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Multiplicity Results for a Fourth Order Quasi-Linear Problems

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ABSTRACT: In this paper we prove the existence of nontrivial solutions to a p-biharmonic elliptic equations with Navier boundary conditions. The results are proved by applying minimax arguments and Morse theory.

Key Words: p-biharmonic operator; Variational method; Critical group; Multiple solutions; Morse theory.

Contents

1. Introduction

We consider the following problem with Navier boundary conditions

$$(\mathcal{P}) \begin{cases} \Delta_p^2 u = f(x,u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 1$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$, is the p-biharmonic operator, $1 < p < \infty$ and
$f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the subcritical growth condition:

(F_0) $\qquad |f(x,t)| \leq c(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,$

for some $c > 0$, and $1 \leq q < p^*$ where $p^* = \frac{Np}{N-2p}$ if $1 < 2p < N$ and $p^* = +\infty$ if $N \leq 2p$.

Observe that, if $f(x,0) \equiv 0$, then the problem $(\mathcal{P})$ has a trivial solution $u \equiv 0$. We are interested in finding multiple nontrivial solutions of $(\mathcal{P})$ in the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, equipped with the norm

$$\| u \| = (\int_\Omega |\Delta u|^p \, dx)^{\frac{1}{p}}.$$

It is well known that the functional $\Phi : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to \mathbb{R}$

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p \, dx - \int_\Omega F(x,u) \, dx,$$

2000 *Mathematics Subject Classification:* 35A15, 37B30 with $F(x,t)=\int_0^t f(x,s)\,ds$, is of class C^1 and

$$\langle \Phi'(u),\varphi\rangle = \int_\Omega |\Delta u|^{p-2}\Delta u \Delta\varphi\,dx - \int_\Omega f(x,u)\varphi\,dx$$

for every $\varphi\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Moreover, the critical points of Φ are weak solutions for $(\mathcal{P})$. Notice that for the eigenvalue problem

$$\begin{cases} \Delta_p^2 u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u=\Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

as for the p-Laplacian eigenvalue problem with Dirichlet boundary data,

$$\lambda_n = \inf_{K\in A_n}\sup_{u\in K}\int_\Omega |\Delta u|^p\,dx, \quad n=1,2,...$$

is the sequence of eigenvalues, where

$$A_n=\{K\subset N : K \quad \text{is compact, symmetric and } \gamma(K)\geq n\}$$

and

$$N=\{u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) : \int_\Omega |u|^p\,dx = 1\}.$$

Here $\gamma(K)$ indicate the genus of K. It has been recently proved by P. Drábek and M. Ôtani [4] that (1.1) has the least eigenvalue

$$\lambda_1(p)=\inf\{\int_\Omega |\Delta u|^p\,dx : u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega), \int_\Omega |u|^p\,dx = 1\}, \tag{1.2}$$

which is simple, positive and has an associated normalized eigenfunction φ_1 which is positive in Ω. It is also known, (see [4]), that there exists $\delta>0$ such that $(\lambda_1(p),\lambda_1(p)+\delta)$ that not contain other eigenvalues.

Remark 1.1. *Let $V=span\{\varphi_1\}$ be the eigenspace associated with λ_1, where $\|\varphi_1\|=1$. Taking a subspace $W\subset W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ complementing V, that is, $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)=V\oplus W$, there exists $\hat\lambda>\lambda_1$ with*

$$\int_\Omega |\Delta u|^p\,dx \geq \hat\lambda\int_\Omega |u|^p\,dx \tag{1.3}$$

for each $u\in W$ (in case $p=2$, one may take $\hat\lambda=\lambda_2$).

The existence of solutions of p-biharmonic equation has been studied by several authors see [1,4,9,11] and the reference therein.
It will be seen that critical groups and Morse Theory, developed by Chang [3] or Mawhin and Willem [10], are the main tools used to solve our problem. The main point in this theory is to introduce the critical groups of an isolated critical point. With this aim, we need to suppose a conditions that give us information about the

behavior of the perturbed function $f(x,t)$ or its primitive $F(x,t) = \int_0^t f(x,s)\,ds$ near infinity and near zero. More precisely, the following conditions are assumed:
(F_1) $\lim\limits_{|t|\to\infty} [tf(x,t) - pF(x,t)] = \infty$ uniformly for a.e. $x \in \Omega$,
(F_2) $\lim\limits_{|t|\to\infty} [tf(x,t) - pF(x,t)] = -\infty$ uniformly for a.e. $x \in \Omega$,
(F_3) there exists $\delta \in (0, \lambda_2 - \lambda_1)$ such that
$\limsup\limits_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_1 + \delta$, uniformly a.e. $x \in \Omega$,
(F_4) $\lim\limits_{|t|\to\infty} [\int_\Omega F(x, t\varphi_1)\,dx - \frac{1}{p}|t|^p) = \infty$,
(F_5) there exist $\mu \in (0,p)$, $\gamma > 0$ and α a constant non positive, such that
$0 < \mu F(x,t) \leq tf(x,t)$, for a.e. $x \in \Omega$, $0 <| t |\leq \gamma$,
and

$$\liminf_{|t|\to 0} \frac{\mu F(x,t) - tf(x,t)}{|t|^p} \geq \alpha > \lambda_1(\frac{\mu}{p} - 1) \quad \text{uniformly a.e. } x \in \Omega.$$

The main result reads as follows.

