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A new characterization of the projective linear groups by the Sylow numbers

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ABSTRACT: Let G be a finite group, let $\pi(G)$ be the set of primes p such that G contains an element of order p and let $n_p(G)$ be the number of Sylow $p-$subgroup of G, that is, $n_p = n_p(G)=|\mathrm{Syl}_p(G)|$. Set $\mathrm{NS}(G) := \{n_p(G)|\ p \in \pi(G)\}$. In this paper we show the projective linear groups $L_2(q)$ are recognizable by NS(G) and order. Also we prove if NS(G)=NS($L_2(8)$), then finite centerless group G is isomorphic to $L_2(8)$ or Aut($L_2(8)$).

Key Words: Finite group, Sylow subgroup, simple group.

Contents

1. Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. Let G be a finite group. Denote by $\pi(G)$ the set of primes p such that G contains an element of order p. A finite group G is called a simple K_n-group, if G is a simple group with $|\pi(G)| = n$. Denote by $n_{2'}$ the largest positive odd divisor of the positive integer n. Also denote by (a, b) the greatest common divisor of positive integers a and b. If G is a finite group, then we denote by n_q the number of Sylow $q-$subgroup of G, that is, $n_q = n_q(G)=|\mathrm{Syl}_q(G)|$. All other notations are standard and we refer to [11], for example.

In 1992, Bi [6] showed that $L_2(p^k)$ can be characterized only by the order of normalizer of its Sylow subgroups. In other words, if G is a group and $|N_G(P)| = |N_{L_2(p^k)}(Q)|$ where $P \in\mathrm{Syl}_r(G)$ and $Q \in\mathrm{Syl}_r(L_2(p^k))$ for every prime r, then $G \cong L_2(p^k)$. Similar characterizations have been found for the following groups: $L_n(q)$ [5], $S_4(q)$ [9], the alternating simple groups [8], $U_n(q)$ [10], the sporadic simple groups [2] and ${}^2D_n(p^k)$ [1].

Set $\mathrm{NS}(G) := \{n_p(G)|\ \ p \in \pi(G)\}$. Let S be one of the above simple groups. It is clear that if $n_p(G) = n_p(S)$ for every prime p and $|G| = |S|$, then $|N_G(P)| = |N_S(Q)|$ where $P \in\mathrm{Syl}_p(G)$ and $Q \in\mathrm{Syl}_p(S)$. By the above references, $G \cong S$.

Now let NS(G) =NS(S) and $|G| = |S|$. In this case we don't know $n_p(G) = n_p(S)$ for any prime p. Thus we can not conclude that $G \cong S$. In this paper first we will show that if NS(G) =NS($L_2(q)$) and $|G| = |L_2(q)|$, then $G \cong L_2(q)$.

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We denote by $k(\mathrm{NS}(G))$ the number of isomorphism classes of the finite centerless groups H satisfying $\mathrm{NS}(G) = \mathrm{NS}(H)$. A finite centerless group G is called n-recognizable by $\mathrm{NS}(G)$ if $k(\mathrm{NS}(G)) = n$.

In [3] it is proved that A_5 is recognizable and A_6 is 5-recognizable by only $\mathrm{NS}(G)$. We will show that $k(\mathrm{NS}(L_2(8))) = 2$.

2. Preliminary Results

Lemma 2.1. [5] *Let G be a finite group. If $|N_G(R_1)| = |N_{L_n(q)}(R_2)|$ for every prime r, where $R_1 \in Syl_r(G)$ and $R_2 \in Syl_r(PSL(n,q))$, then $G \cong L_n(q)$.*

Lemma 2.2. *Let G be solvable. Then when $n_p(G)$ is factored as a product of prime powers, each factor is congruent to 1 mod p.*

