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$\mathcal{D}-$Tangent Surfaces of Timelike Biharmonic $\mathcal{D}-$Helices according to Darboux Frame on Non-degenerate Timelike Surfaces in the Lorentzian Heisenberg Group$\mathbb{H}$

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abstract: In this paper, we study $\mathcal{D}-$tangent surfaces of timelike biharmonic $\mathcal{D}$-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation $\mathcal{D}-$tangent surfaces of timelike biharmonic $\mathcal{D}$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

Key Words: $\mathcal{D}-$Biharmonic curve, $\mathcal{D}-$tangent surfaces, Darboux frame, Heisenberg group.

Contents

1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \operatorname{tr} R\,(\mathcal{T}(\phi), d\phi)\, d\phi,$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study $\mathcal{D}-$tangent surfaces of timelike biharmonic $\mathcal{D}$-helices according to Darboux frame on non-degenerate timelike surfaces in the Lorentzian Heisenberg group $\mathbb{H}$. We obtain parametric equation $\mathcal{D}-$tangent surfaces of timelike biharmonic $\mathcal{D}$-helices in the Lorentzian Heisenberg group $\mathbb{H}$. Moreover, we illustrate the figure of our main theorem.

2. Preliminaries

Heisenberg group plays an important role in many branches of mathematics such as representation theory, harmonic analysis, PDEs or even quantum mechanics, where it was initially defined as a group of 3×3 matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \; x, y, z \in \mathbb{R} \right\}$$

with the usual multiplication rule.

The left-invariant Lorentz metric on $\mathbb{H}$ is

$$\rho = -dx^2 + dy^2 + (xdy + dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{ \mathbf{e}_1 = \frac{\partial}{\partial z}, \; \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \; \mathbf{e}_3 = \frac{\partial}{\partial x} \right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$\rho(\mathbf{e}_1, \mathbf{e}_1) = \rho(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad \rho(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

3. Timelike Biharmonic $\mathcal{D}$-Helices According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group $\mathbb{H}$

Let $\Pi \subset \mathbb{H}$ be a timelike surface with the unit normal vector $\mathbf{n}$ in the Lorentzian Heisenberg group $\mathbb{H}$. If γ is a timelike curve on $\Pi \subset \mathbb{H}$, then we have the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and Darboux frame $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ with spacelike vector $\mathbf{g} = \mathbf{T} \wedge \mathbf{n}$ along the curve γ. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ) and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned} \rho(\mathbf{T}, \mathbf{T}) &= -1,\ \rho(\mathbf{N}, \mathbf{N}) = 1,\ \rho(\mathbf{B}, \mathbf{B}) = 1, \\ \rho(\mathbf{T}, \mathbf{N}) &= \rho(\mathbf{T}, \mathbf{B}) = \rho(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

Let θ be the angle between $\mathbf{N}$ and $\mathbf{n}$. The relationships between $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}, \mathbf{n}, \mathbf{g}\}$ are as follows:

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{N} &= \cos\theta\mathbf{n} + \sin\theta\mathbf{g}, \\ \mathbf{B} &= \sin\theta\mathbf{n} - \cos\theta\mathbf{g}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \mathbf{T} &= \mathbf{T}, \\ \mathbf{g} &= \sin\theta\mathbf{N} - \cos\theta\mathbf{B}, \\ \mathbf{n} &= \cos\theta\mathbf{N} + \sin\theta\mathbf{B}. \end{aligned} \tag{3.4}$$

By differentiating (3.4), using (3.1), (3.3) and Frenet formulas we obtain

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= (\kappa\cos\theta)\,\mathbf{n} + (\kappa\sin\theta)\,\mathbf{g}, \\ \nabla_{\mathbf{T}}\mathbf{g} &= (-\kappa\sin\theta)\,\mathbf{T} + \left(\tau + \frac{d\theta}{ds}\right)\mathbf{n}, \\ \nabla_{\mathbf{T}}\mathbf{n} &= (-\kappa\cos\theta)\,\mathbf{T} - \left(\tau + \frac{d\theta}{ds}\right)\mathbf{g}. \end{aligned} \tag{3.5}$$

If we represent $\kappa\cos\theta$, $\kappa\sin\theta$ and $\tau+\frac{d\theta}{ds}$ with the symbols κ_n, κ_g, and τ_g respectively, then the equations in (3.5) can be written as

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa_g\mathbf{g}+\kappa_n\mathbf{n},\\ \nabla_{\mathbf{T}}\mathbf{g} &= -\kappa_g\mathbf{T}+\tau_g\mathbf{n},\\ \nabla_{\mathbf{T}}\mathbf{n} &= -\kappa_n\mathbf{T}-\tau_g\mathbf{g}.\end{aligned}\tag{3.6}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}\mathbf{T} &= T_1\mathbf{e}_1+T_2\mathbf{e}_2+T_3\mathbf{e}_3,\\ \mathbf{g} &= g_1\mathbf{e}_1+g_2\mathbf{e}_2+g_3\mathbf{e}_3.\\ \mathbf{n} &= n_1\mathbf{e}_1+n_2\mathbf{e}_2+n_3\mathbf{e}_3,\end{aligned}\tag{3.7}$$

To separate a curve according to Darboux frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve as $\mathcal{D}$-curve.

