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Pedal Curves of Tangent Surfaces of Biharmonic $\mathfrak{B}$-General Helices according to Bishop Frame in Heisenberg Group Heis^3

Talat Körpinar and Essin Turhan

ABSTRACT: In this paper, we study pedal curves of tangent surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for $\mathfrak{B}$-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

Key Words: Biharmonic curve, Bishop frame, Heisenberg group, Pedal curve.

Contents

1. Introduction

Developable surfaces, which can be developed onto a plane without stretching and tearing, form a subset of ruled surfaces, which can be generated by sweeping a line through space. There are three types of developable surfaces: cones, cylinders (including planes) and tangent surfaces formed by the tangents of a space curve, which is called the cuspidal edge of this surface, [3].

In this paper, we study pedal curves of tangent surfaces of biharmonic $\mathfrak{B}$-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for $\mathfrak{B}$-general helices to be biharmonic according to Bishop frame. We characterize this pedal curves in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

2. The Heisenberg Group Heis^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z) = (\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \tag{2.1}$$

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Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

We obtain

$$\begin{aligned}
\nabla_{\mathbf{e}_1}\mathbf{e}_1 &= \nabla_{\mathbf{e}_2}\mathbf{e}_2 = \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \\
\nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \\
\nabla_{\mathbf{e}_1}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \\
\nabla_{\mathbf{e}_2}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1.
\end{aligned}$$

3. Biharmonic $\mathfrak{B}$-General Helices with Bishop Frame In The Heisenberg Group Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\
\nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\
\nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
g(\mathbf{T}, \mathbf{T}) &= 1, \ g(\mathbf{M}_1, \mathbf{M}_1) = 1, \ g(\mathbf{M}_2, \mathbf{M}_2) = 1, \\
g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0.
\end{aligned}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures.

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned}$$

To separate a general helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{B}$-general helix.

Theorem 3.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix with non-zero natural curvatures. Then the parametric equation of $\gamma_{\mathfrak{B}}$ are*

$$\begin{aligned} x_{\mathfrak{B}}(s) &= \frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2, \\ y_{\mathfrak{B}}(s) &= -\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3, \\ z_{\mathfrak{B}}(s) &= (\cos\theta)\,s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\ &\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4, \end{aligned} \tag{3.2}$$

where ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 are constants of integration, [10].

4. Pedal Curves In The Heisenberg Group Heis^3

The purpose of this section is to study pedal curves of tangent developable of biharmonic $\mathfrak{B}$-general helices with Bishop frame in the Heisenberg group Heis^3.

The tangent surface of $\gamma_{\mathfrak{B}}$ is a ruled surface

$$\mathcal{R}(s,u) = \gamma_{\mathfrak{B}}(s) + u\mathbf{T}(s). \tag{4.1}$$

Let $\mathcal{R}$ be a developable ruled surface given by equation (4.1) in Heis^3. Since the tangent plane is constant along rulings of $\mathcal{R}$, it is clear that the pedal of $\mathcal{R}$ is a curve. Thus, for the pedal of $\mathcal{R}$, we can write

$$\bar{\gamma}(s) = \gamma_{\mathfrak{B}}(s) + \Pi(s)\mathbf{T}(s),$$

where $\Pi(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

Theorem 4.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix and $\bar{\gamma}$ its pedal curve. Then, the parametric equations of pedal cuve are*

$$
\begin{aligned}
x_{\bar{\gamma}}(s) &= \frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +\Pi(s)\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2, \\
y_{\bar{\gamma}}(s) &= -\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +\Pi(s)\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3, \\
z_{\bar{\gamma}}(s) &= (\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
&\quad -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\Pi(s)\cos\theta \\
&\quad +\frac{\Pi(s)\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin^2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +\Pi(s)\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_4,
\end{aligned}
\tag{4.2}
$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration and $\Pi(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

Proof: From orthonormal basis (2.2) and (3.8), we obtain

$$
\begin{aligned}
\mathbf{T} &= (\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0],\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0], \\
&\quad \cos\theta+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin^2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
&\quad +\zeta_1\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]),
\end{aligned}
\tag{4.3}
$$

where ζ_1 is constant of integration.

Using above equation, we have (4.2), the theorem is proved. □

Theorem 4.2. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic $\mathfrak{B}$-general helix* *and $\bar{\gamma}$ its pedal curve. Then the equation of pedal cuve is*

$$
\begin{aligned}
\bar{\gamma}(s) \quad = \quad & [\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& +\Pi(s)\sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2]\mathbf{e}_1 \\
& +[-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0] \\
& +\Pi(s)\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3]\mathbf{e}_2 \qquad (4.4) \\
& +[-[\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_2] \\
& [-\frac{\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\zeta_3] \\
& +(\cos\theta)s+\frac{\sin^2\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}(\frac{s}{2}-\frac{\sin 2[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]}{4(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}} \\
& -\frac{\zeta_1\sin\theta}{(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}}\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]+\Pi(s)\cos\theta+\zeta_4]\mathbf{e}_3,
\end{aligned}
$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration and $\Pi(s)$ is the distance between the points $\gamma(s)$ and $\bar{\gamma}(s)$.

Proof: From section 3, we immediately arrive at

$$
\begin{aligned}
\mathbf{T} \quad = \quad & \sin\theta\cos[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_1+\sin\theta\sin[(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta)^{\frac{1}{2}}s+\zeta_0]\mathbf{e}_2 \\
& +\cos\theta\mathbf{e}_3. \qquad (4.5)
\end{aligned}
$$

Using above equation and theorem we easily have (4.4). □

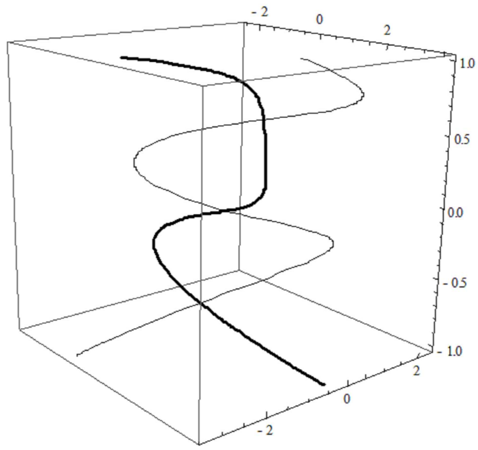

Fig. 1

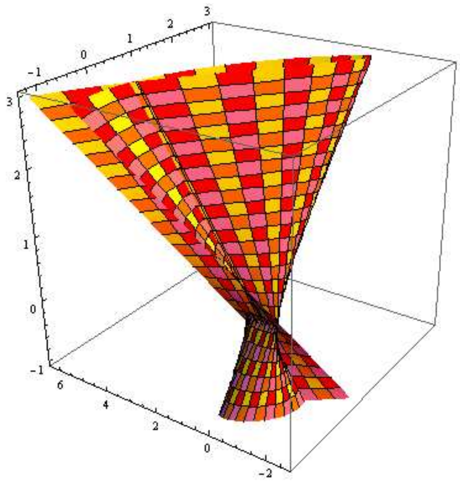

Fig. 2

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Talat Körpinar
Fırat University, Department of Mathematics
23119, Elazığ, Turkey
E-mail address: talatkorpinar@gmail.com

and

Essin Turhan
Fırat University, Department of Mathematics
23119, Elazığ, Turkey
E-mail address: essin.turhan@gmail.com