Proof: Let P be a Sylow p-subgroup of G. By Sylow's theorem, $n_p(G) = |G : N_G(P)|$. Consider a chain of subgroups $N_G(P) = H_0 < ... < H_r = G$, where each subgroup is maximal in the next. By Sylow's theorem we have $|H_i : H_0| \equiv 1 \pmod p$ for all i. In fact, P is a Sylow p-subgroup of H_i and $H_0 = N_G(P) = N_{H_i}(P)$. Thus for all i , $|H_{i+1} : H_0| = |H_{i+1} : H_i| \times |H_i : H_0| \equiv |H_{i+1} : H_i| \equiv 1 \pmod p$. Then $|H_{i+1} : H_i| \equiv 1 \pmod p$, and since G is solvable, each of these indices is a prime power. Since $n_p(G) = |G : N_G(P)| = \prod_{i=0}^{r-1} |H_{i+1} : H_i|$, each factor of $n_p(G)$ therefore is congruent to 1 mod p. □

Lemma 2.3. [13] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.*

Lemma 2.4. [12] *Let G be a finite group and M a normal subgroup. Then $n_p(M)\, n_p(G/M)$ divides $n_p(G)$.*

Lemma 2.5. [14] *Let G be a simple group of order $2^a \cdot 3^b \cdot 5^c \cdot 7^d$, $abcd \neq 0$. Then G is isomorphic to one of the following groups: A_n for $n = 7, 8, 9, 10$; J_2; $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7(2)$, $O_8^+(2)$, $U_3(5)$ and $U_4(3)$.*

By Sylow's theorem implies that if p is prime, then $n_p = 1 + pk$. If $p = 2$, then n_2 is odd. If $p \in \pi(G)$, then

$$\begin{cases} p \mid (n_p - 1) \\ (p, n_p) = 1 \end{cases} \qquad (*)$$

In the proof of the theorem 3.2 and 3.3, we often apply $(*)$ and the above comments.

3. Main Results

Lemma 3.1. *Let $q = p^n$ and $p \neq 2$. Then $NS(L_2(q)) = \{q(q^2-1)_{2'},\ q+1,\ q(q-1)/2,\ q(q+1)/2\}$ or $\{q(q^2-1)/24,\ q+1,\ q(q-1)/2,\ q(q+1)/2\}$. In particular if $q = 2^n$, then $NS(L_2(q)) = \{q+1,\ q(q-1)/2,\ q(q+1)/2\}$.*

Proof: Let $q = p^n$ and $p \neq 2$. To find the number of Sylow p-subgroups in $L_2(q)$ first, look at $\mathrm{SL}_2(q)$. The normalizer of a Sylow p-subgroup is the set of upper triangular matrices with determinant 1, so the order of the normalizer is $q(q-1)$. The order of the whole group $\mathrm{SL}_2(q)$ is $q(q^2-1)$. The number of Sylow p-subgroups, therefore, is $(q+1)$. This will be the same as the number of Sylow p-subgroups of $L_2(q)$ because the canonical homomorphism from $\mathrm{SL}_2(q)$ to $L_2(q)$ yields a bijection on Sylow p-subgroups.

If $r \neq p$ is an odd prime divisor of $|G|$, then r divides exactly one of $\{q+1, q-1\}$ and a Sylow r-subgroup of G is cyclic with normalizer dihedral of order $q-1$ or $q+1$. In particular, the number of Sylow r-subgroups is $q(q+1)/2$ or $q(q+1)/2$. When q is a power of 2, we have $L_2(q) = \mathrm{SL}_2(q)$ and a Sylow 2 -normalizer is a Borel subgroup of order $q(q-1)$. Hence there are $q+1$ Sylow 2 -subgroups as $\mathrm{SL}_2(q)$ has order $(q-1)q(q+1)$. When q is odd, the order of $L_2(q)$ is $q(q-1)(q+1)/2$.

