



Global and Local Controllability*

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ABSTRACT: We discuss three concepts of controllability for open loop systems, namely, complete, local and global controllability, emphasizing that for nonlinear systems, the fact that the system is completely controllable does not mean that it is locally controllable.

Key Words: Open-Loop control system, complete controllability, local controllability, global controllability.

Contents

1 Introduction	27
2 Preliminaries Notions on Controllability	28
3 Local and Global Controllability	29
4 Equivalence between local and global controllability for autonomous linear systems	32
5 Conclusion	34

1. Introduction

In general, mathematical global properties imply local ones. That is the case of a minimum or a maximum global (or absolute) point of a real function which is in particular a local extremum point. Also, in some situations a local property can be used to determine a global property of a mathematical object. Thus, usually there is a connection between the two concepts. Dealing with controllability of dynamical systems, meaning a system of autonomous ordinary differential equations with control, a certain notion of a system being “globally” controllable does not always imply that it is locally controllable. We present with details the example suggested in [2] where for a nonlinear control system it is possible to drive arbitrary states into a state x_0 using large excursions, but if the intention is to control any state closed to the desired x_0 , using a path that does not leave a neighborhood of x_0 , it can not be done. In this sense, what we mean for the moment by “global” controllability is the concept known in literature as complete controllability and it does not imply local controllability for nonlinear systems, although for linear systems both notions are equivalent [2], [1]. Thus, to define the concept of global controllability for

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nonlinear systems, both completely and local controllability notions must be taken in consideration.

2. Preliminaries Notions on Controllability

Definition 2.1. *By an open-loop control system we mean an autonomous system of the form*

$$\frac{dx}{dt} = f(x, u) \quad (1)$$

where $f : D \times U \subset \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$. $x = x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is the output or state variable vector, $u = u(t) = (u_1(t), u_2(t), \dots, u_m(t))$ is the input or control variable vector. We assume that f is a continuous function and locally Lipschitz with respect to variable x , defined in an open and connected subset $D \subset \mathbb{R}^n$, called state space. We observe that if $f \in C^1$, then f is locally Lipschitz. The subset $U \subset \mathbb{R}^m$ is a nonempty set and it is called the set of the admissible controls or control space. We assume that the control functions are piecewise continuous functions, but in a more general context, the controls are considered as being Lebesgue integrable functions. Also, if the control space U is a one-element set, we consider the system as a classical dynamical one, for if it is possible to control the system, there is only one way to do it so.

The state $x = x(t)$ and the control $u = u(t)$ variables are defined on a finite time interval $I = [0, T]$, $T > 0$, or on $I = [0, +\infty)$, depending on the system we are interested in controlling. For the purpose of the definitions of this article, we consider these functions defined only on a finite interval.

Considering the above hypotheses, for every fixed control u , the autonomous system satisfies the Uniqueness and Existence Theorem for Ordinary Differential Equation for any given initial condition $x(0) = x_0$. Let us denote by $x^u = x^u(t)$ the unique solution of $\frac{dx}{dt} = f(x, u)$, such that $x^u(0) = x_0$. We call this solution the trajectory or path associated to the fixed control u .

Definition 2.2 (Admissible process). *An admissible process is a pair (x^u, u) , where u is a control variable that was fixed and $x^u = x^u(t)$ is the unique solution of the autonomous system $\frac{dx}{dt} = f(x, u)$ associated to the fixed control u , satisfying a given initial condition $x^u(0) = x_0$.*

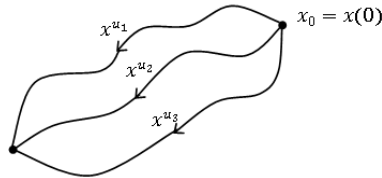


Figure 1: Admissible process (x^u, u) with initial state x_0 .

