



## Sobolev-Type Volterra-Fredholm Functional Integrodifferential Equations In Banach Spaces

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ABSTRACT: This paper deals with the problems such as the existence, uniqueness, continuous dependence and boundedness of mild solution of Volterra-Fredholm functional integrodifferential equations of sobolev type in Banach spaces. Our analysis is based on Banach fixed point principle, the integral inequality established by B. G. Pachpatte, Gronwall-Bellman inequality and the semigroup theory.

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### 1. Introduction

Let  $X \equiv (X, \|\cdot\|_X)$  and  $Y \equiv (Y, \|\cdot\|)$  be Banach spaces. Let  $C = C([-r, 0], X)$ ,  $0 < r < \infty$ , denotes the Banach space of all continuous functions  $\psi : [-r, 0] \rightarrow X$  endowed with supremum norm  $\|\psi\|_C = \sup\{\|\psi(\theta)\|_X : -r \leq \theta \leq 0\}$ . For  $0 < b < \infty$ , let  $D = C([-r, b], X)$  be the Banach space of all continuous functions  $x : [-r, b] \rightarrow X$  with the supremum norm  $\|x\|_D = \sup\{\|x(t)\|_X : -r \leq t \leq b\}$ . For any  $x \in D$  and  $t \in [0, b]$ ,  $x_t$  denotes the element of  $C$  defined by  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-r, 0]$ .

Consider the Volterra-Fredholm functional integrodifferential equations of sobolev type in Banach spaces of the form

$$(Bx(t))' + Ax(t) = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in [0, b], \tag{1.1}$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \tag{1.2}$$

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and

$$\begin{aligned} \frac{d}{dt}[(Bx(t)) - g(t, x_t)] + Ax(t) \\ = f \left( t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds \right), \quad t \in [0, b], \quad (1.3) \end{aligned}$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \quad (1.4)$$

where  $B$  and  $A$  are linear operators with the domains contained in a Banach space  $X$  and ranges contained in a Banach space  $Y$ . The nonlinear functions  $f : [0, b] \times C \times Y \times Y \rightarrow Y$ ,  $g, w, h : [0, b] \times C \rightarrow Y$  are continuous. The kernel functions  $k, l : [0, b] \times [0, b] \rightarrow \mathbb{R}$  are continuous and  $\phi$  is a given element of  $C$ .

Equations of the form (1.1)-(1.2) and (1.3)-(1.4) and their special forms arise in various physical phenomena such as in flow of fluid through fissured rocks, the propagation of long waves of small amplitudes, thermodynamics and shear in second order fluids, see for example, [4,5,9,14] and the references cited therein. Many authors have studied the existence, uniqueness, continuation and other properties of solutions of various special forms of the equations (1.1)-(1.2) and (1.3)-(1.4) by using different techniques, see for example, [7]-[11], [16]-[18], [21] and some of the references given therein.

Brill [6] and Showalter [23] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. Lightbourne and Rankin [15] discussed the solution of partial functional differential equation of Sobolev type. Dhakne and Pachpatte [12] discussed mild and classical solutions of functional integrodifferential equations by using various approaches. Han [13] has studied controllability problem for the special form of (1.1)-(1.2) with  $w = 0$ ,  $h = 0$  and without functional arguments assuming the resolvent operator is compact. Balchandran and coauthors [1,2] established existence and qualitative properties of (1.1)-(1.2) when  $k(t, s)w(s, x_s) = k(t, s, x_s)$  and  $h = 0$  with nonlocal condition. Also, Balachandran et al. [3] have studied existence of solution of special form of (1.3)-(1.4) when  $f = \int_0^t w(s, x_s)ds$  using the Schaefer fixed point theorem.

In the present paper, we prove the existence results of mild solutions for (1.1)-(1.2) and (1.3)-(1.4) using the Banach fixed point principle. Using the tools of Pachpatte's integral inequality and Gronwall-Bellman inequality we establish the results pertaining to continuous dependence, uniqueness, and boundedness of mild solution of (1.1)-(1.2) considering the different cases on the argument  $t$  of  $x(t)$ ,  $t \in [-r, b]$ .

The paper is organized as follows. In section 2, we present the preliminaries and the hypothesis. Section 3 deals with existence results of (1.1)-(1.2) and (1.3)-(1.4). In section 4 we prove continuous dependence and boundedness of mild solution of (1.1)-(1.2). In section 5 we give applications of some of our results obtained in section 3. Finally in section 6 we give example to illustrate the results obtained in section 3.

### 2. Preliminaries and Hypotheses

In this section we shall set forth some preliminaries from [6] and the hypotheses that will be used in our subsequent discussion. Here and hereafter, we assume that  $(H_0)$  the operators  $A : D(A) \subset X \rightarrow Y$  and  $B : D(B) \subset X \rightarrow Y$  satisfy the following conditions

- (i)  $A$  and  $B$  are closed linear operators.
- (ii)  $D(B) \subset D(A)$  and  $B$  is bijective.
- (iii)  $B^{-1} : Y \rightarrow D(B)$  is continuous.