Theorem 1.1. *Suppose (F_0), $(F_3) - (F_5)$ and (F_1) or (F_2). Then the problem $(\mathcal{P})$ has at least nontrivial solution.*

The second purpose of this paper is to show the existence of at least two nontrivial solutions of problem $(\mathcal{P})$ under the following assumptions:
(F_6) $\exists R > 0$, $\bar{\lambda} \in]\lambda_1, \hat{\lambda}[$ such that for all $| t |\leq R$ and $x \in \Omega$

$$\lambda_1 \mid t \mid^p \leq pF(x,t) \leq \bar{\lambda} \mid t \mid^p .$$

(F_7) $\lim\limits_{|t|\to\infty} (F(x,t) - \frac{\lambda_1}{p} \mid t \mid^p) = -\infty$.
Now, we can state the following result.

Theorem 1.2. *Under (F_0), (F_6) and (F_7), the problem $(\mathcal{P})$ has at least two nontrivial solutions.*

Remark 1.2. *In Theorem 1.4 [9], the authors established the existence of at least two nontrivial solutions of problem $(\mathcal{P})$, under (F_0), (F_6) and the following hypothesis.*
(F_7') $\lim\limits_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_1$.
Note that our condition (F_7) is weaker than (F_7').

For finding critical points of Φ, by applying minimax methods, we will use the following compactness condition, introduced by Cerami [2], which is a generalization of the classical Palais-Smale type (PS).

Definition 1.1. *Given $c \in \mathbb{R}$, we say that $\Phi \in C^1(X, \mathbb{R})$ satisfies the condition (C_c), if*
(i) Every bounded sequence $(u_n) \subset X$ such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ has a

convergent subsequence,
(ii) there exists constants δ, R, $\alpha > 0$ *such that*
$$\| \Phi'(u) \|_{X'} \| u \|_X \geq \alpha, \quad \forall u \in \Phi^{-1}([c-\delta, c+\delta]) \quad \text{with } \| u \|_X \geq R.$$
If Φ *satisfies condition* (C_c) *for every* $c \in \mathbb{R}$, *we simply say that* Φ *satisfies* (C).

The paper is organized as follows. In section 2 we introduce some auxiliary results. In Section 3, we will prove the existence of at least nontrivial solution by combining the minimax method and Morse theory. In section 4, we will give the proof of Theorem 1.2.

2. Critical Groups

In this section, we investigate the critical groups at zero and at a mountain pass type. To proceed, some concepts are needed. Let X denote the generalized Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, given a $\Phi \in C^1(X, \mathbb{R})$. For β, $c \in \mathbb{R}$, we set

$$\Phi^\beta = \{x \in X : \Phi(x) \leq \beta\},$$
$$K = \{x \in X : \Phi'(x) = 0\},$$
$$K_c = \{x \in K : \Phi(x) = c\}.$$

Denote by $H_q(A, B)$ the q-th homology group of the topological pair (A, B) with integer coefficient. The critical groups of Φ at an isolated critical point $u \in K_c$ are defined by $C_q(\Phi, u) = H_q(\Phi^c \cap U, (\Phi^c \setminus \{u\}) \cap U), \quad q \in \mathbb{Z}$,
where U is a closed neighborhood of u.
Moreover, it is known that $C_q(\Phi, u)$ is independent of the choice of U due to the excision property of homology. We refer the readers to [3,10] for more information.

Recall that in the case when Φ satisfies the Cerami condition and for $[a, b] \subset \mathbb{R} \cup \{\infty\}$ the critical set $K_a^b = \{x \in Y; a \leq \Phi(x) \leq b, \Phi'(x) = 0\}$ is finite, we have the following Morse relations between the Morse critical groups and homological characterization of subset sets:

$$H_q(\Phi^a, \Phi^b) = \bigoplus_{K_a^b} C_q(\Phi, u). \tag{2.1}$$

Now, we will show that the critical groups of Φ at zero are trivial.

Lemma 2.1. *Assume* (F_0) *and* (F_5). *Then* $C_q(\Phi, 0) \cong 0, \quad \forall q \in \mathbb{Z}$.

