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An Inverse Sturm-Liouviile Problem for a Hill's Equation

Münevver Tuz

abstract: In this paper, we consider Hill's equation $-y'' + q(x)y = \lambda y$, where $q \in L^1[0,\pi]$. A Hill equation defined on a semi-infinite interval will in general have a mixed spectrum. The continuous spectrum will in general consist of an infinite number of disjoint finite intervals. Between these intervals, point eigenvalues can exist. It is shown that under suitable hypotheses on the spectrum a full knowledge ot the spectrum leads to a unique determination of the potential function in the Hill's equation. Moreover , it is shown here that if $q(x)$ is prescribed over the interval $\left[\frac{\pi}{2},\pi\right]$, then a single spectrum suffices to determined $q(x)$ on the interval $\left[0,\frac{\pi}{2}\right]$.

Key Words: Hill's equation, inverse problem, spectrum, potential, uniqueness.

Contents

1. Introduction

Direct and inverse spectral problems for differential operators without discontinuities have been thoroughly studied.There arc at least four different versions of the inverse Sturm-Liouville problem.The best known is the one studied by Gel'fand and Levitan [7], in which the potential and the boundary conditions are uniquely determined by the spectral function. This ease has also been investigated by Marchenko [17], Krein [13] and Zikov [24]. In the second version, the potential and the boundary conditions are uniquely determined by two spectra. This case can be reduced to the previous one as shown by Marchenko [17], Levitan [15], Gasymov and Levitan [6] and Zikov [24]. In the third version, the potential is uniquely determined by the boundary conditions and two-possible reduced-spectra. This ease has been studied by Borg [3], Levinson [14] and Hochstadt [9]. The fact, that the boundary conditions are known implies that the lowest eigenvalue in one of the spectra is superfluous. Borg [3], Levinson [14], and Hochstadt [9] have shown that if the boundary conditions and one-possible reduced-spectrum are given, then the potential is uniquely determined, provided it is an even function around the middle of the interval.In this paper we will present a constructive method for the last case. However, some of the results can be extended to the other versions as well. The basic result is an extension of a formula due to Hochstadt [9] for the difference of two potentials and our proof rests on the technique developed by Hochstadt [9]. This formula leads directly to several uniqueness theorems due to Borg [3], Levinson [14], Hochstadt [9] and Hald [8], as well as a new weIl-posedness result.

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Hochstadt [9] has pointed out that his formula leads to an algorithm for solving the inverse Sturm-Liouville problem. The trick is to reduce the problem to solving a system of ordinary differential equations.

The presence of continuities produces essential qualitative modifications in the investigation of the operators. Some aspects of direct and inverse problems for continuous boundary value problems in various formulations have been considered in [4,20] and other works.If $q\ (x)$ is known on the interval $\left[\frac{1}{2},1\right]$, then q(x) on the interval $\left[0,\frac{1}{2}\right]$ is uniquely determined by the above data.Also, if $q\left(\frac{1}{2}-x\right)=q\left(\frac{1}{2}+x\right)$, then q($x$) is uniquely determined on the interval $[0,1]$ by the above data.In [20] the continuous inverse problem is considered on the half-line. Boundary value problems with singularities have been studied in [1,4] for further discussion see the references therein.A representation with transformation operator of problem was obtained in [2], as in [1].

In this paper we shall be concerned with an inverse problem for the Sturm Liouville equation.In this section we will consider two Sturm-Liouville problems with different potentials and different boundary conditions. We will assume that the potentials are even functions around the middle of the interval. The main result is that if the sum of the absolute value of the differences of the eigenvalues of the two Sturm-Liouville problems is finite, then the potentials differ by a continuous function. We consider the diferential equation

$$-y''+q(x)y=\lambda y, \tag{1.1}$$

over the interval $[0,\pi]$ where λ is a real parametre, which may be extended to the real line by periodicity.We denote by $y_1(x,\lambda)$ and $y_2(x,\lambda)$ solutions of (1.1) satisfying the initial conditions $y_1(0,\lambda)=y_2'(0,\lambda)=1$ and $y_1'(0,\lambda)=y_2(0,\lambda)=0$. We call $\Delta(\lambda)=y_2'(\pi,\lambda)+y_1(\pi,\lambda)$ a Hill (or Ljapunov) function. We have we impose two sets of boundary conditions, namely, the periodic boundary conditions