The assumptions (i) – (ii) and the closed graph theorem imply the boundedness of the linear operator  $AB^{-1} : Y \rightarrow Y$ . Further  $-AB^{-1}$  generates a uniformly continuous semigroup  $T(t)$ ,  $t \geq 0$  on  $Y$  and so  $\max\{\|T(t)\| : t \in [0, b]\}$  is finite. We denote  $R = \max_{t \in [0, b]} \|B^{-1}T(t)\|$  and  $M = \max_{t \in [0, b]} \|B^{-1}T(t)B\|$ .

**Definition 2.1.** A function  $x \in D$  is called mild solution of the initial value problem (1.1)-(1.2) if it satisfies the integral equations

$$x(t) = B^{-1}T(t)B\phi(0) + \int_0^t B^{-1}T(t-s)f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) ds, \\ t \in [0, b], x(t) = \phi(t), t \in [-r, 0].$$

**Definition 2.2.** A function  $x \in D$  is called mild solution of the initial value problem (1.3)-(1.4) if the restriction  $x(\cdot)$  to the interval  $[0, b]$  is continuous, for each  $0 \leq t < b$  the function  $AB^{-1}T(t-s)g(s, u_s)$ ,  $s \in [0, t]$  is integrable,  $Bx(t) \in C([0, b]; Y) \cap C^1([0, b]; Y)$  and the integral equation

$$x(t) = B^{-1}T(t)[B\phi(0) - g(0, \phi)] + B^{-1}g(t, x_t) + \int_0^t B^{-1}AB^{-1}T(t-s)g(s, x_s)ds \\ + \int_0^t B^{-1}T(t-s)f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) ds, \\ t \in [0, b], x(t) = \phi(t), t \in [-r, 0].$$

is satisfied.

The following integral inequalities plays the crucial role in our analysis.

**Lemma 2.1.** ([19], p-47) Let  $z(t)$ ,  $u(t)$ ,  $v(t)$ ,  $w(t) \in C([\alpha, \beta], \mathbb{R}_+)$  and  $k \geq 0$  be a real constant and

$$z(t) \leq k + \int_\alpha^t u(s) \left[ z(s) + \int_\alpha^s v(\sigma)z(\sigma)d\sigma + \int_\alpha^\beta w(\sigma)z(\sigma)d\sigma \right] ds, \text{ for } t \in [\alpha, \beta].$$

If

$$r^* = \int_{\alpha}^{\beta} w(\sigma) \exp \left( \int_{\alpha}^{\sigma} [u(\tau) + v(\tau)] d\tau \right) d\sigma < 1,$$

then

$$z(t) \leq \frac{k}{1-r^*} \exp \left( \int_{\alpha}^t [u(s) + v(s)] ds \right), \quad \text{for } t \in [\alpha, \beta].$$

**Lemma 2.2.** ([20], p-11) Let  $u$  and  $f$  be continuous functions defined on  $\mathbb{R}_+$  and  $c$  be a nonnegative constant. If

$$u(t) \leq c + \int_0^t f(s)u(s)ds, \text{ for } t \in \mathbb{R}_+,$$

then

$$u(t) \leq c \exp \left( \int_0^t f(s)ds \right), \text{ for } t \in \mathbb{R}_+.$$

We list the following hypothesis.

(H<sub>1</sub>) There exists constant  $F$  such that

$$\|f(t, \psi, x, y) - f(t, \chi, u, v)\| \leq F(\|\psi - \chi\|_C + \|x - u\| + \|y - v\|)$$

for every  $t \in [0, b]$ , and  $(\psi, x, y), (\chi, u, v) \in C \times Y \times Y$ .

(H<sub>2</sub>) There exists constants  $W, H$  such that

$$(i) \|w(t, \psi) - w(t, \chi)\| \leq W\|\psi - \chi\|_C$$

$$(ii) \|h(t, \psi) - h(t, \chi)\| \leq H\|\psi - \chi\|_C$$

for every  $t \in [0, b]$ , and  $\psi, \chi \in C$ .

(H<sub>3</sub>) There exists  $N > 0$  such that  $\|AB^{-1}T(t)\| \leq N, t \geq 0$ .

(H<sub>4</sub>) There exists  $G > 0$  such that

$$\|g(t, \psi) - g(t, \chi)\| \leq G\|\psi - \chi\|_C$$

for every  $t \in [0, b]$ , and  $\psi, \chi \in C$ .

