A new characterization of the projective linear groups by the Sylow numbers

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ABSTRACT: Let $G$ be a finite group, let $\pi(G)$ be the set of primes $p$ such that $G$ contains an element of order $p$ and let $n_p(G)$ be the number of Sylow $p$-subgroup of $G$, that is, $n_p = n_p(G) = |\text{Syl}_p(G)|$. Set $\text{NS}(G) := \{n_p(G) \mid p \in \pi(G)\}$. In this paper we show the projective linear groups $L_2(q)$ are recognizable by $\text{NS}(G)$ and order. Also we prove if $\text{NS}(G) = \text{NS}(L_2(8))$, then finite centerless group $G$ is isomorphic to $L_2(8)$ or $\text{Aut}(L_2(8))$.

Key Words: Finite group, Sylow subgroup, simple group.

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1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. A finite group $G$ is called a simple $K_n$-group, if $G$ is a simple group with $|\pi(G)| = n$. Denote by $n_p$ the largest positive odd divisor of the positive integer $n$. Also denote by $(a, b)$ the greatest common divisor of positive integers $a$ and $b$. If $G$ is a finite group, then we denote by $n_q$ the number of Sylow $q$-subgroup of $G$, that is, $n_q = n_q(G) = |\text{Syl}_q(G)|$. All other notations are standard and we refer to [11], for example.

In 1992, Bi [6] showed that $L_2(p^k)$ can be characterized only by the order of normalizer of its Sylow subgroups. In other words, if $G$ is a group and $|N_G(P)| = |\text{Syl}_r(L_2(p^k))|$ for every prime $r$, then $G \cong L_2(p^k)$. Similar characterizations have been found for the following groups: $L_n(q)$ [5], $S_4(q)$ [9], the alternating simple groups [8], $U_n(q)$ [10], the sporadic simple groups [2] and $^2D_n(p^k)$ [1].

Set $\text{NS}(G) := \{n_p(G) \mid p \in \pi(G)\}$. Let $S$ be one of the above simple groups. It is clear that if $n_p(G) = n_p(S)$ for every prime $p$ and $|G| = |S|$, then $|N_G(P)| = |N_S(Q)|$ where $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_p(S)$. By the above references, $G \cong S$.

Now let $\text{NS}(G) = \text{NS}(S)$ and $|G| = |S|$. In this case we don’t know $n_q(G) = n_q(S)$ for any prime $p$. Thus we can not conclude that $G \cong S$. In this paper first we will show that if $\text{NS}(G) = \text{NS}(L_2(q))$ and $|G| = |L_2(q)|$, then $G \cong L_2(q)$.

2000 Mathematics Subject Classification: 20D06, 20D20

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We denote by $k(\text{NS}(G))$ the number of isomorphism classes of the finite centerless groups $H$ satisfying $\text{NS}(G) \cong \text{NS}(H)$. A finite centerless group $G$ is called $n$-recognizable by $\text{NS}(G)$ if $k(\text{NS}(G)) = n$.

In [3] it is proved that $A_5$ is recognizable and $A_6$ is 5-recognizable by only $\text{NS}(G)$. We will show that $k(\text{NS}(L_2(8))) = 2$.

2. Preliminary Results

Lemma 2.1. [5] Let $G$ be a finite group. If $|N_G(R_1)| = |N_{L_n(q)}(R_2)|$ for every prime $r$, where $R_1 \in \text{Syl}_r(G)$ and $R_2 \in \text{Syl}_r(\text{PSL}(n, q))$, then $G \cong L_n(q)$.

Lemma 2.2. Let $G$ be solvable. Then when $n_p(G)$ is factored as a product of prime powers, each factor is congruent to $1 \mod p$.

Proof: Let $P$ be a Sylow $p$-subgroup of $G$. By Sylow’s theorem, $n_p(G) = |G : N_G(P)|$. Consider a chain of subgroups $N_G(P) = H_0 < ... < H_r = G$, where each subgroup is maximal in the next. By Sylow’s theorem we have $|H_i : H_0| \equiv 1 \mod p$ for all $i$. In fact, $P$ is a Sylow $p$-subgroup of $H_i$ and $H_0 = N_G(P) = N_{H_i}(P)$.

Thus for all $i$, $|H_{i+1} : H_0| = |H_{i+1} : H_i| \times |H_i : H_0| \equiv |H_{i+1} : H_i| \equiv 1 \mod p$. Then $|H_{i+1} : H_i| \equiv 1 \mod p$, and since $G$ is solvable, each of these indices is a prime power. Since $n_p(G) = |G : N_G(P)| = \prod_{i=0}^{r-1} |H_{i+1} : H_i|$, each factor of $n_p(G)$ therefore is congruent to $1 \mod p$. \hfill \Box

Lemma 2.3. [13] If $G$ is a simple $K_3$–group, then $G$ is isomorphic to one of the following groups: $A_5$, $A_6$, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.