A Sylow 2 -subgroup of $\mathrm{SL}_2(q)$ is quaternion or generalized quaternion and a Sylow 2 -subgroup of $L_2(q)$ is either a Klein 4 -group or a dihedral 2 -group with at least 8 elements. In all these cases, a Sylow 2 -subgroup of $\mathrm{SL}_2(q)$ contains its centralizer, and the same is true in $L_2(q)$. The outer automorphism group of a dihedral 2 -group with at least 8 elements is a 2 -group. Hence a Sylow 2 -subgroup of $L_2(q)$ is self-normalizing when $q \equiv \pm 1 \pmod 8$, and in that case the number of Sylow 2 -subgroups of $L_2(q)$ is $q(q^2-1)_{2'}$. When $q \equiv \pm 3 \pmod 8$, then a Sylow 2 -normalizer of $L_2(q)$ must have order 12, because a Sylow 2 -subgroup is a self-centralizing Klein 4 -group, but there must be an element of order 3 in its normalizer by Burndisde's transfer theorem. In this case, the number of Sylow 2 -subgroups of $L_2(q)$ is $q(q^2-1)/24$. Therefore $\mathrm{NS}(L_2(q)) = \{q(q^2-1)_{2'},\ q+1,\ q(q-1)/2,\ q(q+1)/2\}$ or $\{q(q^2-1)/24,\ q+1,\ q(q-1)/2,\ q(q+1)/2\}$.

Arguing as above if $q = 2^n$, then $\mathrm{NS}(L_2(q)) = \{q+1,\ q(q-1)/2,\ q(q+1)/2\}$.□

Theorem 3.2. *Let G be a finite group such that $NS(G) = NS(L_2(q))$ and $|G| = |L_2(q)|$. Then $G \cong L_2(q)$.*

Proof: Let $q = p^n$ and $q \equiv \pm 1 \pmod 8$. By Sylow's theorem since $q+1$ is the only Sylow number not divisible by p, $n_p = q+1$. If $r \in \pi(q+1)$, then $n_r = q+1$ or $q(q-1)/2$. Suppose that $n_r = q+1$. By Sylow's theorem $n_r = rk+1 = q+1$ where k is a positive number. Then $rk = q$ and $r = p$, a contradiction. Therefore $n_r = q(q-1)/2$. Similarly if $r \in \pi(q-1)$, then $n_r = q(q+1)/2$. Now it is clear that $n_2 = q(q^2-1)_{2'}$. Thus we have proved that $n_r(G) = n_r(L_2(q))$ for every r. Arguing as above if $q \equiv \pm 3 \pmod 8$, then $n_r(G) = n_r(L_2(q))$ for every r. Since $|G| = |L_2(q)|$, $|N_G(R_1)| = |N_{L_2(q)}(R_2)|$ for every prime r where $R_1 \in \mathrm{Syl}_r(G)$ and $R_2 \in \mathrm{Syl}_r(L_2(q))$. Therefore by Lemma 2.1, $G \cong L_2(q)$.

Arguing as above if $q = 2^n$ and $\mathrm{NS}(G) = \mathrm{NS}(L_2(q))$, then $G \cong L_2(q)$. □

Theorem 3.3. *Let G be finite centerless group and NS(G)=NS($L_2(8)$). Then $G \cong L_2(8)$ or $G \cong Aut(L_2(8))$.*

Proof: We have NS(G)=NS($L_2(8)$)={9, 28, 36}. First we prove that $\pi(G) = \{2, 3, 7\}$. By Sylow's theorem $n_p \mid |G|$ for every p, hence by NS(G), we can conclude that $\{2, 3, 7\} \subseteq \pi(G)$. On the other hand, by (∗) if $p \in \pi(G)$, then $p \mid (n_p - 1)$ and $(p, n_p) = 1$, which implies that $p \in \{2, 3, 5, 7\}$.

Let $\pi(G) = \{2, 3, 5, 7\}$. Then $n_2(G) = 9$, $n_3(G) = 28$ and $n_5(G) = n_7(G) = 36$. We show that G is a nonsolvable group. If G is a solvable group since $n_7(G) = 36$ by Lemma 2.2, $9 \equiv 1 \pmod 7$, a contradiction. Hence G is a nonsolvable group.

Since G is a finite group, it has a chief series. Let $1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \lhd N_{r-1} \trianglelefteq N_r = G$ be a chief series of G. Since G is a nonsolvable group there exists a maximal number of non-negative integer i such that N_i/N_{i-1} is a simple group or the direct product of isomorphic simple groups and N_{i-1} is a maximal solvable normal subgroup of G. Now set $N_i := H$ and $N_{i-1} := N$. Hence G has the following normal series

$$1 \trianglelefteq N \lhd H \trianglelefteq G$$

such that H/N is a non-abelian simple group or H/N is a direct product of isomorphic non-abelian simple groups. Since G is a K_4−group, H/N is a simple K_n−group or H/N is a direct product of simple K_n−groups for $n = 3$ or 4. By Lemma 2.4, $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$. Thus H/N is a simple K_3−group or simple K_4−group.