### 3. Existence results

**Theorem 3.1.** If the hypotheses (H<sub>0</sub>)-(H<sub>2</sub>) are satisfied then initial value problem (1.1)-(1.2) has a unique mild solution on  $[-r, b]$ , provided the condition

$$RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right] < 1,$$

holds, where  $K$  and  $L$  are constants as in condition (3.3).

**Proof:** Define an operator  $\Gamma : D \rightarrow D$  by

$$(\Gamma x)(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ B^{-1}T(t)B\phi(0) + \int_0^t B^{-1}T(t-s) \\ \times f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) ds, & t \in [0, b]. \end{cases} \tag{3.1}$$

Observe that using the definition of the operator  $\Gamma$  in (3.1), the equivalent integral equations of initial value problem (1.1)-(1.2) can be written as  $x = \Gamma x$ , whose fixed point is the mild solution of (1.1)-(1.2). Now we will show that  $\Gamma$  is contraction on  $D$ . Consider,

$$(\Gamma x)(t) - (\Gamma y)(t) = \int_0^t B^{-1}T(t-s) \left[ f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) - f\left(s, y_s, \int_0^s k(s, \tau)w(\tau, y_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, y_\tau)d\tau\right) \right] ds, \tag{3.2}$$

for  $x, y \in D$  and  $t \in [0, b]$ .

We note that, the kernel functions  $k$  and  $l$  are continuous on compact set  $[0, b] \times [0, b]$ , therefore there exists constants  $K$  and  $L$  such that

$$|k(t, s)| \leq K, \text{ for } t \geq s \geq 0 \text{ and } |l(t, s)| \leq L, \text{ for } s, t \in [0, b]. \tag{3.3}$$

Using the condition (3.3), the hypothesis  $(H_1)$  and  $(H_2)$  in equation (3.2), we get

$$\begin{aligned} & \|(\Gamma x)(t) - (\Gamma y)(t)\|_X \\ & \leq \int_0^t \|B^{-1}T(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) - f\left(s, y_s, \int_0^s k(s, \tau)w(\tau, y_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, y_\tau)d\tau\right) \right\| ds, \\ & \leq \int_0^t RF \left[ \|(x-y)_s\|_C + \int_0^s KW\|(x-y)_\tau\|_C d\tau + \int_0^b LH\|(x-y)_\tau\|_C d\tau \right] ds \end{aligned} \tag{3.4}$$

Define the function  $\mu$  by  $\mu(t) = \sup\{\|(x-y)(s)\|_X : s \in [-r, t]\}$ ,  $t \in [0, b]$ . Then

$\|(x - y)_t\|_C \leq \mu(t)$  for all  $t \in [0, b]$ . Hence from (3.4) we have

$$\begin{aligned} \|(\Gamma x)(t) - (\Gamma y)(t)\|_X &\leq \int_0^t RF \left[ \mu(s) + \int_0^s KW\mu(\tau)d\tau + \int_0^b LH\mu(\tau)d\tau \right] ds \\ &\leq \|x - y\|_D \int_0^t RF \left[ 1 + \int_0^s KWd\tau + \int_0^b LHd\tau \right] ds \\ &\leq RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right] \|x - y\|_D \end{aligned} \tag{3.5}$$

Further,

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_X = 0, \text{ for } x, y \in D \text{ and } t \in [-r, 0] \tag{3.6}$$

Therefore, from (3.5) and (3.6) we have

$$\|\Gamma x - \Gamma y\|_D \leq \gamma \|x - y\|_D, \text{ for } x, y \in D, \tag{3.7}$$

where  $\gamma = RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right]$ . Since  $\gamma < 1$  the inequality (3.7) implies that  $\Gamma$  is a contraction on  $D$ . Consequently, the operator  $\Gamma$  satisfies all the assumptions of the Banach contraction theorem. Therefore, there is a unique fixed point for  $\Gamma$  in space  $D$  and this point is the mild solution of the initial problem (1.1)-(1.2).  $\square$

**Theorem 3.2.** *If the hypothesis  $(H_0)$ - $(H_4)$  are satisfied, then the initial value problem (1.3)-(1.4) has a unique solution on  $[-r, b]$ , provided the condition*

$$\|B^{-1}\|G(1 + Nb) + RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right] < 1,$$

holds.