Proof. Let $B_\rho = \{u \in X, \| u \| \leq \rho\}$, $\rho > 0$ which is to be chosen later. The idea of the proof is to construct a retraction of $B_\rho \setminus \{0\}$ to $B_\rho \cap \Phi^0 \setminus \{0\}$ and to prove that $B_\rho \cap \Phi^0$ is contractible in itself. For this purpose, we need to analyze the local properties of Φ near zero. Thus, some technical affirmations must be proved.
Claim 1. Under (F_0) and (F_5), zero is local maximum for the functional $\Phi(su)$, $s \in \mathbb{R}$, for $u \neq 0$.

In fact, it follows from the first condition of (F_5), there exists a constant $c_0 > 0$ such that
$$F(x, t) \geq c_0 \mid t \mid^\mu, \quad \text{for } x \in \Omega, \quad \mid t \mid \leq \gamma. \tag{2.2}$$

Using (F_0) and (2.2), we get

$$F(x,t) \geq c_0 \mid t \mid^{\mu} - c_1 \mid t \mid^{q}, \quad x \in \Omega, \quad t \in \mathbb{R} \tag{2.3}$$

for some $q \in (p, p^*)$ and $c_1 > 0$.
Then, for $u \in X$, $u \neq 0$ and $s > 0$, we have

$$\begin{aligned} \Phi(su) &= \frac{1}{p}s^p \int_{\Omega} |\Delta u|^p \, dx - \int_{\Omega} F(x,(su)) \, dx \\ &\leq \frac{s^p}{p} \parallel u \parallel^p - \int_{\Omega} (c_0 \mid (su) \mid^{\mu} - c_1 \mid (su) \mid^{q}) \, dx \\ &\leq \frac{s^p}{p} \parallel u \parallel^p - c_0 s^{\mu} \parallel u \parallel_{L^{\mu}}^{\mu} + c_1 s^q \parallel u \parallel_{L^q}^{q} . \end{aligned} \tag{2.4}$$

Since $\mu < p < q$, there exists a $s_0 = s_0(u) > 0$ such that

$$\Phi(su) < 0, \quad \text{for all} \;\; 0 < s < s_0. \tag{2.5}$$

Claim 2. There exists $\rho > 0$ such that

$$\frac{d}{ds}\Phi(su) \mid_{s=1} > 0, \tag{2.6}$$

for every $u \in X$ with $\Phi(u) = 0$ and $0 < \parallel u \parallel \leq \rho$.

Indeed, let $u \in X$ be such that $\Phi(u) = 0$. In turn, for (F_5) and (F_0) respectively, we have for $\varepsilon > 0$ sufficiently small that there exists
$r = r(\varepsilon) > 0$ such that

$$\mu F(x,u) - f(x,u)u \geq (\alpha - \varepsilon) \mid u \mid^p, \quad \text{a.e. } x \in \Omega \text{ and } \mid u \mid \leq r,$$

and

$$\mu F(x,u) - f(x,u)u \geq -c_{\varepsilon} \mid u \mid^q, \quad \text{a.e. } x \in \Omega \text{ and } \mid u \mid > r,$$

for some $q \in (p, p^*)$ and $c_{\varepsilon} > 0$.

Define $\Omega_r(u) = \{x \in \Omega : \; \mid u \mid > r\}$ and $\Omega^r(u) = \{x \in \Omega : \; \mid u \mid \leq r\}$.
Denote by $\langle .,. \rangle$ the duality pairing between X and X'. Then, since $\Phi(u) = 0$, by

virtue of (1.2), we find

$$\begin{aligned}
\frac{d}{ds}\Phi(su)\mid_{s=1} &= \langle \Phi'(su),u\rangle \mid_{s=1}\\
&= \int_\Omega |\Delta u|^p\,dx - \int_\Omega f(x,u)u\,dx,\\
&= (1-\frac{\mu}{p})\int_\Omega |\Delta u|^p\,dx + \int_{\Omega^r(u)} (\mu F(x,u)-f(x,u)u)\,dx\\
&+ \int_{\Omega_r(u)} (\mu F(x,u)-f(x,u)u)\,dx,\\
&\geq (1-\frac{\mu}{p})\parallel u\parallel^p + (\alpha-\varepsilon)\int_{\Omega^r(u)} \mid u\mid^p\ dx - c_\varepsilon\int_{\Omega_r(u)}\mid u\mid^q\ dx,\\
&\geq (1-\frac{\mu}{p})\parallel u\parallel^p + \frac{(\alpha-\varepsilon)}{\lambda_1}\int_\Omega \mid\Delta u\mid^p\ dx - c_\varepsilon c^q\parallel u\parallel^q,\\
&\geq (1-\frac{\mu}{p})\parallel u\parallel^p + \frac{(\alpha-\varepsilon)}{\lambda_1}\parallel u\parallel^p - c_\varepsilon c^q\parallel u\parallel^q,\\
&\geq \theta\parallel u\parallel^p - C_\varepsilon\parallel u\parallel^q,
\end{aligned}$$