Definition 2.3 (Controllable state in a finite time). *Let $x_0, x_1 \in D$. The state x_0 is U -controllable to the state x_1 in time $T, T > 0$, if there exists an admissible process (x^u, u) where $u \in U$, and $x^u = x^u(t)$ is the trajectory associated to the control u , both defined on the interval $[0, T]$, such that $x^u(0) = x_0$ and $x^u(T) = x_1$. That is, the state x_0 can be driven into the state x_1 through the application of a suitable control.*

Definition 2.4 (Controllability in a neighborhood). *Let $x_0, x_1 \in D$ and W an open neighborhood of D such that $x_0, x_1 \in W$. We say that x_0 can be controlled to x_1 without leaving W (or inside W), if there exists a time $T, T > 0$, and an admissible process (x^u, u) , where $u \in U$, and $x^u = x^u(t)$ is the trajectory associated to the control u , both defined in the interval $[0, T]$, such that $x^u(0) = x_0$ and $x^u(T) = x_1$ and $x^u(t) \in W, \forall t \in [0, T]$.*

3. Local and Global Controllability

The idea of local controllability related to a state \bar{x} is that it is possible to drive any two states closed to the desired state \bar{x} changing one into the other through the application of a suitable control, in such a way that the trajectory remains close to the state \bar{x} , i.e., without deviating far from it.

Definition 3.1 (Local controllability related to a state). *Let $\bar{x} \in D$. The system (1) is locally controllable at the state \bar{x} (or related to the state \bar{x} , or around \bar{x}), if for each neighborhood W of \bar{x} , there exists a neighborhood V , with $\bar{x} \in V \subset W$, such that for any pair of states $x_0, x_1 \in W$, the state x_0 can be controlled to the state x_1 without leaving W in the sense of definition 1.4. This concept of local controllability is also called strong local controllability around the state \bar{x} [2].*

In the next definition, the property of a system being completely controllable is somehow a “global” notion, for it allows us to control the entire system, although, during the process, the trajectory taking one state into the other can leave a neighborhood or subset $W \subset D$ that was fixed for some purpose.

Definition 3.2 (Complete Controllability). *Let D be the state space and U the set of admissible controls. The system (1) is said to be completely U -controllable in D if for any two arbitrary states $x_0, x_1 \in D$, there exists an admissible process (x^u, u) , where $u \in U$, and $x^u = x^u(t)$ is the trajectory associated to the control u , both defined on an interval $[0, T]$, for some $T > 0$, such that $x^u(0) = x_0$ and $x^u(T) = x_1$.*

The following example [2] shows that a nonlinear system can be completely controllable without being locally controllable related to a state.

Example 3.1. *Let us consider the open-loop control system*

$$\begin{cases} \frac{dx}{dt} = -y + xu \\ \frac{dy}{dt} = x + yu \end{cases} \Leftrightarrow \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + u \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}, \quad (2)$$

onde $u \in \mathbb{R}$.

Introducing polar coordinates $\begin{cases} x(t) = \rho \cos \theta \\ y(t) = \rho \sin \theta \end{cases}$, where $\rho = \rho(t) > 0$ and $\theta = \theta(t)$, and differentiating using the chain rule, we obtain the equivalent system in $\mathbb{R}^2 - \{(0, 0)\}$

$$\begin{cases} \frac{d\rho}{dt} = \rho u \\ \frac{d\theta}{dt} = 1 \end{cases} \quad (3)$$

The trajectory of the system (3) associated to the control u is of the form $\xi^u(t) = (\theta^u(t), \rho^u(t)) = (t + \theta_0, e^{ut} \rho_0)$, where $\theta_0 = \theta(0)$, $\rho_0 = \rho(0)$.

(i) The system is completely controllable when defined in $\mathbb{R}^2 - \{(0, 0)\}$. Let us consider any two states in $\mathbb{R}^2 - \{(0, 0)\}$ with polar coordinates given by (θ_0, ρ_0) , (θ_1, ρ_1) , where $\rho_0, \rho_1 > 0$, the angles are chosen with positive orientation and (θ_0, ρ_0) is a polar representation of the initial state.