If H/N is a simple K_3−group, then by Lemma 2.3 and 2.4, $H/N \cong L_2(8)$. Now set $\overline{H} := H/N \cong L_2(8)$ and $\overline{G} := G/N$. On the other hand, we have

$$L_2(8) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G | xN \in C_{\overline{G}}(\overline{H})\}$, then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$. Hence $L_2(8) \leq G/K \leq \mathrm{Aut}(L_2(8))$. Hence $G/K \cong L_2(8)$ or $G/K \cong \mathrm{Aut}(L_2(8))$.

Let G/K isomorphic to $L_2(8)$ by Lemma 2.4, $n_2(K) = 1$, $n_3(K) = 1$, $n_7(K) = 1$ and $n_5(K) \mid 36$. We show that $K = N$. Suppose that $K \neq N$. Since $N < K$ and N is a maximal solvable normal subgroup G, K is a nonsolvable normal subgroup of G. Therefore K has the following normal series

$$1 \trianglelefteq N_1 \lhd H_1 \trianglelefteq K,$$

such that $H_1/N_1 \cong A_5$, A_6, $L_2(7)$, $L_2(8)$, $U_3(3)$, $U_4(2)$ or S where S is one of the groups: A_n for $n = 7, 8, 9, 10$, J_2, $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7(2)$, $O_8^+(2)$, $U_3(5)$ and $U_4(3)$, by Lemma 2.3 and 2.5. Because $n_2(H_1/N_1) \mid n_2(K) = 1$, we get a contradiction. Thus $N = K$.

Therefore $G/N \cong L_2(8)$, it follows that $5 \in \pi(N)$ and the order of a Sylow 5-subgroup in G and N are equal. As N is normal in G, the number of Sylow 5-subgroups of G and N are equal. Thus the number of Sylow 5-subgroups of N is 36. Since N is solvable by Lemma 2.2, $4 \equiv 1 \pmod 5$, a contradiction.

Arguing as above if $G/K \cong \mathrm{Aut}(L_2(8))$, then we get a contradiction.

If H/N is simple K_4-group, then by Lemma 2.5, H/N is isomorphic to one of the groups: A_n for $n = 7, 8, 9, 10$, J_2, $L_2(49)$, $L_3(4)$, $O_5(7)$, $O_7(2)$, $O_8^+(2)$, $U_3(5)$ or $U_4(3)$. Since $n_p(H/N) \mid n_p(G)$ for every prime $p \in \pi(G)$, we get a contradiction.

Therefore $\pi(G) = \{2, 3, 7\}$. Since G is a nonsolvable group, it has the following normal series

$$1 \trianglelefteq N \triangleleft H \trianglelefteq G$$

such that H/N is a simple K_3-group or H/N is a direct product of simple K_3-groups. By Lemma 2.3 and 2.4, $H/N \cong L_2(8)$. Now set $\overline{H} := H/N \cong L_2(8)$ and $\overline{G} := G/N$. Thus we have

$$L_2(8) \cong \overline{H} \cong \overline{H}C_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \overline{G}/C_{\overline{G}}(\overline{H}) = N_{\overline{G}}(\overline{H})/C_{\overline{G}}(\overline{H}) \leq \mathrm{Aut}(\overline{H}).$$

Let $K = \{x \in G \mid xK \in C_{\overline{G}}(\overline{H})\}$. Then $G/K \cong \overline{G}/C_{\overline{G}}(\overline{H})$ and $L_2(8) \leq G/K \leq \mathrm{Aut}(L_2(8))$. So G/K isomorphic to $L_2(8)$ or $\mathrm{Aut}(L_2(8))$.

