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An extension of Banach fixed point theorem in fuzzy metric space

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abstract: In the present paper, we establish the existence of fixed point of mapping satisfying a general contractive condition depended on another function in a complete fuzzy metric space. In particular, this result is an analogue of T-Banach contraction principle by Beiranvand et al. [2] in fuzzy metric space.

Key Words: Fuzzy metric space, contraction mapping, fixed point

Contents

1. Introduction

Banach contraction principle by S.Banach [1] in 1922 has the pivotal role in the research of fixed point theory. In the general setting of complete metric space, this theorem runs as follows (see Theorem 2.1, [4] or, Theorem 1.2.2, [12]).

Theorem 1.1. (Banach contraction principle) *Let (X, d) be a complete metric space, $\alpha \in (0, 1)$ and $S : X \to X$ be a mapping such that for each $x, y \in X$,*

$$d(Sx, Sy) \leq \alpha d(x, y) \tag{1.1}$$

then S has a unique fixed point in X, and for each $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Since the Banach contraction principle, several type of contraction mappings on metric space have appeared. One can see for details the survey articles by Rhoades ([9], [10], [11]). Recently, Beiranvand et al. [2] addressed a new classes of T-Contraction functions, which are depending on another function and extended the Banach contraction principle successfully.

Definition 1.2. *Let (X, d) be a metric space and $T, S : X \to X$ be two functions. A mapping S is said to be a T-contraction if there exists $\alpha \in (0, 1)$ such that for*

$$d(TSx, TSy) \leq \alpha d(Tx, Ty) \textit{for all } x, y \in X. \tag{1.2}$$

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Note 1. By taking $Tx = x$ (T is identity function) T-contraction and contraction are equivalent.
The following example [2] shows that T-contraction function may not be a contraction.

Example 1.3. *Let $X = [1, +\infty)$ with metric induced by $\Re : d(x, y) = |x - y|$. We consider two mappings $T, S : X \to X$ by $Tx = \frac{1}{x} + 1$ and $Sx = 2x$. Obviously S is not contraction but S is T-contraction, because:*

$$|TSx - TSy| = \left|\frac{1}{2x} + 1 - \frac{1}{2y} - 1\right| = \left|\frac{1}{2x} - \frac{1}{2y}\right| \leq \frac{1}{2}\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{2}\left|\frac{1}{x} + 1 - \frac{1}{y} - 1\right|$$
$$= \frac{1}{2}|Tx - Ty|$$

2. Fuzzy metric space

On the otherhand, the evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [14] in 1965, as a new way to represent the vagueness in every day life. Since its initiation, several mathematicians have worked with fuzzy sets in different branches of mathematics. In 1975, Kramosil and Michalek [8] introduced the concept of fuzzy metric space which opened an avenue for further development of analysis in such spaces. In 1994, George and Veeramani [3] revised the notion of fuzzy metric space with the help of continuous t-norm. Actually, the study of fuzzy metric evolved in two different perspectives. One group of mathematicians following Hu [6], consider a fuzzy metric to be a non-negative real-valued function on the collection of all fuzzy points on a set X, satisfying a list of axioms similar to those of a general metric; while another group imposes the fuzziness on the metric itself rather than the points of the space. In the present paper, we study the existence of fixed point for mapping satisfying a general contractive condition dependent on another function in fuzzy metric space. Our approach of proving results in fuzzy metric space along the first line of development, initiated by George and Veeramani [3].

Definition 2.1. [13] *A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norms if $*$ is satisfying conditions:*
(i) $*$ is an commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0, 1]$;
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0, 1]$.

Examples of t-norms are $a * b = \min\{a, b\}$ (minimum t-norm), $a * b = ab$ (product t-norm), and $a * b = \max\{a + b - 1, 0\}$ (Lukasiewicz t-norm).

Definition 2.2. [3] *A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space (or, briefly FMS) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, such that $t, s > 0$,*
(FM 1) $M(x, y, t) > 0$;

(FM 2) $M(x, y, t) = 1$ *if and only if* $x = y$*;*
(FM 3) $M(x, y, t) = M(y, x, t)$*;*
(FM 4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$*;*
(FM 5) $M(x, y, *) : (0, \infty) \to (0, 1]$ *is continuous.*

Then M is called a fuzzy metric on X and $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. In the following example, we know that every metric induces a fuzzy metric.

Example 2.3. *Let X be a non-empty set and d is a metric on X. Denote $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t > 0$, define*

$$M(x, y, t) = \frac{t}{t + d(x, y)}. \tag{2.1}$$

Then $(X, M, *)$ is a fuzzy metric space (see [3]). We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

In the present section, we invite some more useful definitions and concepts in this space.