**Proof:** Define an operator  $\Omega : D \rightarrow D$  by

$$(\Omega x)(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ B^{-1}T(t)[B\phi(0) - g(0, \phi)] + B^{-1}g(t, x_t) \\ + \int_0^t B^{-1}AB^{-1}T(t-s)g(s, x_s)ds + \int_0^t B^{-1}T(t-s) \\ \times f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) ds, & t \in [0, b]. \end{cases} \tag{3.8}$$

Using the definition of  $\Omega$  in (3.8), the equivalent integral equations of initial value problem (1.3)-(1.4) can be written as  $x = \Omega x$ , whose fixed point is the mild solution of (1.3)-(1.4). Let any  $x, y \in D$  and  $t \in [0, b]$ , then using the condition (3.3), the

hypothesis  $(H_1)$ - $(H_4)$ , from equation (3.8), we have

$$\begin{aligned}
 & \|(\Omega x)(t) - (\Omega y)(t)\|_X \\
 & \leq \|B^{-1}\| \|g(t, x_t) - g(t, y_t)\| + \int_0^t \|B^{-1}\| \|AB^{-1}T(t-s)\| \|g(s, x_s) - g(s, y_s)\| ds \\
 & \quad + \int_0^t \|B^{-1}T(t-s)\| \left\| f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) \right. \\
 & \quad \left. - f \left( s, y_s, \int_0^s k(s, \tau)w(\tau, y_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, y_\tau)d\tau \right) \right\| ds, \\
 & \leq \|B^{-1}\|G\|(x-y)_t\|_C + \int_0^t \|B^{-1}\|NG\|(x-y)_s\|_C ds \\
 & \quad + \int_0^t RF \left[ \|(x-y)_s\|_C + \int_0^s KW\|(x-y)_\tau\|_C d\tau + \int_0^b LH\|(x-y)_\tau\|_C d\tau \right] ds
 \end{aligned} \tag{3.9}$$

Define the function  $\mu$  as in the proof of Theorem 3.1, then from (3.9) we have

$$\begin{aligned}
 \|(\Omega x)(t) - (\Omega y)(t)\|_X & \leq \|B^{-1}\|G\mu(t) + \int_0^t \|B^{-1}\|NG\|\mu(s)\| ds \\
 & \quad + \int_0^t RF \left[ \mu(s) + \int_0^s KW\mu(\tau)d\tau + \int_0^b LH\mu(\tau)d\tau \right] ds \\
 & \leq \|B^{-1}\|G\|x-y\|_D + b\|B^{-1}\|NG\|x-y\|_D \\
 & \quad + \|x-y\|_D \int_0^t RF \left[ 1 + \int_0^s KWd\tau + \int_0^b LHd\tau \right] ds \\
 & \leq \|B^{-1}\|G(1+Nb)\|x-y\|_D \\
 & \quad + RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right] \|x-y\|_D
 \end{aligned} \tag{3.10}$$

Further,

$$\|(\Omega x)(t) - (\Omega y)(t)\|_X = 0, \text{ for } x, y \in D \text{ and } t \in [-r, 0] \tag{3.11}$$

Therefore, from (3.10) and (3.11) we have

$$\|\Omega x - \Omega y\|_D \leq q\|x-y\|_D, \text{ for } x, y \in D,$$

where  $q = \|B^{-1}\|G(1+Nb) + RFb \left[ 1 + \left( \frac{KW}{2} + LH \right) b \right] < 1$ . This shows that  $\Omega$  is a contraction on  $D$ . Therefore, in space  $D$  there is a unique fixed point for  $\Omega$  and this point is the mild solution of the initial problem (1.3)-(1.4).  $\square$

4. Continuous Dependence and Boundedness

**Theorem 4.1.** *Suppose that the hypotheses (H<sub>0</sub>)-(H<sub>2</sub>) are satisfied. Let  $x$  and  $\bar{x}$  be mild solutions of (1.1) with initial functions  $\phi$  and  $\bar{\phi} \in C$  respectively. Then the following inequality*

$$\|x - \bar{x}\|_D \leq [(M + 1)\|\phi - \bar{\phi}\|_C + RFLHb^2\|x - \bar{x}\|_D] \exp(bRF[1 + bKW]),$$

is true. Additionally, if  $RFLHb^2 \exp(bRF[1 + bKW]) < 1$  then

$$\|x - \bar{x}\|_D \leq \frac{(M + 1) \exp(bRF[1 + bKW])}{1 - RFLHb^2 \exp(bRF[1 + bKW])} \|\phi - \bar{\phi}\|_C \tag{4.1}$$

**Proof:** Let  $x$  and  $\bar{x}$  be mild solutions of (1.1) with initial functions  $\phi$  and  $\bar{\phi} \in C$  respectively. Then we have,

$$\begin{aligned} x(t) - \bar{x}(t) &= B^{-1}T(t)B[\phi(0) - \bar{\phi}(0)] \\ &+ \int_0^t B^{-1}T(t-s) \left[ f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) \right. \\ &\left. - f \left( s, \bar{x}_s, \int_0^s k(s, \tau)w(\tau, \bar{x}_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, \bar{x}_\tau)d\tau \right) \right] ds, \quad t \in [0, b] \end{aligned} \tag{4.2}$$

$$x(t) - \bar{x}(t) = \phi(t) - \bar{\phi}(t), \quad t \in [-r, 0]. \tag{4.3}$$

Using the condition (3.3) and the hypotheses (H<sub>1</sub>) and (H<sub>2</sub>), from the equation (4.2) for  $t \in [0, b]$ , we obtain

