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A New Characterization of PSL(2, 27)

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ABSTRACT: Let G be a group and $\pi_e(G)$ be the set of element orders of G. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. Set nse(G):=$\{m_k | k \in \pi_e(G)\}$. In this paper, we prove if G is a group such that nse(G)=nse(PSL(2, 27)), then $G \cong$PSL(2, 27).

Key Words: Element order, set of the numbers of elements of the same order, Sylow subgroup.

Contents

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. Also the set of element orders of G is denoted by $\pi_e(G)$. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Set m_i=$m_i(G)$=$|\{g \in G|$ the order of g is $i\}|$. In fact, m_i is the number of elements of order i in G, and nse(G) := $\{m_i | i \in \pi_e(G)\}$, the set of numbers of elements with the same order. Throughout this paper, we denote by ϕ the Euler totient function. If G is a finite group, then we denote by P_q a Sylow $q-$subgroup of G and $n_q(G)$ is the number of Sylow $q-$subgroup of G, that is, $n_q(G)$=$|Syl_q(G)|$. All further unexplained notation is standard and we refer to [1], for example.
The problem of characterizing groups G by the set nse(G) was first studied by Shao et al. [2] where the authors proved that the simple K_4-group G are characterized by the set nse(G) and the group order $|G|$. In [3], the authors showed that the alternating group A_n for $4 \leq n \leq 6$ are uniquely determined by only the set of numbers of elements of the same order. Later on, it is proved in [4] that the simple groups PSL(2, q) for $q \in \{7, 8, 11, 13\}$ are also characterized by this set and they

2000 *Mathematics Subject Classification:* 20D06 asked whether $G \cong$PSL(2, q) if nse(G) = nse(PSL(2, q)), where q is a prime power. In this paper, we give a positive answer to this question and show that the group PSL(2, q) is characterizable by only nse(G) for $q = 27$. In fact the main theorem of our paper is as follows:

Main Theorem: Let G be a group. Suppose nse(G)=nse(PSL(2, 27)). Then $G \cong$PSL(2, 27).

We note that although we apply the technique used in [4], but by that method, we cannot characterized the group with order more than 2000. Because, they used the GAP program and in the library of GAP, there are only the groups with order less than 2000. In this paper, we use a new technique for the proof of our main result and our method can work for the groups with order more than 2000.

2. Preliminary Results

In this section we present some preliminary lemmas that will be used in the proof of the main theorem.

Lemma 2.1. *[5, Theorem 9.3.1] Let G be a finite solvable group and $|G| = m \cdot n$, where $m = p_1^{\alpha_1}...p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, ..., p_r\}$ and h_m be the number of $\pi-$Hall subgroups of G. Then $h_m = q_1^{\beta_1}...q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, ..., s\}$:*

1. *$q_i^{\beta_i} \equiv 1$ (mod p_j), for some p_j.*
2. *The order of some chief factor of G is divisible by $q_i^{\beta_i}$.*

Lemma 2.2. *[6] If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, PSL(2, 7), PSL(2, 8), PSL(2, 17), PSL(3, 3), PSU(3, 3) or PSU(4, 2).*

Lemma 2.3. *[7] Let G be a simple K_4-group. Then G is isomorphic to one of the following groups:*
(1) A_7, A_8, A_9, A_{10}.
(2) M_{11}, M_{11}, J_2.
(3) (a) $L_2(r)$, where r is a prime and satisfies $r^2 - 1 = 2^a.3^b.v^c$ with $a \geq 1$, $b \geq 1$, $c \geq 1$ and $v > 3$ is a prime;
(b) $L_2(2^m)$, where m satisfies $2^m - 1 = u$, $2^m + 1 = 3t^b$, with $m \geq 2$, u, t are primes, $t > 3$, $b \geq 1$;

(c) $L_2(3^m)$, *where* m *satisfies* $3^m+1=4t$, $3^m-1=2u^c$ *or* $3^m+1=4t^b$, $3^m-1=2u$, *with* $m\geq 2$, u, t *odd primes,* $b\geq 1$, $c\geq 1$*;*
(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$.