$$y(0)=y(\pi),\quad y'(0)=y'(\pi), \tag{1.2}$$

and the semi-periodic boundary conditions

$$y(0)=-y(\pi),\ \ y'(0)=-y'(\pi). \tag{1.3}$$

Let $\lambda_n\ (n=0,1,..)$ denote the periodic eigenvalues of (1.1) with the boundray conditions (1.2) and $\lambda_0,\lambda_1,\lambda_2,...$ the zeros of $2-\Delta(\lambda)$. $\mu_n\ (n=0,1,...)$ denote the semi-periodic eigenvalues of (1.1) with the boundray conditions (1.3) and also $\mu_1,\mu_2,\mu_3,..$ is the zeros of $2+\Delta(\lambda)$. It is well known that

$$\lambda_0<\mu_0\leq\mu_1<\lambda_1\leq\lambda_2<\mu_2\leq\mu_3<...\ . \tag{1.4}$$

Along with the problem (1.2) and (1.3), we consider yet another problem

$$\begin{gathered}-y''+q(x)y=\lambda y,\\ y(0)=y(\pi)=0. \end{gathered}\tag{1.5}$$

We denote the spectrum of the problem (1.5) by $\nu_1 < \nu_2 < \nu_3 < ...$ It is well known that $\lambda_0 < \mu_0 \leq \nu_1 \leq \mu_1 < \lambda_1 \leq \nu_2 \leq \lambda_2 < \mu_2 \leq \nu_3 \leq \mu_3 < ....$It is known that $4 - \Delta^2(\lambda)$ has only a finite number of simple zeros, we also assume, without loss of generality

$$\int q(x)dx = 0.$$

We shall consider a variation of the abouve inverse problem in that we shall not require any information about a second spectrum but rather suppose $q(x)$ is known almost eweywhere on $\left[\frac{\pi}{2}, \pi\right]$.

In this paper, we consider Hill's equation $-y'' + q(x)y = \lambda y$, where $q \in L^1[0, \pi]$. A Hill equation defined on a semi-infinite interval will in general have a mixed spectrum. The continuous spectrum will in general consist of an infinite number of disjoint finite intervals. Between these intervals, point eigenvalues can exist. It is shown that under suitable hypotheses on the spectrum a full knowledge ot the spectrum leads to a unique determination of the potential function in the Hill's equation. Moreover , it is shown here that if $q(x)$ is prescribed over the interval $\left[\frac{\pi}{2}, \pi\right]$, then a single spectrum suffices to determined $q(x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

2. Statement and Proof of Results

In this section, we state and prove our main results.

Theorem 2.1. *Consider the operator*

$$Lv = -v'' + q(x)v = \lambda v, \tag{2.1}$$

where $q(x + \pi) = q(x)$. First $q(x)$ will be assumed to be integrable on the interval $(0, \pi)$, that is $\int_0^\pi |q(x)|\, dx < \infty$. Let $\{\lambda_n\}$ be the spectrum of L subject to (1.2) and (1.3).

Consider a second operator

$$\tilde{L}v = -v'' + \tilde{q}(x)v = \lambda v, \tag{2.2}$$

where $\tilde{q}$ integrable on $(0, \pi)$ and satisfy

$$\tilde{q}(x) = q(x) \quad on \left(\frac{\pi}{2}, \pi\right). \tag{2.3}$$

Suppose that the spectrum of $\tilde{L}$ subject to (1.2) is also $\{\lambda_n\}$. Then $\tilde{q}(x) = q(x)$ almost everywhere on $(0, \pi)$.