$$\begin{aligned} &\|x(t) - \bar{x}(t)\|_X \\ &\leq \|B^{-1}T(t)B\| \|\phi(0) - \bar{\phi}(0)\|_X \\ &+ \int_0^t \|B^{-1}T(t-s)\| \left\| f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) \right. \\ &\quad \left. - f \left( s, \bar{x}_s, \int_0^s k(s, \tau)w(\tau, \bar{x}_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, \bar{x}_\tau)d\tau \right) \right\| ds \\ &\leq M\|\phi - \bar{\phi}\|_C \\ &+ \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds. \end{aligned} \tag{4.4}$$

Case 1: Suppose  $t \geq r$ . Then for every  $\theta \in [-r, 0]$  we have  $t + \theta \geq 0$ . For such  $\theta$ 's



from (4.4) we get,

$$\begin{aligned} & \|(x - \bar{x})(t + \theta)\|_X \\ & \leq M\|\phi - \bar{\phi}\|_C \\ & \quad + \int_0^{t+\theta} RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds \\ & \leq M\|\phi - \bar{\phi}\|_C \\ & \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds, \end{aligned}$$

which yields,

$$\begin{aligned} & \|(x - \bar{x})_t\|_C \leq M\|\phi - \bar{\phi}\|_C \\ & \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds. \end{aligned} \tag{4.5}$$

Case 2: Suppose  $0 \leq t < r$ . Then for every  $\theta \in [-r, -t)$  we have  $t + \theta < 0$ . For such  $\theta$ 's we get,

$$\|(x - \bar{x})(t + \theta)\|_X = \|x(t + \theta) - \bar{x}(t + \theta)\|_X = \|\phi(t + \theta) - \bar{\phi}(t + \theta)\|_X$$

which yields,

$$\|(x - \bar{x})_t\|_C = \|\phi - \bar{\phi}\|_C \tag{4.6}$$

For  $\theta \in [-t, 0]$ ,  $t + \theta \geq 0$  then from (4.4) we get as in the case 1,

$$\begin{aligned} & \|(x - \bar{x})_t\|_C \\ & \leq M\|\phi - \bar{\phi}\|_C \\ & \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds. \end{aligned} \tag{4.7}$$

Thus for every  $\theta \in [-r, 0]$ , ( $0 \leq t < r$ ), from (4.6) and (4.7), we obtain

$$\begin{aligned} & \|(x - \bar{x})_t\|_C \\ & \leq (M + 1)\|\phi - \bar{\phi}\|_C \\ & \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW\|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH\|(x - \bar{x})_\tau\|_C d\tau \right] ds. \end{aligned} \tag{4.8}$$

Therefore, for every  $t \in [0, b]$ , from (4.5) and (4.8) we have

$$\begin{aligned}
& \|(x - \bar{x})_t\|_C \\
& \leq (M + 1)\|\phi - \bar{\phi}\|_C \\
& \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW \|(x - \bar{x})_\tau\|_C d\tau + \int_0^b LH \|(x - \bar{x})_\tau\|_C d\tau \right] ds \\
& = (M + 1)\|\phi - \bar{\phi}\|_C \\
& \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW \|(x - \bar{x})_\tau\|_C d\tau \right. \\
& \qquad \qquad \qquad \left. + \int_0^b LH \sup_{-r \leq \theta \leq 0} \|(x - \bar{x})_\tau(\theta)\|_X d\tau \right] ds \\
& \leq (M + 1)\|\phi - \bar{\phi}\|_C + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW \|(x - \bar{x})_\tau\|_C d\tau \right. \\
& \quad \left. + \int_0^b LH \sup_{-r \leq \tau + \theta \leq b} \|(x - \bar{x})(\tau + \theta)\|_X d\tau \right] ds \\
& \leq (M + 1)\|\phi - \bar{\phi}\|_C \\
& \quad + \int_0^t RF \left[ \|(x - \bar{x})_s\|_C + \int_0^s KW \|(x - \bar{x})_\tau\|_C d\tau + LHb\|x - \bar{x}\|_D \right] ds. \\
& \leq (M + 1)\|\phi - \bar{\phi}\|_C + b^2 RFLH\|x - \bar{x}\|_D \\
& \quad + \int_0^t RF \|(x - \bar{x})_s\|_C ds + \int_0^t \int_0^s RFKW \|(x - \bar{x})_\tau\|_C d\tau ds.
\end{aligned}$$

Thus for every,  $t \in [0, b]$ , we get

$$\begin{aligned}
\|(x - \bar{x})_t\|_C & \leq (M + 1)\|\phi - \bar{\phi}\|_C + b^2 RFLH\|x - \bar{x}\|_D \\
& \quad + RF(1 + bKW) \int_0^t \|(x - \bar{x})_s\|_C ds. \tag{4.9}
\end{aligned}$$