Definition 2.4. (see [8]) *Let $(X, M, *)$ be a fuzzy metric space:*
(1) *A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.*
(2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.*
(3) *A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.*

Definition 2.5. (see [8]) *The fuzzy metric space $(X, M, *)$ is called complete if every Cauchy sequence in X is convergent.*

Definition 2.6. (see [8]) *The fuzzy metric space $(X, M, *)$ is called compact if every sequence contains a convergent subsequence.*

Lemma 2.7. (Grabiec [5]) *$M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.8. (Grabiec [5]) *Let $\{y_n\}$ be a sequence in an FM-space X. If there exists a positive number $k < 1$ such that $M(y_{n+2}, y_{n+1}, kt) \geq M(Y_{n+1}, y_n, t)$, $t > 0$, $n \in N$ then $\{y_n\}$ is a Cauchy sequence in X.*

Lemma 2.9. (see [7]) *If for two points x, y in X and a positive number $k < 1$,*

$$M(x, y, kt) \geq M(x, y, t), \text{ then } x = y.$$

Remark 2.10. (see [5]) *Since $*$ is continuous, it follows from (FM-4) that the limit of the sequence in fuzzy metric space is uniquely determined.*

3. T-Banach contraction principle in fuzzy metric space

In this section, we first formulate the idea of a fuzzy T-contraction map and then prove T-Banach contraction theorem in fuzzy metric space. Finally, the theorem has been validated by proper examples. For this purpose, we introduce some necessary definitions and a proposition in this space.

Definition 3.1. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be sequentially convergent, if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then $\{y_n\}$ is also convergent.*

Definition 3.2. *Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be subsequentially convergent, if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then $\{y_n\}$ has a convergent subsequence.*

Proposition 3.3. *Let $(X, M, *)$ be a compact fuzzy metric space, then every function $T : X \to X$ is subsequentially convergent and every continuous function $T : X \to X$ is sequentially convergent.*

Definition 3.4. *(Fuzzy T-contraction) Let $(X, M, *)$ be a fuzzy metric space and $T, S : X \to X$ be two mappings. A mapping S is said to be a Fuzzy T-contraction if there exists $k \in (0, 1)$ such that*

$$M(TSx, TSy, kt) \geq M(Tx, Ty, t) \tag{3.1}$$

for all $x, y \in X$ and $t > 0$.

Now following is the Fuzzy T-Banach fixed point theorem which is our main result of this paper .

Theorem 3.5. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be one to one, continuous and subsequentially convergent mapping. Also let for all $x, y \in X$, $M(x, y, t) \to 1$ as $t \to \infty$. Then for every Fuzzy T-contraction mapping, a continuous function $S : X \to X$, S has a unique fixed point. Also if T is a sequentially convergent, then for each $x_0 \in X$, the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.*

Proof: Define an iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$, (or, $x_n = S^n x_0$), $n = 1, 2, 3,...$, starting from $x_0 \in X$. Then for any integer $n \in N$ and $t > 0$, by simple induction we get,

$$M(Tx_n, Tx_{n+1}, kt) = M(TSx_{n-1}, TSx_n, kt) \geq M(Tx_0, Tx_1, \frac{t}{k^{n-1}}) \tag{3.2}$$

Thus for any positive integer p, we have by (3.2)

$$\begin{aligned} M(Tx_n, Tx_{n+p}, t) &\geq M(Tx_n, Tx_{n+1}, \frac{t}{p}) * ... * M(Tx_{n+p-1}, Tx_{n+p}, \frac{t}{p}) \\ &\geq M(Tx_0, Tx_1, \frac{t}{pk^n}) * ... * M(Tx_0, Tx_1, \frac{t}{pk^n}) \end{aligned} \tag{3.3}$$

Since $M(x, y, t) \to 1$ as $t \to \infty$, (3.3) gives

$$\lim_{n} M(Tx_n, Tx_{n+p}, t) \geq 1 * \dots * 1 = 1$$

Then $\{Tx_n\}$ or, $\{TS^n x_0\}$ is a Cauchy sequence and hence is convergent. So there exists $a \in X$ such that

$$\lim_{n\to\infty} TS^n x_0 = a \tag{3.4}$$

Since T is a subsequentially convergent, $\{S^n x_0\}$ has a convergent subsequence. So there exists $b \in X$ and $\{n_k\}_{k=1}^{\infty}$ such that $\lim\limits_{k\to\infty} S^{n_k} x_0 = b$. Hence $\lim\limits_{k\to\infty} TS^{n_k} x_0 = Tb$. Then by (3.4) we conclude that