Lemma 2.4. [12] Let $G$ be a finite group and $M$ a normal subgroup. Then $n_p(M)n_p(G/M)$ divides $n_p(G)$.

Lemma 2.5. [14] Let $G$ be a simple group of order $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, $abcd \neq 0$. Then $G$ is isomorphic to one of the following groups: $A_n$ for $n = 7, 8, 9, 10$; $J_4$; $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7(2)$, $O_8^+(2)$, $U_3(5)$ and $U_4(3)$.

By Sylow’s theorem implies that if $p$ is prime, then $n_p = 1 + pk$. If $p = 2$, then $n_2$ is odd. If $p \in \pi(G)$, then

$$
\begin{align*}
\begin{cases}
p \mid (n_p - 1) \\
pn_p & = 1
\end{cases}
\end{align*}
\tag{*}
$$

In the proof of the theorem 3.2 and 3.3, we often apply $(*)$ and the above comments.
3. Main Results

Lemma 3.1. Let \( q = p^n \) and \( p \neq 2 \). Then \( NS(L_2(q)) = \{q(q^2 - 1)/2, q + 1, q(q - 1)/2, q(q + 1)/2 \} \) or \( \{q(q^2 - 1)/24, q + 1, q(q - 1)/2, q(q + 1)/2 \} \). In particular if \( q = 2^n \), then \( NS(L_2(q)) = \{q + 1, q(q - 1)/2, q(q + 1)/2 \} \).

Proof: Let \( q = p^n \) and \( p \neq 2 \). To find the number of Sylow \( p \)-subgroups in \( L_2(q) \) first, look at \( SL_2(q) \). The normalizer of a Sylow \( p \)-subgroup is the set of upper triangular matrices with determinant 1, so the order of the normalizer is \( q(q - 1) \). The order of the whole group \( SL_2(q) \) is \( q(q^2 - 1) \). The number of Sylow \( p \)-subgroups, therefore, is \( (q + 1) \). This will be the same as the number of Sylow \( p \)-subgroups of \( L_2(q) \) because the canonical homomorphism from \( SL_2(q) \) to \( L_2(q) \) yields a bijection on Sylow \( p \)-subgroups.

If \( r \neq p \) is an odd prime divisor of \( |G| \), then \( r \) divides exactly one of \( \{q + 1, q - 1\} \) and a Sylow \( r \)-subgroup of \( G \) is cyclic with normalizer dihedral of order \( q - 1 \) or \( q + 1 \). In particular, the number of Sylow \( r \)-subgroups is \( q(q + 1)/2 \) or \( q(q + 1)/2 \). When \( q \) is a power of 2, we have \( L_2(q) = SL_2(q) \) and a Sylow 2-normalizer is a Borel subgroup of order \( q(q - 1) \). Hence there are \( q + 1 \) Sylow 2-subgroups as \( SL_2(q) \) has order \( (q - 1)(q + 1) \). When \( q \) is odd, the order of \( L_2(q) \) is \( q(q - 1)(q + 1)/2 \).

A Sylow 2-subgroup of \( SL_2(q) \) is quaternion or generalized quaternion and a Sylow 2-subgroup of \( L_2(q) \) is either a Klein 4-group or a dihedral 2-group with at least 8 elements. In all these cases, a Sylow 2-subgroup of \( SL_2(q) \) contains its centralizer, and the same is true in \( L_2(q) \). The outer automorphism group of a dihedral 2-group with at least 8 elements is a 2-group. Hence a Sylow 2-subgroup of \( L_2(q) \) is self-normalizing when \( q \equiv \pm 1 \) (mod 8), and in that case the number of Sylow 2-subgroups of \( L_2(q) \) is \( q(q^2 - 1)/2 \). When \( q \equiv \pm 3 \) (mod 8), then a Sylow 2-normalizer of \( L_2(q) \) must have order 12, because a Sylow 2-subgroup is a self-centralizing Klein 4-group, but there must be an element of order 3 in its normalizer by Burnside’s transfer theorem. In this case, the number of Sylow 2-subgroups of \( L_2(q) \) is \( q(q^2 - 1)/24 \). Therefore \( NS(L_2(q)) = \{q(q^2 - 1)/2, q + 1, q(q - 1)/2, q(q + 1)/2 \} \) or \( \{q(q^2 - 1)/24, q + 1, q(q - 1)/2, q(q + 1)/2 \} \).

Arguing as above if \( q = 2^n \), then \( NS(L_2(q)) = \{q + 1, q(q - 1)/2, q(q + 1)/2 \} \). □

Theorem 3.2. Let \( G \) be a finite group such that \( NS(G) \neq NS(L_2(q)) \) and \( |G| = |L_2(q)| \). Then \( G \cong L_2(q) \).