where $\theta = (1-\frac{\mu}{p}+\frac{\alpha}{\lambda_1}-\frac{\varepsilon}{\lambda_1})$, $c>0$ is the embedding constant for $X\hookrightarrow L^q(\Omega)$ and $C_\varepsilon = c_\varepsilon c^q$.
Since $p<q$, the inequality (2.6) follows for ε small enough such that $\theta>0$.
Claim 3. For all $u\in X$ with $\Phi(u)\leq 0$ and $\parallel u\parallel\leq\rho$, we have

$$\Phi(su)\leq 0, \quad \text{for all} \quad s\in(0,1). \tag{2.7}$$

Indeed, given $\parallel u\parallel\leq\rho$ with $\Phi(u)\leq 0$, assume by contradiction that there exists some $s_0\in(0,1]$ such that $\Phi(s_0u)>0$. Thus, by the continuity of Φ, there exists an $s_1\in(s_0,1]$ such that $\Phi(s_1u)=0$. Choose $s_2\in(s_0,1]$ such that $s_2=\min\{s\in[s_0,1];\Phi(su)=0\}$. It is easy to see that $\Phi(su)\geq 0$ for each $s\in[s_0,s_2]$. Taking $u_1=s_2u$, it is clear that

$$\Phi(su)-\Phi(s_2u)\geq 0 \quad \text{implies that} \quad \frac{d}{ds}\Phi(su)\mid_{s=s_2} = \frac{d}{ds}\Phi(su_1)\mid_{s=1}\leq 0.$$

This is a contradiction with (2.6). The proof of the claim is completed.

Let us fix $\rho>0$ such that zero is the unique critical point of Φ in B_ρ. First, by taking the mapping $h:[0,1]\times(B_\rho\cap\Phi^0)\to B_\rho\cap\Phi^0$ as

$$h(s,u)=(1-s)u,$$

we have that $B_\rho\cap\Phi^0$ is contractible in itself.

Now, we prove that $(B_\rho\cap\Phi^0)\setminus\{0\}$ is contractible in itself too. For this purpose, define a mapping $T:B_\rho\setminus\{0\}\to(0,1]$ by

$$\begin{gathered}
T(u)=1, \quad \text{for} \quad u\in(B_\rho\cap\Phi^0)\setminus\{0\},\\
T(u)=s, \quad \text{for} \quad u\in B_\rho\setminus\Phi^0 \quad \text{with} \quad \Phi(su)=0,\ s<1.
\end{gathered} \tag{2.8}$$

From the relations (2.5)-(2.7), the mapping T is well defined and if $\Phi(u) > 0$ then there exists an unique $T(u) \in (0, 1)$ such that

$$\begin{aligned} \Phi(su) &< 0, \quad \forall s \in (0, T(u)), \\ \Phi(T(u)u) &= 0, \\ \Phi(su) &> 0, \quad \forall s \in (T(u), 1). \end{aligned} \tag{2.9}$$

Thus, using (2.6) and (2.9) and the Implicit Function Theorem we get that the mapping T is continuous.

Next, we define a mapping $\eta : B_\rho \setminus \{0\} \to (B_\rho \cap \Phi^0) \setminus \{0\}$ by

$$\begin{aligned} \eta(u) &= T(u)u, \quad u \in B_\rho \setminus \{0\} \text{ with } \Phi(u) \geq 0, \\ \eta(u) &= u, \quad u \in B_\rho \setminus \{0\} \text{ with } \Phi(u) < 0. \end{aligned}$$

Since $T(u) = 1$ as $\Phi(u) = 0$, the continuity of η follows from the continuity of T.

Obviously, $\eta(u) = u$ for $u \in (B_\rho \cap \Phi^0) \setminus \{0\}$. Thus, η is a retraction of $B_\rho \setminus \{0\}$ to $(B_\rho \cap \Phi^0) \setminus \{0\}$. Since X is infinite dimensional, $B_\rho \setminus \{0\}$ is contractible in itself. By the fact that retracts of contractible space are also contractible, $(B_\rho \cap \Phi^0) \setminus \{0\}$ is contractible in itself.

From the homology exact sequence, one has

$$H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \quad \forall q \in \mathbb{Z}.$$

Hence

$$C_q(\Phi, 0) = H_q(B_\rho \cap \Phi^0, (B_\rho \cap \Phi^0) \setminus \{0\}) = 0, \quad \forall q \in \mathbb{Z}.$$

The proof of lemma 2.1 is completed. □

We will use the following lemma, which is proved with (PS) condition see for example [10].