(i. 1) If $\rho_0 \neq \rho_1$, it is always possible to take $\theta_1 > \theta_0$.

In fact, if the two states are in the same radius but their distance from the origin are not the same ($\rho_0 \neq \rho_1$), we take $\theta_1 = \theta_0 + 2K\pi$, for some $K \in \mathbb{N}$, noticing that we can choose $\theta_1 = \theta_0 + 2\pi$. Similarly, if the states lay in different radii and $\theta_1 < \theta_0$, we take $\theta_1 = \theta_0 + 2K\pi$, for some $K \in \mathbb{N}$, in order to have $\theta_0 < \theta_1$. Taking

$$T = \theta_1 - \theta_0 \quad \text{and the control } u = \frac{1}{\theta_1 - \theta_0} \text{Ln} \left(\frac{\rho_1}{\rho_0} \right) \quad (4)$$

where $\rho_0, \rho_1 > 0$ and $\theta_1 > \theta_0$, the associated trajectory $\xi^u(t) = (\theta^u(t), \rho^u(t))$ satisfies $\xi^u(0) = (\theta^u(0), \rho^u(0)) = (\theta_0, \rho_0)$ and $\xi^u(T) = (\theta^u(T), \rho^u(T)) = (\theta_1, e^{u(\theta_1 - \theta_0)} \rho_0) = \left(\theta_1, \left[e^{\text{Ln}(\frac{\rho_1}{\rho_0})} \right] \rho_0 \right) = (\theta_1, \rho_1)$.

(i. 2) If $\rho_0 = \rho_1$, we choose the null control. For if $u = 0$, the system (2) turns out to be a linear system without control, whose matrix has pure imaginary eigenvalues. So, one state is driven to the other one by using the own closed orbit (a circle) of the linear system that passes by these two states.

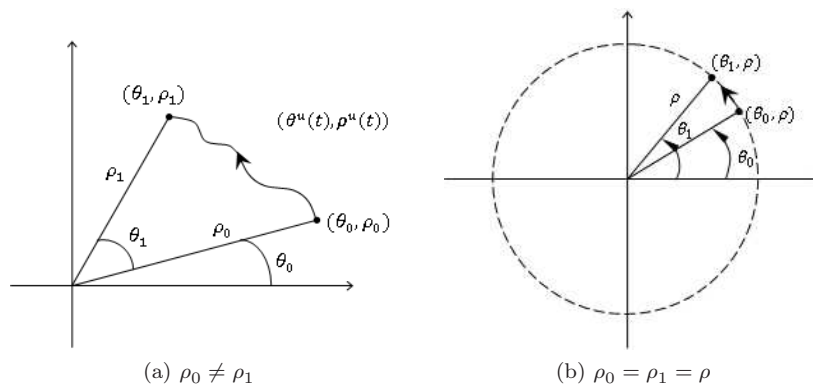


Figure 2

(ii) The system is not locally controllable related to the state with cartesian coordinates $(x_0, y_0) = (1, 0)$.

Consider a neighborhood of $(x_0, y_0) = (1, 0)$ of radius $\varepsilon = 1$. For any δ , such that $0 < \delta < 1$, take $(x_1, y_1) = (1 + \frac{\delta}{2}, 0)$. We notice that

$$\| (x_1, y_1) - (x_0, y_0) \| = \frac{\delta}{2} < \delta < 1.$$

In order to drive the state (x_0, y_0) to the state (x_1, y_1) , it is necessary to consider a finite time at least equal to 2π and a control u given in (4). Using polar coordinates, the trajectory associated to the control u is $(\theta^u(t), \rho^u(t)) = (t, [(1 + \frac{\delta}{2})^{\frac{t}{2\pi}}])$. It satisfies $(\theta^u(0), \rho^u(0)) = (0, 1)$, and $(\theta^u(2\pi), \rho^u(2\pi)) = (2\pi, 1 + \frac{\delta}{2})$, which are polar representations of the states whose cartesian coordinates are $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1 + \frac{\delta}{2}, 0)$, respectively. Furthermore, by the expression of $\rho^u(t)$, the trajectory leaves the neighborhood of radius $\varepsilon = 1$ and center in the point with cartesian coordinates $(x_0, y_0) = (1, 0)$.