Lemma 2.4. *[2] Let* G *be a finite group and let* $P\in Syl_p(G)$, *where* $p\in\pi(G)$. *Let* G *have a normal series* $K\trianglelefteq L\trianglelefteq G$. *If* $P\leq L$ *and* $p\nmid |K|$, *then the following hold:*
(1) $N_{G/K}(PK/K)=N_G(P)K/K$*;*
(2) $|G:N_G(P)|=|L:N_L(P)|$, *that is,* $n_p(G)=n_p(L)$*;*
(3) $|L/K:N_{L/K}(PK/K)|t=|G:N_G(P)|=|L:N_L(P)|$, *that is,* $n_p(L/K)t=n_p(G)=n_p(L)$ *for some positive integer* t, *and* $|N_K(P)|t=|K|$.

Lemma 2.5. *[8] Let* G *be a finite group and* m *be a positive integer dividing* $|G|$. *If* $L_m(G)=\{g\in G|g^m=1\}$, *then* $m\mid |L_m(G)|$.

Lemma 2.6. *[3] Let* G *be a group containing more than two elements. Let* $k\in\pi_e(G)$ *and* m_k *be the number of elements of order* k *in* G. *If* $s=sup\{m_k|k\in\pi_e(G)\}$ *is finite, then* G *is finite and* $|G|\leq s(s^2-1)$.

Lemma 2.7. *[9] Let* G *be a finite group and* $p\in\pi(G)$ *be odd. Suppose that* P *is a Sylow* $p-$*subgroup of* G *and* $n=p^s m$, *where* $(p,m)=1$. *If* P *is not cyclic and* $s>1$, *then the number of elements of order* n *is always a multiple of* p^s.

Lemma 2.8. *[10] Let* G *be a finite group and* M *be normal subgroup of* G. *Then both the Sylow* $p-$*number* $n_p(M)$ *and the Sylow* $p-$*number* $n_p(G/M)$ *of the quotient* G/M *divide the Sylow* $p-$*number* $n_p(G)$ *of* G *and moreover* $n_p(M)\ n_p(G/M)\mid n_p(G)$.

Let G be a group such that nse(G)=nse(PSL(2, 27)). By Lemma 2.6, we can assume that G is finite. Let m_n be the number of elements of order n. We note that $m_n= k\phi(n)$, where k is the number of cyclic subgroups of order n in G. Also we note that if $n>2$, then $\phi(n)$ is even. If $n\in\pi_e(G)$, then by Lemma 2.5 and the above notation we have:

$$\begin{cases} \phi(n) \mid m_n \\ \\ n \mid \sum_{d|n} m_d \end{cases} \qquad (*)$$

In the proof of the main theorem, we apply $(*)$ and the above comments.

3. Proof of the Main Theorem

Let G be a group such that nse(G)=nse(PSL(2, 27))=$\{1, 351, 728, 2106, 4536\}$. At first, we prove that $\pi(G) \subseteq \{2, 3, 7, 13\}$. Since $351 \in \text{nse}(G)$, it follows from $(*)$ that $2 \in \pi(G)$ and $m_2 = 351$. Let $2 \neq p \in \pi(G)$, by $(*)$, $p \mid (1 + m_p)$ and $(p - 1) \mid m_p$, which implies that $p \in \{3, 7, 13\}$. Therefore, $\pi(G) \subseteq \{2, 3, 7, 13\}$. If 3, 7 and $13 \in \pi(G)$, then $m_3 = 728$, $m_7 = 2106$ and $m_{13} = 4536$, by $(*)$. Suppose that $13 \in \pi(G)$. Because $\phi(13^2) = 156$ and $\phi(13^3) = 2028$, by $(*)$ we can see easily that G does not contain any elements of order 13^2 and 13^3. Thus $\exp(P_{13}) = 13$ and $|P_{13}| \mid (1 + m_{13}) = 4537$ by Lemma 2.5. Hence $|P_{13}| = 13$ and $n_{13} = m_{13}/\phi(13) = 378 \mid |G|$. Therefore if $13 \in \pi(G)$, because $n_{13} \mid |G|$ this implies that 3 and $7 \in \pi(G)$. As $\phi(16) = 8$, $\phi(49) = 42$ and $\phi(729) = 486$, it is easy to check that G does not contain any elements of order 16, 49 and 729. If $7 \in \pi(G)$, then $|P_7| \mid (1 + m_7) = 2107$. Hence $|P_7| \mid 49$. Also since $16 \notin \pi_e(G)$, we have $|P_2| \mid 16$. We know that if $13 \in \pi(G)$, then 3 and $7 \in \pi(G)$. So if we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2, 3\}$, $\{2, 7\}$ and $\{2, 3, 7\}$, then $\pi(G)$ must be equal to $\{2, 3, 7, 13\}$. We consider the following cases:

Case a. $\pi(G) = \{2\}$. We have $\pi_e(G) \subseteq \{1, 2, 4, 8\}\}$ and so $|\pi_e(G)| \leq 4$, which is a contradiction since $|\text{nse}(G)|= 5$. Thus this case impossible.

Case b. $\pi(G) = \{2, 3\}$. Since $729 \notin \pi_e(G)$, we have $\exp(P_3) = 3, 9, 27, 81$ or 243. If $\exp(P_3) = 3$, then $|P_3| \mid (1 + m_3) = 729$. Hence $|P_3| \mid 3^6$. Let $|P_3| = 3$. Then $n_3 = m_3/\phi(3) = 364 \mid |G|$ since $7 \notin \pi(G)$, we get a contradiction. So $|G| = 2^m \times 3^n$ where $m \leq 4$ and $2 \leq n \leq 6$, on the other hand, $7722 \leq |G|$ and so $m = 4$ and $n = 6$. Since $\pi_e(G) \subseteq \{1, 2, 4, 8\} \bigcup \{3, 3\times 2, 3\times 4, 3\times 8\}$ and the sum of all the numbers in nse(G) is 7722, we have $|G| = 11664 = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1+k_2+k_3 = |\pi_e(G)|-|\text{nse}(G)|\leq 3$. Therefore, $3942 = 728k_1+2106k_2+4536k_3$. It is easy to check that this equation has no solution.
If $\exp(P_3) = 9$, then $|P_3| \mid (1 + m_3 + m_9)$ by Lemma 2.5. Since $m_9 \in \{2106, 4536\}$, we have $|P_3| \mid 3^4$. On the other hand, $|P_2| \mid 16$ and $7722 \leq |G|$, a contradiction. Similarly if $\exp(P_3) = 27$, then $|P_3| \mid 3^4$, a contradiction.

If $\exp(P_3) = 81$, $|P_3| \mid (1 + m_3 + m_9 + m_{27} + m_{81})$, so $|P_3| \mid 3^6$. It is clear that $|G| = 11664 = 3^6 \times 16$. Since $\pi_e(G) \subseteq \{1, 2, 4, 8\} \bigcup \{3, 3 \times 2, 3 \times 4, 3 \times 8\} \bigcup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \bigcup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \bigcup \{81, 81 \times 2, 81 \times 4, 81 \times 8\}$, we have $|G| = 11664 = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\mathrm{nse}(G)| \leq 15$. It is easy to check that this equation has no solution.
If $\exp(P_3) = 243$, then $|P_3| = 3^n$ where $n \geq 5$. If $n = 5$ since $m_{243} \in \{2106, 4536\}$, we have $n_3 = m_{243}/\phi(243) = 13$ or 28. As the group P_3 is cyclic of order 243, it has two elements of order 3. Since every element of order 3 lies in one or more of Sylow 3-subgroups, $m_3 \leq 2 \times 28 = 56$, a contradiction. If $n > 5$, then by Lemma 2.7, $243 \mid m_{243}$, a contradiction.

Case c. $\pi(G) = \{2, 7\}$. Since $49 \notin \pi_e(G)$, we have $\exp(P_7) = 7$. Then $|P_7| \mid (1 + m_7) = 2107$. Hence $|P_7| \mid 49$. Assume $|P_7| = 7$, so $n_7 = m_7/\phi(7) = 351 \mid |G|$ since $13 \notin \pi(G)$, we get a contradiction. If $|P_7| = 49$, then by $|P_2| \mid 16$ and $7722 \leq |G|$, we get a contradiction.