Lemma 2.2. *Assume $\Phi \in C^1(X, \mathbb{R})$, there exists $u_0 \in X$, $u_1 \in X$ and a bounded open neighborhood Ω of u_0 such that $u_1 \in X \setminus \bar{\Omega}$ and*

$$\max(\Phi(u_0), \Phi(u_1)) < \inf \Phi_{\partial\Omega}.$$

Let $\Gamma = \{g \in C([0, 1], X) : g(0) = u_0,\ g(1) = u_1\}$ *and*

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} \Phi(g(t)).$$

If Φ satisfies the (C) condition over X and if each critical point of Φ in K_c is isolated in X, then there exists $u \in K_c$ such that $\dim C_1(\Phi, u) \geq 1$.

Proof. Let $\varepsilon > 0$ be such that $c - \varepsilon > \max(\Phi(u_0), \Phi(u_1))$ and c is the only critical value of Φ in $[c - \varepsilon, c + \varepsilon]$. Consider the exact sequence

$$\ldots \to H_1(\Phi^{c+\varepsilon}, \Phi^{c-\varepsilon}) \xrightarrow{\partial} H_0(\Phi^{c-\varepsilon}, \emptyset) \xrightarrow{i_*} H_0(\Phi^{c+\varepsilon}, \emptyset) \to \ldots$$

where ∂ is the boundary homomorphism and i_* is induced by the inclusion mapping $i : (\Phi^{c-\varepsilon}, \emptyset) \to (\Phi^{c+\varepsilon}, \emptyset)$. The definition of c implies that u_0 and u_1 are path connected in $\Phi^{c+\varepsilon}$ but not in $\Phi^{c-\varepsilon}$. Thus, $\ker i_* \neq \{0\}$ [3,10] and, by exactness, $H_1(\Phi^{c+\varepsilon}, \Phi^{c-\varepsilon}) \neq \{0\}$. Using (2.1), we deduce that $\dim C_1(\Phi, u) \geq 1$. The lemma 2.2 is proved. □

3. Proof of Theorem 1.1

The proof is based on the following minimax theorem due to the first author [5, Theorem 3.5], with Cerami condition.

Theorem 3.1. *Let Φ be a C^1 functional on X satisfying (C), let Q be a closed connected subset of X such that $\partial Q \cap \partial(-Q) \neq \emptyset$ and $\beta \in \mathbb{R}$. Assume that*

1. *for every $K \in A_2$, there exists v_K such that*
   $\Phi(v_K) \geq \beta$ *and* $\Phi(-v_K) \geq \beta$,
2. $a = \sup_{\partial Q} \Phi < \beta$,
3. $\sup_{\partial Q} \Phi < \infty$.

Then Φ has a critical value $c \geq \beta$ given by

$$c = \inf_{h \in \Gamma} \sup_{x \in Q} \Phi(h(x)),$$

where $\Gamma = \{h \in C(X, X) : h(x) = x \text{ for every } x \in \partial\Omega\}$.

We will establish the compactness condition under the condition (F_0), (F_3) and (F_1). The proof is similar for (F_0), (F_3) and (F_2).

Lemma 3.1. *Assume (F_0), (F_3) and (F_1). Then Φ satisfies the condition (C).*

Proof. (i) First, we verify that the Palais-Small condition is satisfied on the bounded subsets of X. Let $(u_n) \subset X$ be bounded such that

$$\Phi'(u_n) \to 0 \ \text{ and } \ \Phi(\mathrm{u_n}) \to \mathrm{c}, \quad \mathrm{c} \in \mathbb{R}. \tag{3.1}$$

Passing if necessary to a subsequence, we may assume that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } X, \\ u_n &\to u \quad \text{strongly in } L^p(\Omega), \\ u_n(x) &\to u(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we have $\langle \Phi'(u_n), u_n - u \rangle \to 0$, or equivalently

$$\int_\Omega |\Delta u_n|^{p-2} \Delta u_n \Delta(u_n - u)\, dx - \int_\Omega f(x, u_n)(u_n - u)\, dx \to 0. \tag{3.3}$$

By the Hölder inequality, we obtain

$$\int_\Omega f(x,u_n)(u_n-u)\,dx \to 0. \tag{3.4}$$

Thus, it follows from (3.3) and (3.4) that $\langle \Delta_p^2 u_n, u_n-u\rangle \to 0$. Since , Δ_p^2 is of type S^+ (see [4]), we deduce that $u_n \to u$ strongly in X.