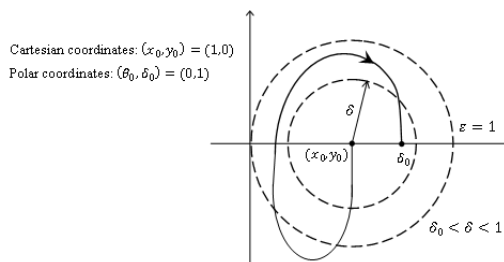


Figure 3

Definition 3.3 (Global controllability related to a state). Let D be the space state and $\bar{x} \in D$. The system (1) is globally controllable to the state \bar{x} , if

- (i) The system is locally controllable related to the state \bar{x} ;
- (ii) Every $x \in D$ can be U -controllable to the state \bar{x} for some time T , $T > 0$.

Definition 3.4 (Global controllability). *Let D be the space state. The system (1) is globally controllable in D , if*

- (i) For each $x \in D$, the system is locally controllable related to the state \bar{x} ;
- (ii) The system is completely controllable in D .

In the definitions above, if all pairs of given states x_0 and x_1 can be controlled in a very same finite interval of time $[0, T]$, $T > 0$, then we say that the system is locally controllable at time T , completely controllable at time T and so on.

4. Equivalence between local and global controllability for autonomous linear systems

An autonomous linear control system is a system of the form

$$\frac{dx}{dt} = Ax + Bu,$$

where A and B are matrices, $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $x = x(t) \in \mathbb{R}^n$ and $u = u(t) \in \mathbb{R}^m$. We denote it by (A, B) . We suppose $\det A \neq 0$ in order to have the origin as the only equilibrium point of the linear system with no controls $\frac{dx}{dt} = Ax$. Also we consider the control set U with no constraints, which means that we do not impose restrictions on the images of the controls. So we can consider any piecewise continuous function, $u : I \rightarrow \mathbb{R}^m$, as a control input, where $I = [0, T]$ or $I = [0, +\infty)$, depending on whether we are interested in controlling the system in a finite time T or not.

Theorem 4.1. *Let (A, B) be an autonomous linear control system. The following properties are equivalent:*

- (i) The system is locally controllable at the state $\bar{x} = 0$.
- (ii) The system is controllable to $\bar{x} = 0$ in \mathbb{R}^n .

Proof:

- (i) Suppose that the linear system (A, B) is locally controllable at the state $\bar{x} = 0$. For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that if x is a state satisfying $\|x\| < \delta$, then there is an admissible process (x^u, u) , where $x^u = x^u(t)$ and $u = u(t)$ are defined on an interval $[0, T]$, for some $T > 0$, such that $x^u(0) = x$, $x^u(T) = 0$ and $\|x^u(t)\| < \varepsilon$, $\forall t \in [0, T]$.
Let $y \in \mathbb{R}^n$, $y \neq 0$. We claim that the state y is controllable to the origin in time T . In fact, let us take $x = \frac{\delta}{2\|y\|}y$. We notice that $\|x\| = \frac{\delta}{2\|y\|}\|y\| = \frac{\delta}{2} < \delta$.
Let (x^u, u) be a process given above and consider the control $w(t) = \frac{2\|y\|}{\delta}u(t)$.

By linear property of the transition map of a linear control system ([2], p. 26), the solution associated to the control $w = w(t)$ is $x^w(t) = \frac{2\|y\|}{\delta}x^u(t)$.

$$x(0) = \frac{2\|y\|}{\delta}x^u(0) = \frac{2\|y\|}{\delta}x = \frac{2\|y\|}{\delta} \cdot \frac{\delta}{2\|y\|}y = y$$

and

$$x^w(T) = \frac{2\|y\|}{\delta}x^u(T) = \frac{2\|y\|}{\delta}0 = 0$$

Thus the control $w = w(t)$ take the state y to the origin. By definition 2.3 the system is globally controllable to the origin.