By applying Grownwall-Bellman inequality given in Lemma 2.2 to the inequality (4.9), we obtain

$$\begin{aligned}
\|(x - \bar{x})_t\|_C & \leq [(M + 1)\|\phi - \bar{\phi}\|_C \\
& \quad + RFLHb^2\|x - \bar{x}\|_D] \exp \left( \int_0^t RF(1 + bKW) ds \right), \quad t \in [0, b].
\end{aligned}$$

This gives that,

$$\|(x - \bar{x})(t)\|_X \leq [(M + 1)\|\phi - \bar{\phi}\|_C + RFLHb^2\|x - \bar{x}\|_D] \exp(bRF(1 + bKW)),$$

for every  $t \in [-r, b]$ . Therefore,

$$\|x - \bar{x}\|_D \leq [(M + 1)\|\phi - \bar{\phi}\|_C + RFLHb^2\|x - \bar{x}\|_D] \exp(bRF(1 + bKW)),$$

and hence, the inequality (4.1) holds. Hence the proof is complete. □

**Remark 4.1.** We note that estimates obtained in the Theorem 4.1 yields not only the continuous dependence of mild solution on initial functions  $\phi$  and  $\bar{\phi}$ , but also gives the criterion to prove the uniqueness of mild solution of (1.1)-(1.2). It follows by putting  $\phi = \bar{\phi}$ .

In the following theorem we give condition for the boundedness of mild solution of the initial value problem (1.1)-(1.2)

**Theorem 4.2.** Assume that there exists  $p, q, r \in C([0, b], [0, \infty))$  such that

$$\begin{aligned} \|f(t, \psi, x, y)\| &\leq p(t)(\|\psi\|_C + \|x\| + \|y\|), \\ \|w(t, \psi)\| &\leq q(t)\|\psi\|_C, \end{aligned}$$

and

$$\|h(t, \psi)\| \leq r(t)\|\psi\|_C,$$

for every  $t \in [0, b]$ ,  $\psi \in C$  and  $x, y \in Y$ . Then any mild solution of (1.1)-(1.2) is bounded on  $[-r, b]$  if

$$\lambda = \int_0^b Lr(\sigma) \exp\left(\int_0^\sigma [Rp(\tau) + Kq(\tau)d\tau]\right) d\sigma < 1.$$

**Proof:** Let

$$\begin{aligned} x(t) &= B^{-1}T(t)B\phi(0) \\ &+ \int_0^t B^{-1}T(t-s)f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) ds, \quad t \in [0, b], \end{aligned} \tag{4.10}$$

$$x(t) = \phi(t), \quad t \in [-r, 0] \tag{4.11}$$

be the mild solution of (1.1)-(1.2). Using condition (3.3) and the assumption of theorem, in the equation (4.10), we get

$$\begin{aligned} \|x(t)\|_X &\leq \|B^{-1}T(t)B\|\|\phi(0)\|_X \\ &+ \int_0^t \|B^{-1}T(t-s)\| \left\| f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) \right\| ds \\ &\leq M\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds \end{aligned} \tag{4.12}$$

*Case1:* Suppose  $t \geq r$ . Then for every  $\theta \in [-r, 0]$  we have  $t + \theta \geq 0$ . For such  $\theta$ 's from (4.12) we have,

$$\begin{aligned} \|x(t + \theta)\|_X &\leq M\|\phi\|_C + \int_0^{t+\theta} Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau \right. \\ &\quad \left. + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds \\ &\leq M\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau \right. \\ &\quad \left. + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds, \end{aligned}$$

which yields,

$$\|x_t\|_C \leq M\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds. \quad (4.13)$$

*Case2:* Suppose  $0 \leq t < r$ . Then for every  $\theta \in [-r, -t]$  we have  $t + \theta < 0$ . For such  $\theta$ 's we get,

$$\|x(t + \theta)\|_X = \|\phi(t + \theta)\|_X$$

which yields,

$$\|x_t\|_C = \|\phi\|_C \quad (4.14)$$

For  $\theta \in [-t, 0]$ ,  $t + \theta \geq 0$  then we have as in the case 1,

$$\|x_t\|_C \leq M\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds. \quad (4.15)$$

Thus for every  $\theta \in [-r, 0]$ , ( $0 \leq t < r$ ), from (4.14) and (4.15) we have

$$\begin{aligned} \|x_t\|_C &\leq (M + 1)\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau \right. \\ &\quad \left. + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds. \end{aligned} \quad (4.16)$$