Proof: Let \( q = p^n \) and \( q \equiv \pm 1 \) (mod 8). By Sylow’s theorem since \( q + 1 \) is the only Sylow number not divisible by \( p \), \( n_p = q + 1 \). If \( r \in \pi(q + 1) \), then \( n_r = q + 1 \) or \( q(q - 1)/2 \). Suppose that \( n_r = q + 1 \). By Sylow’s theorem \( n_r = rk + 1 = q + 1 \) where \( k \) is a positive number. Then \( rk = q \) and \( r = p \), a contradiction. Therefore \( n_r = q(q - 1)/2 \). Similarly if \( r \in \pi(q - 1) \), then \( n_r = q(q + 1)/2 \). Now it is clear that \( n_r = q(q^2 - 1)/2 \). Thus we have proved that \( n_r(G) = n_r(L_2(q)) \) for every \( r \).

Arguing as above if \( q \equiv \pm 3 \) (mod 8), then \( n_r(G) = n_r(L_2(q)) \) for every \( r \). Since \( |G| = |L_2(q)| \), \( |NG(R)| = |NL_2(q)(R)| \) for every prime \( r \) where \( R_1 \in Syl_r(G) \) and \( R_2 \in Syl_r(L_2(q)) \). Therefore by Lemma 2.1, \( G \cong L_2(q) \).

Arguing as above if \( q = 2^n \) and \( NS(G) = NS(L_2(q)) \), then \( G \cong L_2(q) \). □
**Theorem 3.3.** Let $G$ be finite centerless group and $NS(G)=NS(L_2(8))$. Then $G \cong L_2(8)$ or $G \cong Aut(L_2(8))$.

**Proof:** We have $NS(G)=NS(L_2(8))=\{9, 28, 36\}$. First we prove that $\pi(G) = \{2, 3, 7\}$. By Sylow’s theorem $n_p \mid |G|$ for every $p$, hence by $NS(G)$, we can conclude that $\{2, 3, 7\} \subseteq \pi(G)$. On the other hand, by $(\ast)$ if $p \in \pi(G)$, then $p \mid (n_p-1)$ and $(p, n_p) = 1$, which implies that $p \in \{2, 3, 5, 7\}$.

Let $\pi(G) = \{2, 3, 5, 7\}$. Then $n_2(G) = 9$, $n_3(G) = 28$ and $n_5(G) = n_7(G) = 36$. We show that $G$ is a nonsolvable group. If $G$ is a solvable group since $n_7(G) = 36$ by Lemma 2.2, $9 \equiv 1 \pmod{7}$, a contradiction. Hence $G$ is a nonsolvable group.

Since $G$ is a finite group, it has a chief series. Let $1 = N_0 \leq N_1 \leq \ldots \leq N_{r-1} \leq N_r = G$ be a chief series of $G$. Since $G$ is a nonsolvable group there exists a maximal number of non-negative integer $i$ such that $N_i/N_{i-1}$ is a simple group or the direct product of isomorphic simple groups and $N_{r-1}$ is a maximal solvable normal subgroup of $G$. Now set $N_i := H$ and $N_{i-1} := N$. Hence $G$ has the following normal series

$$1 \leq N \triangleleft H \leq G$$

such that $H/N$ is a non-abelian simple group or $H/N$ is a direct product of isomorphic non-abelian simple groups. Since $G$ is a $K_4$–group, $H/N$ is a simple $K_n$–group or $H/N$ is a direct product of simple $K_n$–groups for $n = 3$ or $4$. By Lemma 2.4, $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$. Thus $H/N$ is a simple $K_3$–group or simple $K_4$–group.

If $H/N$ is a simple $K_3$–group, then by Lemma 2.3 and 2.4, $H/N \cong L_2(8)$. Now set $\overline{H} := H/N \cong L_2(8)$ and $\overline{G} := G/N$. On the other hand, we have

$$L_2(8) \cong \overline{H} \cong \overline{H}/C_{\overline{G}^{(1)}(\overline{H})}/C_{\overline{G}^{(1)}(\overline{H})} \leq \overline{G}/C_{\overline{G}^{(1)}(\overline{H})} = \frac{\mathbb{C}_{\overline{G}^{(1)}(\overline{H})}}{C_{\overline{G}^{(1)}(\overline{H})}} \leq Aut(\overline{H}).$$

Let $K = \{x \in G \mid xN \in C_{\overline{G}^{(1)}(\overline{H})}\}$, then $G/K \cong \overline{G}/C_{\overline{G}^{(1)}(\overline{H})}$. Hence $L_2(8) \leq G/K \leq Aut(L_2(8))$. Hence $G/K \cong L_2(8)$ or $G/K \cong Aut(L_2(8))$.