(ii) Assume now that (A, B) is controllable to the origin.

It means that given a state $x \in \mathbb{R}^n$, there exists a process (x^u, u) , defined on some interval $[0, T]$, $T > 0$, such that $x^u(0) = x$ and $x^u(T) = 0$. We observe that for the equilibrium state $x = 0$, we take $u(t) = 0, t \geq 0$ and the stationary solution $x^0(t) = 0$ defined for all $t \geq 0$. Indeed, $(\bar{x}, \bar{u}) = (0, 0)$ is an equilibrium pair of the autonomous control linear system. Considering the fact that the system is a linear one and any solution $x^u = x^u(t)$ achieves the origin after a certain time T , using a suitable control $u = u(t)$ defined for $t \in [0, T]$, we can consider any admissible process defined for $t \geq 0$. In fact, if we take the control $\tilde{u} = \tilde{u}(t)$ defined by $\tilde{u}(t) = u(t)$, if $0 \leq t \leq T$ and $\tilde{u}(t) = 0$, if $t > T$, then the solution associated to control \tilde{u} is $x^{\tilde{u}}(t) = x^u(t)$, if $0 \leq t \leq T$ and $x^{\tilde{u}}(t) = 0$, if $t > T$. In that way all solutions converge to the origin and the origin is an attractor. So all eigenvalues of matrix A have negative real part and there are constants $K > 0$ and $\lambda > 0$ such that $\|x^{\tilde{u}}(t)\| \leq Ke^{-\lambda t}\|x^{\tilde{u}}(0)\|, \forall t \geq 0$. As for $\lambda > 0$, we have $e^{-\lambda t} < 1, \forall t \geq 0$ and $x^{\tilde{u}}(0) = x^u(0)$, then $\|x^{\tilde{u}}(t)\| \leq K\|x^u(0)\|, \forall t \geq 0$.

Given $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{K}$. If $\|x\| = \|x^u(0)\| < \delta$, then $\|x^{\tilde{u}}(t)\| < \varepsilon, \forall t \geq 0$. We conclude the system is locally controllable at the origin.

□

Remark 4.1. For an autonomous linear control system (A, B) , if the state $x_1 \in \mathbb{R}^n$ is driven to the state $x_2 \in \mathbb{R}^n$ using the control $u = u(t)$ during a period of time T , then $x_2 = e^{tA} \left(x_1 + \int_0^T e^{-sA} Bu(s) ds \right)$. By this formula and considering controls with no constraints, it is not difficult to prove that the system (A, B) is completely controllable in time T if and only if every state x is controlled to the origin in time T ([3], p. 41). Also the fact that an autonomous linear system is completely controllable for some time T is equivalent to it being completely controllable for every time T . Indeed, if the state x_0 is controlled to the state x_1 in time T_1 , using a control $u(t)$, we can take one state to the other in a time T_2 , for $T_2 > T_1$, using a control $\tilde{u}(t) = u(t)$, if $0 \leq t \leq T_1$ and $\tilde{u}(t) = 0$, if $T_1 < t \leq T_2$.

Remark 4.2. *If a state x is controlled to the origin when $t \rightarrow +\infty$, we have the analogous definition of a state being asymptotically controllable to the origin. Just like in the theorem above, for autonomous linear control systems, the concepts of global and local asymptotic controllability to the origin coincide ([2], p. 213).*

5. Conclusion

For autonomous linear control systems (A, B) , local and global controllability are equivalent. But the same is not true for nonlinear systems. Thus, for nonlinear systems the definition of global controllability must include the property of the system being locally controllable as a first condition plus an extra one, which is the complete controllability condition. When the admissible processes are defined for $t \geq 0$, similar results are valid for local and global asymptotic controllability notions.

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