Hence for every  $t \in [0, b]$ , from (4.13) and (4.16), we obtain

$$\begin{aligned} \|x_t\|_C &\leq (M + 1)\|\phi\|_C + \int_0^t Rp(s) \left[ \|x_s\|_C + \int_0^s Kq(\tau)\|x_\tau\|_C d\tau \right. \\ &\quad \left. + \int_0^b Lr(\tau)\|x_\tau\|_C d\tau \right] ds. \end{aligned} \quad (4.17)$$

Thanks to Pachpatte integral inequality given in Lemma 2.1 and applying it to the inequality (4.17) with  $z(t) = \|x_t\|_C$ , we get

$$\begin{aligned} \|x_t\|_C &\leq \frac{(M + 1)\|\phi\|_C}{1 - \lambda} \exp\left(\int_0^t [Rp(s) + Kq(s)]ds\right) \\ &\leq \frac{(M + 1)\|\phi\|_C}{1 - \lambda} \exp(b\alpha[R + K]), \quad t \in [0, b], \end{aligned}$$

where,  $\alpha = \max_{t \in [0, b]} \{p(t), q(t)\}$ . This gives that,

$$\|x(t)\|_X \leq \frac{(M + 1)\|\phi\|_C}{1 - \lambda} \exp(b\alpha[R + K]), \quad \text{for every } t \in [-r, b],$$

Therefore,

$$\|x\|_D \leq \frac{(M + 1)\|\phi\|_C}{1 - \lambda} \exp(b\alpha[R + K]).$$

Hence the proof is complete. □

**Remark 4.2.** We observe that, by making similar argument and suitable modifications, one can study the continuous dependence and boundedness of solutions of initial value problem (1.3)-(1.4).

### 5. Applications

As an application of the Theorem 3.1, we shall consider the system (1.1)-(1.2) with the control parameter

$$(Bx(t))' + Ax(t) = Eu(t) + f\left(t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds\right), \quad t \in [0, b], \quad (5.1)$$

$$x(t) = \phi(t), \quad t \in [-r, 0], \quad (5.2)$$

where  $E$  is a bounded linear operator from a Banach space  $U$  into  $Y$  and  $u \in L^2([0, b], U)$ . Then the mild solution of (5.1)-(5.2) is given by

$$\begin{aligned} x(t) &= B^{-1}T(t)B\phi(0) + \int_0^t B^{-1}T(t-s)Eu(s)ds \\ &+ \int_0^t B^{-1}T(t-s)f\left(s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau\right) ds, \quad t \in [0, b], \\ x(t) &= \phi(t), \quad t \in [-r, 0]. \end{aligned}$$

**Definition 5.1.** *The system (5.1)-(5.2) is said to be controllable to the origin if for any given initial function  $\phi \in C$ , there exists a control  $u \in L^2([0, b], U)$  such that the mild solution  $x(t)$  of (5.1)-(5.2) satisfies  $x(b) = 0$ .*

To establish the controllability result we need the following additional hypothesis:

(H<sub>5</sub>) The linear operator  $Q : L^2([0, b], U) \rightarrow Y$  defined by

$$Qu = \int_0^b B^{-1}T(b-s)Eu(s)ds$$

induces an inverse operator  $\tilde{Q}^{-1}$  defined on  $L^2([0, b], U)/kerQ$ , such that the operator  $E\tilde{Q}^{-1}$  is bounded. (For the construction of the operator  $Q$  and its inverse, see [22]).

(H<sub>6</sub>) The condition  $RFb [1 + (\frac{KW}{2} + LH) b] (Rb\|E\tilde{Q}^{-1}\| + 1) < 1$  holds.

**Theorem 5.1.** *If the hypothesis (H<sub>0</sub>) – (H<sub>2</sub>), (H<sub>5</sub>) and (H<sub>6</sub>) are satisfied, then system (5.1)-(5.2) is controllable.*

**Proof:** In the view of the hypothesis (H<sub>5</sub>), for any arbitrary function  $x(\cdot)$  define the control

$$u(t) = -\tilde{Q}^{-1} \left[ B^{-1}T(b)B\phi(0) + \int_0^b B^{-1}T(t-s)f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) ds \right] (t).$$

Using this control we show that the operator  $\Psi$  defined by

$$(\Psi x)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ B^{-1}T(t)B\phi(0) + \int_0^t B^{-1}T(t-s)Eu(s)ds + \int_0^t B^{-1}T(t-s) \\ \times f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) ds, & t \in [0, b]. \end{cases}$$

has a fixed point  $x(\cdot)$  by applying the Banach fixed point theorem. This fixed point is then a mild solution of system (5.1)-(5.2). Substituting  $u(t)$  in the above equation we get,

$$(\Psi x)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ B^{-1}T(t)B\phi(0) - \int_0^t B^{-1}T(t-s)E\tilde{Q}^{-1} [B^{-1}T(b)B\phi(0) \\ + \int_0^b B^{-1}T(b-\tau)f \left( \tau, x_\tau, \int_0^\tau k(\tau, \sigma)w(\sigma, x_\sigma)d\sigma, \right. \\ \left. \int_0^b l(\tau, \sigma)h(\sigma, x_\sigma)d\sigma \right) d\tau] (s)ds \\ + \int_0^t B^{-1}T(t-s) \\ \times f \left( s, x_s, \int_0^s k(s, \tau)w(\tau, x_\tau)d\tau, \int_0^b l(s, \tau)h(\tau, x_\tau)d\tau \right) ds, & t \in [0, b]. \end{cases}$$

