, this implies that $JI = (0)$. Also, $(J \cap I)^2 \subset JI = 0$. By the semiprimeness of R we get $J \cap I = 0$. But if $J \cap I = 0$. Then $J \subseteq (R_{11}+R_{22})$. Now let $J = J_{11}+J_{22}$, J_{11} and J_{22} are ideals in R. For any $j \in J_{11}$, $T(aj+bj)-T(aj)-T(bj) = T(aj)+T(bj)-T(aj)-T(bj) = 0$ because $RJ_{11} \subseteq R_{11}$ and T is additive in R_{11}. So we get $JJ_{11} = 0$ which implies $(J_{11} + J_{22})J_{11} = J_{11}^2 = 0$. Because of the semiprimeness of R we get $J_{11} = 0$. This means $J = J_{22} \subset R_{22}$. □

Now we delete the semiprimeness hypothesis and obtain a similar result:

Proposition 2.7. *Let R be a ring. Let $T : R \to R$ be a multiplicative left centralizer, i.e., $T(xy) = T(x)y$ for all $x, y \in R$. For all integers $n \geq 1$, let J^* be an ideal generated by*

$$S = \{T^n(a+b) - T^n(a) - T^n(b) : \forall a, b \in R\}.$$

Then T^n is again a multiplicative centralizer. Moreover, $T(J^*) \subset J^*$ and T is additive on R/J^*.

Proof: Of course T^n is a left centralizer on R, since for all $x, y \in R$ we have that

$$T^n(xy) = T^{n-1}(T(x)y) = T^{n-2}(T^2(x)y) = \dots\dots\dots = T(T^{n-1}(x)y) = T^n(x)y.$$

The set S is a right ideal, so $J^* = S + RS$ and we have

$$T(RS) \subset T(R)S \subset RS \subset J^*, \text{ which implies } T(J^*) \subset J^*$$

and

$$T[T^n(a+b) - T^n(a) - T^n(b)] \equiv 0 \mod J$$

$$T[T^n(a+b) - T^n(a)] - T^{n+1}(b) \equiv 0 \mod J$$

$$T[T^n(a+b)] - T^{n+1}(a) - T^{n+1}(b) \equiv 0 \mod J$$

$$T^{n+1}(a+b) - T^{n+1}(a) - T^{n+1}(b) \equiv 0 \mod J$$

Since J^* is invariant, T is a left centralizer on the quotient ring R/J^*. Since $T(a+b) - T(a) - T(b) \equiv 0$ in R/J^*, T is additive left centralizer on R/J^*. □

From Propositions 2.6 and 2.7 it follows that:

Proposition 2.8. *Let R be a semiprime ring. If $T : R \to R$ be a multiplicative left centralizer, i.e., $T(xy) = T(x)y$ for all $x, y \in R$. Let J^* be an ideal generated by*

$$S = \{T^n(a+b) - T^n(a) - T^n(b) : \forall a, b \in R\}.$$

Then $J^* \subset R_{22}$.

In light of Proposition 2.7, T^n is a left centralizer for all $n \geq 1$. Hence, by Lemmas 2.1, 2.2 and 2.3, the following lemmas also hold:

Lemma 2.9. *Let* $n \neq 1$ *be an integer. Then for any* $x_{ij} \in R_{ij}$, *we have*

$$[T^n(x_{11}+x_{12}+x_{21}+x_{22}) - T^n(x_{11}+x_{21}) - T^n(x_{12}+x_{22})]R = (0).$$

Lemma 2.10. *Let* $n \neq 1$ *be an integer. For any* $x_{ij} \in R_{ij}$, $*$ *means any value of 1, 2 can be used, we have:*

(I) $T^n(x_{*1}x_{12}x_{2*} + y_{*1}y_{12}y_{2*}) = T^n(x_{*1}x_{12}x_{2*}) + T^n(y_{*1}y_{12}y_{2*})$.

(II) $T^n(x_{*2}x_{21}x_{1*} + y_{*2}y_{21}y_{1*}) = T^n(x_{*2}x_{21}x_{1*}) + T^n(y_{*2}y_{21}y_{1*})$.

Lemma 2.11. *Let* $n \neq 1$ *be an integer. For any* $x_{ij} \in R_{ij}$, *we have*
(1) $[T^n(x_{11}+x_{21}) - T^n(x_{11}) - T^n(x_{21})]R_{12}R = (0)$.
(2) $[T^n(x_{12}+x_{22}) - T^n(x_{12}) - T^n(x_{22})]R_{21}R = (0)$.
(3) $[T^n(x_{21}+y_{21}) - T^n(x_{21}) - T^n(y_{21})]R_{12}R = (0)$.
(4) $[T^n(x_{22}+y_{22}) - T^n(x_{22}) - T^n(y_{22})]R_{21}R = (0)$.
(5) $T^n(x_{11}+y_{11}) = T^n(x_{11}) + T^n(y_{11})$.
(6) $T^n(x_{12}+y_{12}) = T^n(x_{12}) + T^n(y_{12})$.

Theorem 2.12. *Let* R *be a semiprime ring and* $n \geq 1$ *be an integer. Let* J^* *be an ideal generated by* $S = \{T^n(a+b) - T^n(a) - T^n(b), \quad \forall a, b \in R\}$. *If* R *satisfies any of the following conditions then* T^n *is additive on* R:
(1) $J^* = (0)$.
(2) T^n *is additive on* R_{22}.
(3) $R_{22} = R_{21}R_{12}$.

Proof: (1) Since $T^n(a+b) - T^n(a) - T^n(b) \in J^* \ \forall\ a, b \in R$, if $J^* = (0)$ then T^n is additive.
(2) Since T^n is additive on R_{22}, then by Proposition 2.8 it is additive on J^*. Then for any $j \in J^*$, $[T^n(a+b) - T(a) - T(b)]j = T(aj+bj) - T(aj) - T(bj) = 0$. Therefore $(J^*)^2 = (0)$. Since R is semiprime, $J^* = (0)$ and we conclude by the previous argument.
(3) Assume the ideal $I = R_{12}R_{21} + R_{12} + R_{21} + R_{21}R_{12}$. If $R_{22} = R_{21}R_{12}$ then by Lemmas 2.4 and 2.6 we have $J^*IR = (0)$ which implies $J^*I = (0)$. Thus $(J^* \cap I)^2 \subseteq J^*I = (0)$ and by the semiprimeness of R it follows $J^* \cap I = (0)$ which implies $J^* = (0)$ and again conclude by argument in (1). $\square$

As a particular case of the previous theorem we obtain the last result of the paper:

Theorem 2.13. *Let* R *be a semiprime ring and* J *be an ideal generated by* $S = \{T(a+b) - T(a) - T(b), \quad \forall a, b \in R\}$. *If* R *satisfies any of the following conditions*

then T is additive on R:
(1) $J = (0)$.
(2) T *is additive on* R_{22}.
(3) $R_{22} = R_{21}R_{12}$.

Corollary 2.14. *Let R be a semiprime ring which satisfies the conditions of the above Theorem 2.13 then, any multiplicative generalized derivation on R, is additive.*

Proof: If $G(xy) = G(x)y + xD(y)$, where D is an (additive) derivation, then $G - D$ is a multiplicative left centralizer. So $G - D$ is additive and G is additive. $\square$

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