Case d. $\pi(G) = \{2, 3, 7\}$. With the same argument as in Case c, since $13 \notin \pi(G)$ we obtain that $|P_7| = 49$. Hence $|G| = 2^m \times 3^n \times 49$ where $m \leq 4$ and $n \leq 6$. We know that $\pi_e(G) \subseteq \{1, 2, 4, 8\} \bigcup \{3, 3 \times 2, 3 \times 4, 3 \times 8\} \bigcup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \bigcup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \bigcup \{81, 81 \times 2, 81 \times 4, 81 \times 8\} \bigcup \{243, 243 \times 2, 243 \times 4, 243 \times 8\} \bigcup \{7, 2 \times 7, 4 \times 7, 8 \times 7\} \bigcup \{3 \times 7, 9 \times 7, 27 \times 7, 81 \times 7, 243 \times 7\} \bigcup \{2 \times 3 \times 7, 2 \times 9 \times 7, 2 \times 27 \times 7, 2 \times 81 \times 7, 4 \times 3 \times 7, 4 \times 9 \times 7, 4 \times 27 \times 7, 4 \times 81 \times 27, 8 \times 3 \times 7, 8 \times 9 \times 7, 8 \times 27 \times 7, 8 \times 81 \times 7\}$, then $|\pi_e(G)| \leq 45$. Therefore, $|G| = 2^m \times 3^n \times 49 = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\mathrm{nse}(G)| \leq 40$, $m \leq 4$ and $n \leq 6$. By an easy computer calculation we can see that if $n = 6$ then this equation has no solution. If $n < 6$, then $n_7 = 1, 8$ or $2^i \times 3^j$ where $1 \leq i \leq 4$ and $1 \leq j \leq 5$. If $n_7 = 1, 8$, since every element of order 7 lies in one or more of Sylow 7-subgroups, we have $m_7 \leq 48 \times 8$, a contradiction. So $n_7 = 2^i \times 3^j$ where $1 \leq i \leq 4$ and $1 \leq j \leq 5$.
We show that G is a nonsolvable group. Suppose that G is a solvable group. Then by Lemma 2.1, $3^j \equiv 1$ (mode 7), a contradiction. Hence we conclude that G is a finite nonsolvable group. Let N be the solvable radical subgroup of G and let H/N be a chief factor of G. Then H/N is non-abelian and so it is isomorphic to a direct product of isomorphic non-abelian simple groups. We know that G is a K_3-group, thus H/N is a simple K_3-group or H/N is a direct product of simple K_3-groups. By Lemma 2.2, $H/N \cong$PSL(2, 7), PSL(2, 7)$\times$PSL(2, 7), PSL(2, 8) or PSL(2, 8)$\times$PSL(2, 8). On the other hand, by Lemma 2.8 $n_p(H/N) \mid n_p(G)$ for

every prime $p \in \pi(G)$. Hence $H/N \cong$PSL(2, 7) or PSL(2, 8). Let $H/N \cong$PSL(2, 7). Since n_7(PSL(2, 7)) = 8, by Lemma 2.8 we have $8 \mid n_7(G)$, so $n_7(G) = 16 \times 81$. Therefore, $|G| = 16\times81\times49$ or $16\times243\times49$. On the other hand, if $|G| = 16\times81\times49$ or $16 \times 243 \times 49$, then the equation $|G| = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 40$ has no solution, a contradiction.