Let $G/K$ isomorphic to $L_2(8)$ by Lemma 2.4, $n_2(K) = 1$, $n_3(K) = 1$, $n_7(K) = 1$ and $n_5(K) = 36$. We show that $K = N$. Suppose that $K \neq N$. Since $N < K$ and $N$ is a maximal solvable normal subgroup, $K$ is a nonsolvable normal subgroup of $G$. Therefore $K$ has the following normal series

$$1 \leq N_1 \triangleleft H_1 \leq K,$$

such that $H_1/N_1 \cong A_5$, $A_6$, $L_2(7)$, $L_2(8)$, $U_3(3)$, $U_4(2)$ or $S$ where $S$ is one of the groups: $A_n$ for $n = 7, 8, 9, 10$, $J_2$, $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7(2)$, $O_9^+(2)$, $U_3(5)$ and $U_3(3)$, by Lemma 2.3 and 2.5. Because $n_2(H_1/N_1) \mid n_2(K) = 1$, we get a contradiction. Thus $N = K$.

Therefore $G/N \cong L_2(8)$, it follows that $5 \in \pi(N)$ and the order of a Sylow 5-subgroup in $G$ and $N$ are equal. As $N$ is normal in $G$, the number of Sylow 5-subgroups of $G$ and $N$ are equal. Thus the number of Sylow 5-subgroups of $N$ is 36. Since $N$ is solvable by Lemma 2.2, $4 \equiv 1 \pmod{5}$, a contradiction.
Arguing as above if $G/K \cong \text{Aut}(L_2(8))$, then we get a contradiction.

If $H/N$ is simple $K_3$-group, then by Lemma 2.5, $H/N$ is isomorphic to one of the groups: $A_n$ for $n = 7$, 8, 9, 10, $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7^+(2)$, $O_5^-(2)$, $U_3(5)$ or $U_4(3)$. Since $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$, we get a contradiction.

Therefore $\pi(G) = \{2, 3, 7\}$. Since $G$ is a nonsolvable group, it has the following normal series

$$1 \leq N < H \leq G$$

such that $H/N$ is a simple $K_3$-group or $H/N$ is a direct product of simple $K_3$-groups. By Lemma 2.3 and 2.4, $H/N \cong L_2(8)$. Now set $\overline{H} := H/N \cong L_2(8)$ and $\overline{G} := G/N$. Thus we have

$$L_2(8) \cong \overline{H} \cong \overline{H}(\overline{G}(\overline{H})/\overline{G}(\overline{H})) \leq \overline{G}(\overline{G}(\overline{H})) = \overline{G}(\overline{G}(\overline{H})) \leq \text{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xK \in \overline{G}(\overline{H})\}$. Then $G/K \cong \overline{G}(\overline{H})$ and $L_2(8) \leq G/K \cong \text{Aut}(L_2(8))$. So $G/K$ isomorphic to $L_2(8)$ or $\text{Aut}(L_2(8))$.

Let $G/K$ isomorphic to $L_2(8)$. By Lemma 2.3, $n_p(K) = 1$ for every prime $p \in \pi(G)$. Thus $K$ is a nilpotent subgroup of $G$.

We claim that $K = 1$. Let $Q$ be a Sylow $q$-subgroup of $K$, since $K$ is nilpotent, $Q$ is normal in $G$. Now if $P \in \text{Syl}_q(G)$, then $P$ normalizes $Q$ and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that $KP/K$ is a Sylow $p$-subgroup of $G/K$. On the other hand, if $R/K = N_{G/K}(KP/K)$, then $R = N_{G}(P)K$. We know that $n_p(G) = n_p(G/K)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore $K \leq N_G(P)$. So $Q \leq N_G(P)$. Since $P \leq N_G(Q)$ and $Q \leq N_G(P)$ by Lemma 2.4, this implies that $[P, Q] \leq P$ and $[P, Q] \leq Q$, then $[P, Q] \leq P \cap Q = 1$. So $P \leq C_G(Q)$ and $Q \leq C_G(P)$, in other words $P$ and $Q$ centralize each other. Let $C = C_G(Q)$, then $C$ contains a Sylow $p$-subgroup of $G$ for all primes $p$ different from $q$, and thus $|G : C|$ is a power of $q$. Now let $S$ be a Sylow $q$-subgroup of $G$. Then $G = CS$. Also if $Q > 1$, then $C_Q(S)$ is nontrivial, so $C_Q(S) \leq Z(G)$. Since by assumption $Z(G) = 1$, it follows that $Q = 1$. Since $q$ is arbitrary, $K = 1$, as claimed. Therefore $G$ is isomorphic to $L_2(8)$.

Arguing as above if $G/K$ isomorphic to $\text{Aut}(L_2(8))$, then $G$ is isomorphic to $\text{Aut}(L_2(8))$. □

Acknowledgments

The author is thankful to the referee for carefully reading the paper and for his suggestions and remarks.

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