$$Tb = a. \tag{3.5}$$

Since S is continuous and $\lim\limits_{k\to\infty} S^{n_k} x_0 = b$, then $\lim\limits_{k\to\infty} S^{n_k+1} x_0 = Sb$ and so $\lim\limits_{k\to\infty} TS^{n_k+1} x_0 = TSb$. Again by (3.4), $\lim\limits_{k\to\infty} TS^{n_k+1} x_0 = a$, and therefore $TSb = a$. Since T is one to one and by (3.5) $Sb = b$. So, S has a fixed point.
Uniqueness: Let $c \in X$ be another fixed point of S. Then

$$\begin{aligned} 1 \geq M(Tb, Tc, kt) = M(TSb, TSc, kt) &\geq M(Tb, Tc, \frac{t}{k}) = M(TSb, TSc, \frac{t}{k}) \\ &\geq M(Tb, Tc, \frac{t}{k^2}) \geq \dots \geq M(Tb, Tc, \frac{t}{k^n}) \to 1 \text{ as } n \to \infty \end{aligned}$$

Then by (FM-2), $Tb = Tc$, and since T is one to one, this implies that $b = c$. So S has a unique fixed point. □

Remark 3.6. (I) *Taking $Tx = x$ (T is identity function) in the above theorem, we have the fuzzy version of Theorem 1.1.*
(II) *Theorem 3.5 is an important extension and generalization of Theorem 2.6 of Beiranvand et al.* [2] *in complete fuzzy metric space.*

We now give an example to illustrate the Theorem 3.5.

Example 3.7. *Let $X = [1, 500]$ with product t-norm. Let M be the standard fuzzy metric induced by d, where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Define $S : X \to X$ by $Sx = x^2$ and $T : X \to X$ by $Tx = 1 + \frac{1}{x}$, then T, S satisfy (3.1) with $k = \frac{1}{200} \in (0, 1)$. Also S has a unique fixed point $x = 1$.*

Remark 3.8. *It is also to be noted in the above example that S is not a contraction mapping but a T-contraction mapping.*

The following example clearly shows that the condition of subsequential convergence of T in Theorem 3.5 can not be dropped.

Example 3.9. *Let $X = [0, \infty)$ and M be the standard fuzzy metric induced by d, where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Define $S : X \to X$ by $Sx = x + 1$ and $T : X \to X$ by $Tx = exp(-x)$. Then one can easily check that T is one-to-one and S is T-contraction with $k = \frac{2}{e} \in (0, 1)$. But T is not subsequentially convergent (since $Tn \to 0$ as $n \to \infty$, but $\{n\}_1^\infty$ has no convergent subsequence) and also S has no fixed point.*

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References

1. S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux Ã©quations intÃ©grales, *Fund. Math.*,**3**, (1922), 133-181 (French).
2. A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, Two fixed-point theorem for special mappings, arxiv: 0903.1504v1.
3. A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets Systems*, **64**, (1994), 395-399.
4. K. Goebel and W. A. Kirk, Topiqs in Metric Fixed Point Theory, Combridge *University Press,* New York, (1990).
5. M. Grabiec, Fixed points in fuzzy metric space, *Fuzzy Sets Systems*, **27**(1988), 385-389.
6. C. Hu, Fuzzy topological space, *J. Math. Anal. Appl.*, **110**, (1985), 141-178.
7. M. S. Khan, H. K. Pathak and R. George, Compatible Mappings of Type (A-1) and Type (A-2) and Common Fixed Points in Fuzzy Metric Spaces, *International Mathematical Forum*, **2**(11), (2007), 515-524
8. I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika,* **11**, (1975), 336-344.
9. B. E. Rhoades, A Comparison of Various Definitions of Contractive Mappings, *Trans. Amer. Math. Soc.* **226** (1977), 257-290.
10. B. E. Rhoades, Contractive definitions revisited, Topological Methods in Nonlinear Functional Analysis (Toronto, Ont.,1982), *Contemp. Math.,* **21**, *American Mathematical Society, Rhode Island,* (1983), 189-203.
11. B. E. Rhoades, Contractive Definitions, *Nonlinear Analysis, World Science Publishing,* Singapore, 1987, 513-526.
12. O. R. Smart, Fixed Point Theorems, *Cambridge University Press,* London, (1974).
13. B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.*, **10**, (1960), 313-334.
14. L. A. Zadeh, Fuzzy sets, *Inform. Acad Control*, **8**, (1965), 338-353.

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