Now, we will show that (ii) is satisfied for every $c \in \mathbb{R}$. By contradiction, let $(u_n) \subset X$ such that

$$\Phi(u_n) \to c, \quad \langle \Phi'(u_n), u_n\rangle \to 0 \ \text{ and } \ \| \mathrm{u_n} \| \to +\infty. \tag{3.5}$$

Therefore,

$$\lim_n \int_\Omega (u_n f(x,u_n) - pF(x,u_n))\,dx = pc. \tag{3.6}$$

Taking $v_n = \frac{u_n}{\|u_n\|}$, clearly v_n is bounded in X. So, there is a function $v \in X$ and a subsequence still denote by (v_n) such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } X,\\ v_n &\to v \quad \text{strongly in } L^p(\Omega),\\ v_n(x) &\to v(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.7}$$

On the other hand, in view (F_0) and (F_3), it follows that

$$F(x,s) \le \frac{\lambda_1+\delta}{p} \mid s \mid^p + b, \quad \forall s \in \mathbb{R}, \quad b \in L^p(\Omega). \tag{3.8}$$

Combining relations (3.5) and (3.8), we obtain

$$\frac{1}{p} \| u_n \|^p - \frac{\lambda_1+\delta}{p} \| u_n \|^p_{L^p} - b \le C, \quad C \in \mathbb{R}.$$

Dividing by $\| u_n \|$ and passing to the limit, we conclude

$$\frac{1}{p} - \frac{\lambda_1+\delta}{p} \| v \|^p_{L^p} - b \le 0,$$

and consequently $v \neq 0$.

Let $\Omega_0 = \{x \in \Omega : v(x) \neq 0\}$, via the result above we have $\mid \Omega_0 \mid > 0$ and

$$\mid u_n(x) \mid \to +\infty, \quad \text{a.e. x} \in \Omega_0. \tag{3.9}$$

Furthermore, (F_0) and (F_1) implies that there exists $M > 0$ and $d \in L^1(\Omega)$ such that

$$sf(x,s) - pF(x,s) \ge -M + d(x), \quad \forall s \in \mathbb{R}, \quad \text{a.e. x} \in \Omega.$$

Hence,

$$\int_\Omega (u_n f(x,u_n) - pF(x,u_n))\,dx \ge \int_{\Omega_0} (u_n f(x,u_n) - pF(x,u_n))\,dx - M \mid \Omega \backslash \Omega_0 \mid - \| d \|_{L^1}.$$

Using (3.9) and Fatou's lemma, one deduce

$$\lim_n \int_\Omega (u_n f(x,u_n) - pF(x,u_n))\, dx = +\infty.$$

This contradicts (3.6). $\square$

Next, we will prove the geometric conditions of Theorem 3.1. Let denote $E(\lambda_1)$ the eigenspace associated to the eigenvalue λ_1

Lemma 3.2. *Under the hypothesis (F_0), (F_3) and (F_4), we have:*
(i) Φ is anticoercive on $E(\lambda_1)$.
(ii) For all $K \in A_2$, there exists $v_K \in K$ and $\beta \in \mathbb{R}$ such that $\Phi(v_K) \geq \beta$ and $\Phi(-v_K) \geq \beta$.

Proof. (i) For every $v \in E(\lambda_1)$, there exists $t \in \mathbb{R}$ such that $v = t\varphi_1$. Therefore, using (F_4), we write

$$\begin{aligned} \Phi(v) &= \frac{|\, t\,|^p}{p} \int_\Omega |\, \Delta\varphi_1 \,|^p\ dx - \int_\Omega F(x,t\varphi_1)\, dx \\ &= -[\int_\Omega F(x,t\varphi_1)\, dx - \frac{|\, t\,|^p}{p}] \to -\infty, \quad \text{as } |\text{ t }| \to \infty. \end{aligned}$$

(ii) By the Ljusternik-Schnirelmann theory, we write

$$\lambda_2 = \inf_{K\in A_2} \sup\{\int_\Omega |\, \Delta u\,|^p\ dx, \int_\Omega |\, u\,|^p\ dx = 1 \text{ and u} \in \text{K}\}.$$

Then, for all $K \in A_2$, and all $\varepsilon > 0$, there exists $v_K \in K$ such that

$$(\lambda_2 - \varepsilon)\int_\Omega |\, v_K \,|^p\ dx \leq \int_\Omega |\, \Delta v_K\,|^p\ dx. \tag{3.10}$$

Indeed, if $0 \in K$, we take $v_K = 0$.
Otherwise, we consider the odd mapping

$$g: K \to K', \quad v \to \frac{v}{\|\, v\,\|_{L^p}}.$$

By the genus properties, we have $\gamma(g(K)) \geq 2$, and by the definition of λ_2, there exist $\omega_K \in K'$ such that

$$\int_\Omega |\, \omega_K\,|^p\ dx = 1 \ \text{ and } \ (\lambda_2 - \varepsilon) \leq \int_\Omega |\, \Delta\omega_\text{K}\,|^\text{p}\ \text{dx}.$$