Clearly  $x(b) = \Psi(x)(b) = 0$ , which means that the control  $u$  steers the system from the initial function  $\phi$  to the origin in time  $b$ , provided we can obtain a fixed point of the operator  $\Psi$ . The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted.  $\square$

### 6. Example

Consider the following nonlinear mixed Volterra-Fredholm partial integrodifferential equation of the form

$$\begin{aligned} & \frac{\partial}{\partial t}[w(u, t) - w_{uu}(u, t)] - \frac{\partial^2}{\partial u^2}w(u, t) \\ & = P \left( t, w(u, t - r), \int_0^t k_1(t, s)g_1(s, w(u, s - r))ds, \right. \\ & \qquad \qquad \qquad \left. \int_0^b l_1(t, s)h_1(s, w(u, s - r))ds \right), \quad t \in [0, b] \end{aligned} \tag{6.1}$$

$$w(0, t) = w(\pi, t) = 0, \quad 0 \leq t \leq b, \tag{6.2}$$

$$w(u, t) = \phi(u, t), \quad 0 \leq u \leq \pi, \quad -r \leq t \leq 0, \tag{6.3}$$

where  $P : [0, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $k_1, l_1 : [0, T] \times [0, b] \rightarrow \mathbb{R}$ ,  $g_1, h_1 : [0, b] \times \mathbb{R} \rightarrow \mathbb{R}$ ,  $\phi : [0, \pi] \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions.

Take  $X = Y = L^2[0, \pi]$ . Define the operators  $A : D(A) \subset X \rightarrow Y$  and  $B : D(B) \subset X \rightarrow Y$  by

$$Aw = -w'' \text{ and } Bw = w - w'',$$

where domain  $D(A)$  and  $D(B)$  is given by

$$\{w \in X : w, w' \text{ are absolutely continuous, } w'' \in X, w(0) = w(\pi) = 0\}.$$

Then  $A$  and  $B$  can be written respectively as

$$Aw = \sum_{n=1}^{\infty} n^2(w, w_n)w_n, \quad w \in D(A),$$

$$Bw = \sum_{n=1}^{\infty} (1 + n^2)(w, w_n)w_n, \quad w \in D(A),$$

where  $w_n(u) = (\sqrt{2/\pi})\sin nu$ ,  $n = 1, 2, 3, \dots$  is the orthogonal set of vectors of  $A$ . Further, for  $w \in X$  we have

$$B^{-1}w = \sum_{n=1}^{\infty} \frac{1}{1 + n^2}(w, w_n)w_n,$$

$$AB^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2} (w, w_n) w_n,$$

$$T(t)w = \sum_{n=1}^{\infty} \exp\left(\frac{-n^2 t}{1+n^2}\right) (w, w_n) w_n.$$

Define the functions  $f : [0, b] \times C \times Y \times Y \rightarrow Y$ ,  $w, h : [0, b] \times C \rightarrow Y$ ,  $k, l : [0, b] \times [0, b] \rightarrow \mathbb{R}$  as follows

$$\begin{aligned} f(t, \psi, x, y)(u) &= P(t, \psi(-r)u, x(u), y(u)), \\ w(t, \psi)(u) &= g_1(t, \psi(-r)u), \\ h(t, \psi)(u) &= h_1(t, \psi(-r)u), \\ k(t, s) &= k_1(t, s), \quad l(t, s) = l_1(t, s), \end{aligned}$$

for  $t \in [0, b]$ ,  $x, y \in Y$ ,  $\psi \in C$  and  $0 \leq u \leq \pi$ . With these choices of the functions, the equations (6.1)-(6.3) can be formulated as an abstract nonlinear mixed Volterra-Fredholm integrodifferential equation in Banach space  $Y$ :

$$\begin{aligned} (Bx(t))' + Ax(t) &= f\left(t, x_t, \int_0^t k(t, s)w(s, x_s)ds, \int_0^b l(t, s)h(s, x_s)ds\right), \quad t \in [0, b], \\ x(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned}$$

Hence, the Theorem 3.1 can be applied to guarantee the existence of mild solution of the nonlinear mixed Volterra-Fredholm partial integrodifferential equation (6.1)-(6.3).

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