Pedal Curves of Tangent Surfaces of Biharmonic $B$-General Helices according to Bishop Frame in Heisenberg Group $\text{Heis}^3$

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ABSTRACT: In this paper, we study pedal curves of tangent surfaces of biharmonic $B$-general helices according to Bishop frame in the Heisenberg group $\text{Heis}^3$. We give necessary and sufficient conditions for $B$-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group $\text{Heis}^3$. Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, Bishop frame, Heisenberg group, Pedal curve.

Contents

1. Introduction 231

2. The Heisenberg Group $\text{Heis}^3$ 231

3. Biharmonic $B$-General Helices with Bishop Frame In The Heisenberg Group $\text{Heis}^3$ 232

4. Pedal Curves In The Heisenberg Group $\text{Heis}^3$ 233

1. Introduction

Developable surfaces, which can be developed onto a plane without stretching and tearing, form a subset of ruled surfaces, which can be generated by sweeping a line through space. There are three types of developable surfaces: cones, cylinders (including planes) and tangent surfaces formed by the tangents of a space curve, which is called the cuspidal edge of this surface, [3].

In this paper, we study pedal curves of tangent surfaces of biharmonic $B$-general helices according to Bishop frame in the Heisenberg group $\text{Heis}^3$. We give necessary and sufficient conditions for $B$-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group $\text{Heis}^3$. Additionally, we illustrate our main theorem.

2. The Heisenberg Group $\text{Heis}^3$

Heisenberg group $\text{Heis}^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \frac{1}{2} \overline{x}y + \frac{1}{2} \overline{x}z)$$

(2.1)
Heis$^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric $g$ is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$  

The Lie algebra of Heis$^3$ has an orthonormal basis

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z},$$  

for which we have the Lie products

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = [e_3, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$  

We obtain

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = \frac{1}{2} e_3,$$

$$\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = -\frac{1}{2} e_2,$$

$$\nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2} e_1.$$  

### 3. Biharmonic B-General Helices with Bishop Frame In The Heisenberg Group Heis$^3$

Let $\gamma : I \rightarrow \text{Heis}^3$ be a non geodesic curve on the Heisenberg group Heis$^3$ parametrized by arc length. Let $\{T, N, B\}$ be the Frenet frame fields tangent to the Heisenberg group Heis$^3$ along $\gamma$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\nabla_T T = k_1 M_1 + k_2 M_2,$$

$$\nabla_T M_1 = -k_1 T,$$

$$\nabla_T M_2 = -k_2 T,$$

where

$$g(T, T) = 1, \ g(M_1, M_1) = 1, \ g(M_2, M_2) = 1,$$

$$g(T, M_1) = g(T, M_2) = g(M_1, M_2) = 0.$$  

Here, we shall call the set $\{T, M_1, M_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures.
With respect to the orthonormal basis \( \{e_1, e_2, e_3\} \) we can write

\[
T = T^1 e_1 + T^2 e_2 + T^3 e_3,
M_1 = M_1^1 e_1 + M_1^2 e_2 + M_1^3 e_3,
M_2 = M_2^1 e_1 + M_2^2 e_2 + M_2^3 e_3.
\]

To separate a general helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as \( B \)-general helix.

**Theorem 3.1.** Let \( \gamma_B : I \rightarrow \text{Heis}^3 \) be a unit speed biharmonic \( B \)-general helix with non-zero natural curvatures. Then the parametric equation of \( \gamma_B \) are

\[
x_B(s) = \frac{\sin \theta}{(k_1^2 + k_2^2 - \cos \theta)^{\frac{1}{2}}} \sin \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} \sin \left[ \left( k_1^2 + k_2^2 \right) \frac{s}{2} + \zeta_0 \right] + \zeta_2,

y_B(s) = -\frac{\sin \theta}{(k_1^2 + k_2^2 - \cos \theta)^{\frac{1}{2}}} \cos \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} \sin \left[ \left( k_1^2 + k_2^2 \right) \frac{s}{2} + \zeta_0 \right] + \zeta_3,

z_B(s) = (\cos \theta) s + \frac{\sin^2 \theta}{(k_1^2 + k_2^2 - \cos \theta)^{\frac{1}{2}}} \cdot \frac{s^2}{2} - \frac{\sin 2 \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right) \frac{s}{2} + \zeta_0}{4 (k_1^2 + k_2^2 - \cos \theta)^{\frac{1}{2}}}