Thus (3.10) is satisfied by setting $v_K = g^{-1}(\omega_K)$.
On the other hand, the two assumptions (F_0) and (F_3) implies

$$F(x,s) \leq (\frac{\lambda_1 + \delta - 2\varepsilon}{p}) \,|\, s\,|^p + C, \quad \forall s \in \mathbb{R}, \tag{3.11}$$

for some constant $C > 0$. Consequently, one deduce from (3.10) and (3.11) that

$$\begin{aligned}
\Phi(v_K) &\geq \frac{1}{p}\int_\Omega \mid \Delta v_K \mid^p \, dx - (\frac{\lambda_1+\delta-2\varepsilon}{p})\int_\Omega \mid v_K \mid^p \, dx - C \mid \Omega \mid \\
&\geq \frac{1}{p}\int_\Omega \mid \Delta v_K \mid^p \, dx - (\frac{\lambda_2-2\varepsilon}{p})\int_\Omega \mid v_K \mid^p \, dx - C \mid \Omega \mid \\
&\geq \frac{1}{p}(1-\frac{\lambda_2-2\varepsilon}{\lambda_2-\varepsilon})\int_\Omega \mid \Delta v_K \mid^p \, dx - C \mid \Omega \mid .
\end{aligned}$$

The argument is similar for

$$\Phi(-v_K) \geq \frac{1}{p}(1-\frac{\lambda_2-2\varepsilon}{\lambda_2-\varepsilon})\int_\Omega \mid \Delta v_K \mid^p \, dx - C \mid \Omega \mid . \tag{3.12}$$

Finally, for every $K \in A_2$, we have $\Phi(\pm v_K) \geq \beta = -C \mid \Omega \mid$, which completes the proof. □

Proof of Theorem 1.1. Putting $Q = \{t\varphi_1 :\mid t \mid \leq R\}$ for $R > 0$, clearly, Q is closed and compact. In view of lemma 3.2, we can find $t_0 > 0$ such that $\Phi(\pm t_0\varphi_1) < \beta$. In return for lemma 3.2, we may apply Theorem 3.1 to get that Φ has a critical value given by

$$c = \inf_{h\in\Gamma}\sup_{x\in\Omega}\Phi(h(x)) \geq \beta,$$

where $\Gamma = \{h \in C([0,1], X) : h(0) = -t_0\varphi_1, h(1) = t_0\varphi_1\}$. Therefore, there exists at least one critical point u^* of Φ. More precisely, u^* is a mountain Pass type. However, by lemma 2.2, we have $C_1(\Phi, u^*) \ncong 0$. Using lemma 2.1, one deduces $u^* \neq 0$.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2 via the following abstract critical point theorem.

Theorem 4.1 ([8]). *Let X be a real Banach pace and let $\Phi \in C^1(X, \mathbb{R})$ be bounded from below an satisfying the Palais-Smale condition. Assume that Φ has a critical point u which is homologically nontrivial, that is, $C_j(\Phi, u) \neq \{0\}$ for some j, and it is not a minimizer for Φ. Then Φ admits at least three critical points.*

In order to apply Theorem 4.1, we need the following lemmas.
First, we recall the notion of "Local Linking", which was initially introduced by Liu and Li [7].

Definition 4.1. *Let X be a real Banach space such that $X = V \oplus W$, where V and W are closed subspace of X. Let $\Phi : X \to \mathbb{R}$ be a C^1-functional. We say that Φ has a local linking near the origin 0 (with respect to the decomposition $X = V \oplus W$), if there exists $\rho > 0$ such that*

$$\begin{aligned} u \in V :\parallel u \parallel \leq \rho \Longrightarrow \Phi(u) \leq 0, \\ u \in W : 0 <\parallel u \parallel \leq \rho \Longrightarrow \Phi(u) > 0. \end{aligned} \tag{4.1}$$

We now show that our functional Φ has a local linking near the origin with respect to the space decomposition $X = V \oplus W$, according to Remark 1.1.