First of all we recall the following well known result (cf. [8]).

Theorem 3.1. *Let $\gamma: I\longrightarrow\Pi\subset\mathbb{H}$ be a non-geodesic unit speed timelike curve on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$. γ is a unit speed timelike biharmonic curve on Π if and only if*

$$\begin{aligned}\kappa_n^2+\kappa_g^2 &= \text{constant}\neq 0, &(3.8)\\ \kappa_n''-\kappa_n^3+\kappa_g\tau_g-\kappa_g^2\kappa_n+\kappa_g'\tau_g+\kappa_g\tau_g'-\tau_g^2\kappa_n &= \kappa_n(1-4g_1^2)-4\kappa_g n_1 g_1,\\ \kappa_g''-\kappa_g^3-2\kappa_n'\tau_g-\kappa_n^2\kappa_g-\kappa_n\tau_g'-\kappa_g\tau_g^2 &= 4\kappa_n n_1 g_1+\kappa_g(1-4n_1^2).\end{aligned}$$

Theorem 3.2. *Let $\gamma: I\longrightarrow\Pi\subset\mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- helix on timelike surface Π in the Lorentzian Heisenberg group $\mathbb{H}$. Then parametric equations of timelike biharmonic $\mathcal{D}$- helix are*

$$\begin{aligned}\boldsymbol{x}(s) &= \frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s])+\wp_1,\\ \boldsymbol{y}(s) &= \frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s])+\wp_2,\\ \boldsymbol{z}(s) &= \sinh\aleph s+\frac{(\wp+\mathcal{R}s)}{2\mathcal{R}^2}\cosh^2\aleph-\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\cosh[\wp]\cosh[\mathcal{R}s]\\ &\quad-\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\sinh[\wp]\sinh[\mathcal{R}s]-\frac{1}{4\mathcal{R}^2}\cosh^2\aleph\sinh 2[\mathcal{R}s+\wp]+\wp_3,\end{aligned}\tag{3.9}$$

where $\wp,\wp_1,\wp_2,\wp_3,$ are constants of integration and

$$\mathcal{R}=\operatorname{sech}\aleph\sqrt{\kappa_n^2+\kappa_g^2}-2\sinh\aleph.$$

Using Mathematica in above system we have Fig.1:

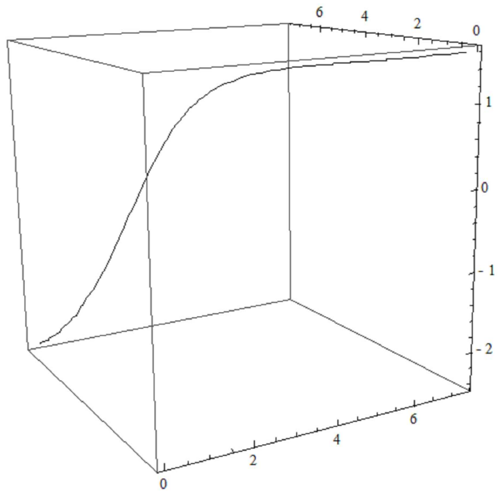

Fig. 1

4. $\mathcal{D}-$Tangent Surfaces According to Darboux Frame on a Non-Degenerate Timelike Surface in the Lorentzian Heisenberg Group $\mathbb{H}$

A ruled surface is formed by a continuous family of straight line segments.

The $\mathcal{D}-$tangent surface of γ is a ruled surface

$$\mathcal{O}\left(s,u\right)=\gamma\left(s\right)+u\mathbf{T}\left(s\right). \tag{4.1}$$

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathcal{O} \subset \mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- helix on timelike surface $\mathcal{O}$ in the Lorentzian Heisenberg group $\mathbb{H}$. Then,* *equation of $\mathcal{D}-$tangent surface of timelike biharmonic $\mathcal{D}$- helix is*

$$\begin{aligned}
\mathcal{O}(s,u) &= [\sinh\aleph s+\frac{(\wp+\mathcal{R}s)}{2\mathcal{R}^2}\cosh^2\aleph-\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\cosh[\wp]\cosh[\mathcal{R}s] \\
&\quad -\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\sinh[\wp]\sinh[\mathcal{R}s]-\frac{1}{4\mathcal{R}^2}\cosh^2\aleph\sinh 2[\mathcal{R}s+\wp] \\
&\quad +\wp_3+[\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s])+\wp_1] \qquad (4.2)\\
&\quad [\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s])+\wp_2]+u\sinh\mathcal{P}]\mathbf{e}_1 \\
&\quad +[\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s]) \\
&\quad +u\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]+\wp_2]\mathbf{e}_2 \\
&\quad +[\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s]) \\
&\quad +u\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]+\wp_1]\mathbf{e}_3,
\end{aligned}$$

where $\wp,\wp_1,\wp_2,\wp_3$, are constants of integration and

$$\mathcal{R}=\operatorname{sech}\aleph\sqrt{\kappa_n^2+\kappa_g^2}-2\sinh\aleph.$$