Now let $H/N \cong$PSL(2, 8). By Lemma 2.8 $36 \mid n_7(G)$, because n_7(PSL(2, 8)) = 36, so $n_7(G) = 36$ or 16×81. Therefore, $|G| = 4 \times 27 \times 49$, $4 \times 81 \times 49$, $4 \times 243 \times 49$, $8 \times 9 \times 49$, $8 \times 27 \times 49$, $8 \times 81 \times 49$, $8 \times 243 \times 49$, $16 \times 9 \times 49$, $16 \times 27 \times 49$, $16 \times 81 \times 49$ or $16 \times 243 \times 49$. As $7722 \leq |G|$, so $|G| \neq 4 \times 27 \times 49$, $8 \times 9 \times 49$ and $16 \times 9 \times 49$. Let $|G| = 4 \times 81 \times 49$, $8 \times 9 \times 49$, $8 \times 81 \times 49$, $8 \times 243 \times 49$, $16\times9\times49$, $16\times81\times49$ or $16\times243\times49$, then it is easy to check that the equation $|G| = 7722+728k_1+2106k_2+4536k_3$ where $0 \leq k_1+k_2+k_3 = |\pi_e(G)|-|\text{nse}(G)| \leq 40$ has no solution. Also if $|G| = 4\times243\times49$, then $\exp(P_2) = 2$ or 4, so $|\pi_e(G)| \leq 34$. Now it is easy to check that the equation $|G| = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 29$ has no solution. Hence this case is impossible.

Therefore, $\pi(G) = \{2, 3, 7, 13\}$. We know that $|P_{13}| = 13$, we will show that $91 \notin \pi_e(G)$. Suppose that $91 \in \pi_e(G)$. We know that if P and Q are Sylow $13-$subgroups of G, then P and Q are conjugate, which implies that $C_G(P)$ and $C_G(Q)$ are conjugate in G. Therefore, $m_{91} = \phi(91)\cdot n_{13}\cdot k$, where k is the number of cyclic subgroups of order 7 in $C_G(P_{13})$. Since $n_{13} = 378$, we have $4536 \mid m_{91}$. On the other hand, $91 \mid (1 + m_{13} + m_7 + m_{91})$, which is a contradiction. Hence $91 \notin \pi_e(G)$.

Since $91 \notin \pi_e(G)$, the group P_7 acts fixed point freely on the set of elements of order 13, and so $|P_7| \mid m_{13} = 4536$, which implies that $|P_7| = 7$. Also we can prove that 26 and $21 \notin \pi_e(G)$. As $21 \notin \pi_e(G)$, the group P_3 acts fixed point freely on the set of elements of order 7, and so $|P_3| \mid m_7 = 2106$, which implies that $|P_3| \mid 81$. Since $26 \notin \pi_e(G)$, the group P_2 acts fixed point freely on the set of elements of order 13, and so $|P_2| \mid m_{13} = 4536$, which implies that $|P_2| \mid 8$. Therefore, $|G| = 2^n \times 3^m \times 7 \times 13$, where $n \leq 3$ and $m \leq 4$.

We claim that G is a nonsolvable group. Suppose G is a solvable group. Since $n_{13} = 378$, we have $7 \equiv 1 \pmod{13}$ by Lemma 2.1, which is a contradiction. Hence G is a nonsolvable group. As G is a nonsolvable group and $p \parallel |G|$, where $p \in \{7, 13\}$, G has a normal series $1 \trianglelefteq N \trianglelefteq H \trianglelefteq G$ such that N is a maximal solvable normal subgroup of G and H/N is a nonsolvable minimal normal subgroup of G/N. Then H/N is a non-abelian simple K_3-group or K_4-group.

Let H/N be a non-abelian simple K_3-group. By Lemma 2.2, $H/N \cong$PSL(2, 7) or PSL(2, 8). Let $H/N \cong$PSL(2, 7). Assume $P_7 \in Syl_7(G)$. Then $P_7N/N \in$