Lemma 4.1. *Under the condition (F_6), Φ has a local linking near the origin 0.*

Proof. Take $u \in V$. Since V is finite dimensional, it is easily seen that $\| u \| \le \rho \Rightarrow | u(x) | \le R$, $\forall x \in \Omega$ for $\rho > 0$ small. So it follows from (F_6) that for $\| u \| \le \rho$,

$$\begin{aligned} \Phi(u) &= \frac{1}{p}\int_\Omega | \Delta u |^p \ dx - \int_\Omega F(x,u)\,dx \\ &= \frac{\lambda_1}{p}\int_\Omega | u |^p \ dx - \int_\Omega F(x,u)\,dx \\ &= \int_{|u|\le R} [\frac{\lambda_1}{p} | u |^p - F(x,u)]\,dx \le 0. \end{aligned}$$

To prove the second assertion, take $u \in W$. Using (F_0) and (1.3) we have

$$\begin{aligned} \Phi(u) &= \frac{1}{p}\int_\Omega | \Delta u |^p \ dx - \int_\Omega F(x,u)\,dx \\ &= \frac{1}{p}\int_\Omega (| \Delta u |^p - \bar{\lambda} | u |^p)\,dx - \int_{|u|\le R} [F(x,u) - \frac{\bar{\lambda}}{p} | u |^p]\,dx \\ &- \int_{|u|> R} [F(x,u) - \frac{\bar{\lambda}}{p} | u |^p]\,dx \\ &\ge \frac{1}{p}(1 - \frac{\bar{\lambda}}{\hat{\lambda}}) \| u \|^p - c\int_\Omega | u |^s \ dx \\ &\ge \frac{1}{p}(1 - \frac{\bar{\lambda}}{\hat{\lambda}}) \| u \|^p - C \| u \|^s, \end{aligned}$$

where $p < s \le p^*$ and c, C are positive constants. Since $s > p$, it follows that $\Phi(u) > 0$ for $\rho > 0$ sufficiently small. This completes the proof. □

Since $\dim V = 1 < +\infty$, by combining Lemma 4.1 and Theorem 2.1 in [6], we obtain the following result.

Lemma 4.2. *The point 0 is a critical point of Φ and $C_1(\Phi, 0) \neq \{0\}$.*

Lemma 4.3. *If f satisfies (F_7), then Φ is coercive on X; that is $\Phi(u) \to +\infty$ as $\| u \| \to \infty$.*

Proof. Let $G(x,t) = F(x,t) - \frac{\lambda_1}{p} | t |^p$.
Then, from (F_7) we conclude that, for every $M > 0$, there is $R_M > 0$ such that

$$G(x,t) \le -M, \quad \forall \, | t | \ge R_M, \quad \text{a.e. x} \in \Omega. \tag{4.2}$$

By contradiction, let $K \in \mathbb{R}$ and $(u_n) \subset X$ be such that $\| u_n \| \to \infty$ and $\Phi(u_n) \le K$.

Putting $v_n = \frac{u_n}{\|u_n\|}$, one has $\| v_n \| = 1$. For a subsequence, we may assume that for some $v_0 \in X$, we have $v_n \rightharpoonup v_0$ weakly in X, $v_n \to v_0$ strongly in $L^p(\Omega)$, $v_n(x) \to v_0(x)$ a.e. in Ω.

Now, using (4.2) it follows that

$$\begin{aligned}\frac{K}{\| u_n \|^p} &\geq \frac{\Phi(u_n)}{\| u_n \|^p} = \frac{1}{p}\int_\Omega |\Delta v_n|^p \, dx - \int_\Omega \frac{F(x,u_n)}{\| u_n \|^p}\, dx \\ &\geq \frac{1}{p}\int_\Omega (|\Delta v_n|^p - \lambda_1 |v_n|^p)\, dx - \int_\Omega \frac{G(x,u_n)}{\| u_n \|^p}\, dx \\ &\geq \frac{1}{p}\int_\Omega (|\Delta v_n|^p - \lambda_1 |v_n|^p)\, dx + \frac{M_1}{\| u_n \|^p}.\end{aligned}$$

Where $M_1 > 0$. Letting $n \to \infty$, we get

$$1 = \limsup_{n\to\infty} \int_\Omega |\Delta v_n|^p \, dx \leq \lambda_1 \int_\Omega |v_0|^p \, dx.$$

Consequently, $v_0 \neq 0$. Let $\Omega_0 = \{x \in \Omega : v_0(x) \neq 0\}$, via the result above we have $|\Omega_0| > 0$ and

$$|u_n(x)| \to +\infty, \quad \text{a.e. x} \in \Omega_0.$$

Thus, from (F_7) and (1.2) we deduce that

$$\begin{aligned}K \geq \Phi(u_n) &= \frac{1}{p}\int_\Omega (|\Delta u_n|^p - \lambda_1 |u_n|^p)\, dx - \int_\Omega G(x,u_n)\, dx \\ &\geq -\int_\Omega G(x,u_n)\, dx \to +\infty.\end{aligned}$$

This is a contradiction. Hence Φ is coercive on X. □

Proof of Theorem 1.2. By lemma 4.3, Φ satisfies the (PS) condition and bounded from below. By lemma 4.2, the trivial solution $u = 0$ is homological nontrivial and is not a minimizer. The conclusion follows from Theorem 4.1. □

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