Let G/K isomorphic to $L_2(8)$. By Lemma 2.3, $n_p(K) = 1$ for every prime $p \in \pi(G)$. Thus K is a nilpotent subgroup of G.

We claim that $K = 1$. Let Q be a Sylow $q-$ subgroup of K, since K is nilpotent, Q is normal in G. Now if $P \in \mathrm{Syl}_p(G)$, then P normalizes Q and so if $p \neq q$, then $P \leq N_G(Q) = G$. Also we note that KP/K is a Sylow p-subgroup of G/K. On the other hand, if $R/K = N_{G/K}(\ KP/K)$, then $R = N_G(P)K$. We know that $n_p(G) = n_p(G/K)$, so $|G : R| = |G : N_G(P)|$. Thus $R = N_G(P)$ and therefore $K \leq N_G(P)$. So $Q \leq N_G(P)$. Since $P \leq N_G(Q)$ and $Q \leq N_G(P)$ by Lemma 2.4, this implies that $[P, Q] \leq P$ and $[P, Q] \leq Q$, then $[P, Q] \leq P \cap Q = 1$. So $P \leq C_G(Q)$ and $Q \leq C_G(P)$, in other words P and Q centralize each other. Let $C = C_G(Q)$, then C contains a full Sylow $p-$subgroup of G for all primes p different from q, and thus $|G : C|$ is a power of q. Now let S be a Sylow $q-$subgroup of G. Then $G = CS$. Also if $Q > 1$, then $C_Q(S)$ is nontrivial, so $C_Q(S) \leq Z(G)$. Since by assumption $Z(G) = 1$, it follows that $Q = 1$. Since q is arbitrary, $K = 1$, as claimed. Therefore G is isomorphic to $L_2(8)$.

Arguing as above if G/K isomorphic to $\mathrm{Aut}(L_2(8))$, then G is isomorphic to $\mathrm{Aut}(L_2(8))$. □

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References

1. A. Iranmanesh, N. Ahanjideh, *A characterization of* ${}^2D_n(p^k)$ *by order of normalizer of Sylow subgroups*, International journal of Algebra, 2(8), 853-865 (2008).

2. A. Khosravi, B. Khosaravi, *Two new characterization of sporadic simple groups*, PU. M. A, 16, 287-293 (2005).

3. A. R. Khallili Asboei, *A New Characterization of Alternating groups* A_5 *and* A_6, New Zealand J. Math (to appear).

4. A. R. Khalili Asboei, S. S. Salehi Amiri, A. Iranmanesh and A. Tehranian, *A characterization of Symmetric group S_r, where r is prime number*, Ann. Math et Informaticae, 40, 13-23 (2012).

5. J. Bi, *A characterization of $L_n(q)$ by the normalizers' orders of their Sylow subgroups*, Acta. Math. Sinica (New Ser) 11(3), 300-306 (1995).

6. J. Bi, *A characterization of $L_2(q)$,* J. Liaoning Univ (Natural Sciences Edition), 19(2), 1-4 (1992) (Chinese).

7. J. Bi, *A characterization of finite projective special unitary group $U_6(q)$*, J. Liaoning Univ (Natural Sciences Edition), 26(4), 295-298 (1990) (Chinese).

8. J. Bi, C*haracterization of Alternating groups by orders of normalizers of Sylow subgroups*, Algebra Colloq, 8(3), 249-256 (2001).

9. J. Bi, *On the group with the same orders of Sylow normalizers as the finite simple group $S_4(q)$*, Algebras Groups and Geom, 18(3), 349-355 (2001).

10. J. Bi, *On the groups with the same orders of Sylow normalizers as the finite projective special unitary group*, Sci. China, Ser A, 47(6), 801-811 (2004).

11. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Oxford, Clarendon, Press (1985).

12. J. Zhang, *Sylow numbers of finite groups*, J. Algebra, 176(10), 111-123 (1995).

13. M. Herzog, *On finite simple groups of order divisible by three primes only*, J. Algebra, 120(10), 383-388 (1968).

14. W. Shi, *The simple groups of order $2^a3^b5^c7^d$ and Janko's simple groups*, J. Southest-China Teachers University, 4, 1-8 (1987) (Chinese).

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