Proof: From the assumption we get

$$\mathbf{T}=\sinh\mathcal{P}\mathbf{e}_1+\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]\,\mathbf{e}_2+\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]\,\mathbf{e}_3. \qquad (4.3)$$

Substituting Eq.(3.9) and Eq.(4.3) into Eq.(4.1) , we obtain Eq.(4.2). This completes the proof. □

Theorem 4.2. *Let $\gamma: I\longrightarrow\mathcal{O}\subset\mathbb{H}$ be a non-geodesic unit speed timelike biharmonic $\mathcal{D}$- helix on timelike surface $\mathcal{O}$ in the Lorentzian Heisenberg group $\mathbb{H}$. Then, equation of $\mathcal{D}-$tangent surface of timelike biharmonic $\mathcal{D}$- helix are*

$$\begin{aligned}
x_{\mathcal{O}} &= [\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s])+u\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]+\wp_1] \\
y_{\mathcal{O}} &= [\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s])+u\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]+\wp_2] \\
z_{\mathcal{O}} &= [\sinh\aleph s+\frac{(\wp+\mathcal{R}s)}{2\mathcal{R}^2}\cosh^2\aleph-\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\cosh[\wp]\cosh[\mathcal{R}s] \\
&\quad -\frac{\wp_1}{\mathcal{R}}\cosh^2\aleph\sinh[\wp]\sinh[\mathcal{R}s]-\frac{1}{4\mathcal{R}^2}\cosh^2\aleph\sinh 2[\mathcal{R}s+\wp] \\
&\quad +\wp_3+[\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s])+\wp_1] \\
&\quad [\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s])+\wp_2]+u\sinh\mathcal{P}] \\
&\quad -[\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\wp]\cosh[\mathcal{R}s]+\sinh[\wp]\sinh[\mathcal{R}s])+u\cosh\mathcal{P}\sinh[\mathcal{M}s+\mathcal{N}]+\wp_2] \\
&\quad [\frac{\cosh\aleph}{\mathcal{R}}(\cosh[\mathcal{R}s]\sinh[\wp]+\cosh[\wp]\sinh[\mathcal{R}s])+u\cosh\mathcal{P}\cosh[\mathcal{M}s+\mathcal{N}]+\wp_1],
\end{aligned}$$

where $\wp, \wp_1, \wp_2, \wp_3,$ *are constants of integration and*

$$\mathcal{R} = \operatorname{sech}\aleph\sqrt{\kappa_n^2 + \kappa_g^2} - 2\sinh\aleph.$$

Proof: From Theorem 4.1, we easily have above system, which completes the proof. □

In the light of Theorem 4.2, we give the following figures for the $\mathcal{D}-$tangent surface of timelike biharmonic $\mathcal{D}$- helix.

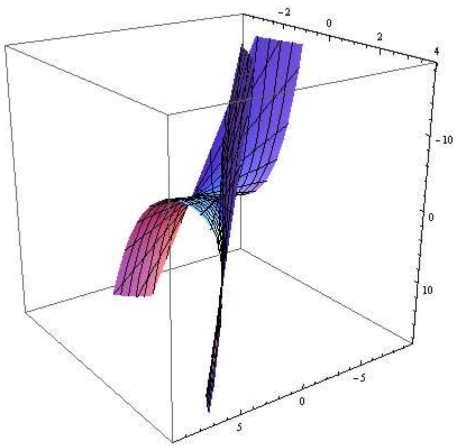

Fig. 2

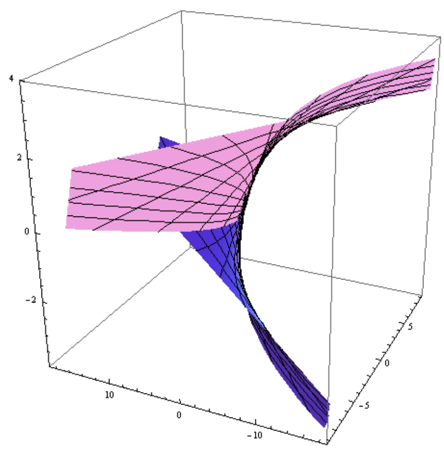

Fig. 3

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