- \frac{\zeta_4 \sin \theta}{(k_1^2 + k_2^2 - \cos \theta)^{\frac{1}{2}}} \cos \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} \sin \left[ \left( k_1^2 + k_2^2 \right) \frac{s}{2} + \zeta_0 \right] + \zeta_4,
\]

where \( \zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \) are constants of integration, \([10]\).

**4. Pedal Curves In The Heisenberg Group Heis\(^3\)**

The purpose of this section is to study pedal curves of tangent developable of biharmonic \( B \)-general helices with Bishop frame in the Heisenberg group \( \text{Heis}^3 \).

The tangent surface of \( \gamma_B \) is a ruled surface

\[
\mathcal{R}(s,u) = \gamma_B(s) + uT(s).
\]

Let \( \mathcal{R} \) be a developable ruled surface given by equation (4.1) in \( \text{Heis}^3 \). Since the tangent plane is constant along rulings of \( \mathcal{R} \), it is clear that the pedal of \( \mathcal{R} \) is a curve. Thus, for the pedal of \( \mathcal{R} \), we can write

\[
\check{\gamma}(s) = \gamma_B(s) + \Pi(s) T(s),
\]

where \( \Pi(s) \) is the distance between the points \( \gamma(s) \) and \( \check{\gamma}(s) \).
Theorem 4.1. Let $\gamma_0 : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix and $\hat{\gamma}$ its pedal curve. Then, the parametric equations of pedal curve are

\[ x_\gamma(s) = \frac{\sin \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \sin[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] \]
\[ + \Pi(s) \sin \theta \cos[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] + \zeta_2, \]

\[ y_\gamma(s) = -\frac{\sin \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \cos[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] \]
\[ + \Pi(s) \sin \theta \sin[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] + \zeta_3, \]

\[ z_\gamma(s) = (\cos \theta) s + \frac{\sin^2 \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \sin\left(\frac{1}{2} \sin 2[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] \right) \]
\[ - \frac{\zeta_1 \sin \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \cos[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] + \Pi(s) \cos \theta \]
\[ + \frac{\Pi(s) \sin^2 \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \sin^2[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] \]
\[ + \Pi(s) \zeta_4 \sin \theta \sin[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0] + \zeta_4, \]

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration and $\Pi(s)$ is the distance between the points $\gamma(s)$ and $\hat{\gamma}(s)$.

Proof: From orthonormal basis (2.2) and (3.8), we obtain

\[ T = (\sin \theta \cos[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0], \sin \theta \sin[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0], \]
\[ \cos \theta + \frac{\sin^2 \theta}{(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}}} \sin^2[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0], \]

\[ + \zeta_4 \sin \theta \sin[(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta)^{\frac{1}{2}} s + \zeta_0]), \]

where $\zeta_1$ is constant of integration.

Using above equation, we have (4.2), the theorem is proved. □

Theorem 4.2. Let $\gamma_0 : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix
and $\bar{\gamma}$ its pedal curve. Then the equation of pedal curve is

$$
\bar{\gamma}(s) = \left[ \sin \theta \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} \sin \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0
$$

$$
+ \Pi(s) \sin \theta \cos \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0] + \zeta_2 e_1
$$

$$
+ \left[ - \sin \theta \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} \cos \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_4 e_2
$$

$$
+ \left[ - \sin \theta \left( \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right)^{\frac{1}{2}} \sin \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0 \right] + \zeta_4 e_2
$$

Using above equation and theorem we easily have (4.4).

**Proof:** From section 3, we immediately arrive at

$$
T = \sin \theta \cos \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0] e_1 + \sin \theta \sin \left[ \frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta \right]^{\frac{1}{2}} s + \zeta_0] e_2
$$

$$
+ \cos \theta e_3.
$$

Using above equation and theorem we easily have (4.4).
Fig. 1

Fig. 2

References


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