$Syl_7(H/N)$. By Lemma 2.4, $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$. Since $n_7(H/N) = n_7(\mathrm{PSL}(2,\, 7)) = 8$, we have $351 = 8t$, which is a contradiction. Now let $H/N \cong \mathrm{PSL}(2,\, 8)$. Assume $P_7 \in Syl_7(G)$. Then $P_7N/N \in Syl_7(H/N)$. By Lemma 2.4, $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$. Since $n_7(H/N) = n_7(\mathrm{PSL}(2,\, 7)) = 36$, we have $351 = 36t$, which is a contradiction.
Hence H/N is a non-abelian simple K_4-group. By Lemma 2.3, $H/N \cong$PSL(2, 13) or PSL(2, 27). Assume that $H/N \cong$PSL(2, 13) and let $P_7 \in Syl_7(G)$. Thus $P_7N/N \in Syl_7(H/N)$ and $n_7(H/N)t = n_7(G)$ for some positive integer t and $7 \nmid t$. Since $n_7(H/N) = n_7(\mathrm{PSL}(2,\, 13)) = 78$, we have $351 = 78t$, which is a contradiction. Hence $H/N \cong$PSL(2, 27).
Let $K/N = C_{G/N}(H/N)$. Then $H/N \trianglelefteq G/K \trianglelefteq \mathrm{Aut}(H/N)$, i.e., G/K is an almost simple group with socle H/N. Thus $G/K \cong$PSL(2, 27), PGL(2, 27), PΓL(2, 27) or PΣL(2, 27). Therefore, $|G| = 2^n \times 3^m \times 7 \times 13$ where $2 \le n \le 3$ and $3 \le m \le 4$. We know that $N \le K$. Since $|K| \mid 6$ and N is a maximal solvable normal subgroup of G, we have $N = K$. Hence G/N is isomorphic to one of the groups: PSL(2, 27), PGL(2, 27), PΓL(2, 27) or PΣL(2, 27).
Assume $|G| = 4 \times 81 \times 7 \times 13$. As G does not contain any elements of order 16, 21, 26, 39, 49, 91, 169 and 243, we have $\pi_e(G) \subseteq \{1,\, 2,\, 4\} \bigcup \{3,\, 3\times 2,\, 3\times 4\} \bigcup \{9, 9\times 2, 9\times 4\} \bigcup \{27,\, 27\times 2,\, 27\times 4\} \bigcup \{81,\, 81\times 2,\, 81\times 4\} \bigcup \{7,\, 2\times 7,\, 4\times 7,\, 8\times 7\} \bigcup \{13\}$. Hence $|\pi_e(G)| \le 20$. Therefore, $|G| = 8 \times 27 \times 7 \times 13 = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \le k_1 + k_2 + k_3 = |\pi_e(G)| - |\mathrm{nse}(G)| \le 15$. By an easy computer calculation we can get that this equation has no solution.
Let $|G| = 8 \times 27 \times 7 \times 13$. Since $\pi_e(G) \subseteq \{1,\, 2,\, 4,\, 8\} \bigcup \{3,\, 3 \times 2,\, 3 \times 4,\, 3 \times 8\} \bigcup \{9, 9\times 2, 9\times 4, 9\times 8\} \bigcup \{27,\, 27\times 2,\, 27\times 4,\, 27\times 8\} \bigcup \{7,\, 2\times 7,\, 4\times 7,\, 8\times 7\} \bigcup \{13\}$, we have $|\pi_e(G)| \le 21$. Therefore, $|G| = 8\times 27\times 7\times 13 = 7722+728k_1+2106k_2+4536k_3$ where $0 \le k_1 + k_2 + k_3 = |\pi_e(G)| - |\mathrm{nse}(G)| \le 16$. By an easy computer calculation we can get that this equation has no solution. Assume $|G| = 8 \times 81 \times 7 \times 13$. Since $\pi_e(G) \subseteq \{1,\, 2,\, 4,\, 8\} \bigcup \{3,\, 3\times 2,\, 3\times 4,\, 3\times 8\} \bigcup \{9,\, 9\times 2,\, 9\times 4,\, 9\times 8\} \bigcup \{27,\, 27\times 2,\, 27\times 4,\, 27\times 8\} \bigcup \{81,\, 81\times 2,\, 81\times 4,\, 81\times 8\} \bigcup \{7,\, 2\times 7,\, 4\times 7,\, 8\times 7\} \bigcup \{13\}$, we have $|\pi_e(G)| \le 25$. Therefore, $|G| = 8 \times 81 \times 7 \times 13 = 7722 + 728k_1 + 2106k_2 + 4536k_3$ where $0 \le k_1 + k_2 + k_3 = |\pi_e(G)| - |\mathrm{nse}(G)| \le 20$. By an easy computer calculation we can get that this equation has no solution. Therefore, $|G| = 4 \times 27 \times 7 \times 13$. By [2], since PSL(2, 27) is a simple K_4-group, we can conclude that $G \cong$PSL(2, 27), and the proof is complete.

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