Proof: The operator to be studied here is defined by

$$Lu = -u'' + q(x)u,$$

$$u(0)\cos\alpha + u'(0)\sin\alpha = 0, \tag{2.4}$$

$$\tilde{L}\ u = -u'' + \tilde{q}(x)u,$$

$$u(0)\cos\tilde{\alpha} + u'(0)\sin\tilde{\alpha} = 0, \tag{2.5}$$

where $q(x), \tilde{q}(x) \in L^1[0,\pi]$ and also satisfies the πperiodicity conditions. The angles $\alpha, \tilde{\alpha}$ satisfy

$$0 < \frac{\sin\alpha}{\sin\tilde{\alpha}} < \infty. \tag{2.6}$$

We also assume that the associated Hill's equations

$$u'' + (\lambda - q)\,u = 0, \tag{2.7}$$

and

$$\tilde{u}'' + (\lambda - \tilde{q})\,\tilde{u} = 0$$

have the same discriminant $\Delta(\lambda)$. The discriminant $\Delta(\lambda)$ of (2.4) is defined by $\Delta(\lambda) = u_2'(\pi,\lambda) + u_1(\pi,\lambda)$. u and $\tilde{u}$ denote the corresponding solution $\Delta(\lambda) = 2$. With (2.4) we can associate two linearly independent solutions defined by the initial conditions

$$u_1(0) = \quad u_2'(0) = 1,$$

$$u_1'(0) = \quad u_2(0) = 0. \tag{2.8}$$

According to Floquet's theorem, if λ does not belong to the simple spectrum then (2.7) has solutions satisfying

$$u_+\ (x+\pi) = \rho_+ u_+\ (x)\,,\ u_-\ (x+\pi) = \rho_- u_-\ (x)\,,$$

where ρ_+ and ρ_- are solutions of

$$\rho^2 - \Delta(\lambda)\rho + 1 = 0.$$

Then, u satisfies the Volterra integral equation

$$u(x) = \cos\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}x + \frac{1}{\sqrt{\lambda}}\int_0^\pi \sin\sqrt{\lambda}\,(x-t)\,q(t)u(t)dt. \tag{2.9}$$

From (2.9) follows that for each x, $u(x,\lambda)$ is an entire function of λ of order $\frac{1}{2}$ and asymptotically, we have

$$u = \cos\left(\sqrt{\lambda}x\right) + O\left(\frac{e^{\left|\operatorname{Im}\sqrt{\lambda}\right|x}}{\lambda}\right), \tag{2.10}$$

$$u' = -\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + O\left(e^{\left|\operatorname{Im}\sqrt{\lambda}\right|x}\right), \tag{2.11}$$

(see Titchmarsh [22]). In addition it can be shown that there exists a Kernel $K(x,t)$ continuous on $(0,\pi)$ x $(0,\pi)$ such that every solution of (2.4) can he expressed in the form

$$u(x,\lambda) = \cos\left(\sqrt{\lambda}x\right) + \int_0^x K(x,t)\cos\left(\sqrt{\lambda}t\right)dt. \tag{2.12}$$

The Kernel $K(x,t)$ is obtained from the solution of a certain Goutsat problem associated with the equation

$$K_{xx} - K_{tt} - qK = 0.$$

Results analogous to (2.12) hold for $\tilde{u}$, the solution to (2.7) and (2.8) in terms of a Kernel $\tilde{K}(x,t)$ which has properties similar to $\tilde{K}(x,t)$. It should be mentioned that both K and $\tilde{K}$ depend upon α. I fowever, since we shall assume α is fixed throughout the problem, we shall not denote this dependence explicitly.

Using equation (2.12) and its analogue for $\tilde{u}$, we find that

$$\begin{aligned} u\,\tilde{u} &= \cos^2\left(\sqrt{\lambda}x\right) + \int_0^x \left(K(x,t) + \tilde{K}(x,t)\right)\cos\left(\sqrt{\lambda}t\right)\cos\left(\sqrt{\lambda}x\right)dt \qquad (2.13) \\ &+ \int_0^x K(x,t)\cos\left(\sqrt{\lambda}t\right)dt \int_0^x \tilde{K}(x,s)\cos\left(\sqrt{\lambda}s\right)ds. \end{aligned}$$

By extending the range of $K(x,t)$ and $\tilde{K}(x,t)$ evenly with respect to the second argument, we can rewrite (2.13) as

$$u\,\tilde{u} = \frac{1+\cos\left(2\sqrt{\lambda}x\right)}{2} + \int_0^x \tilde{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau, \tag{2.14}$$

where

$$\begin{aligned} \tilde{K}(x,t) &= 2[K(x,x-2\tau) + \tilde{K}(x,x-2\tau) \qquad (2.15) \\ &+ \int_{-x+2\tau}^{\tau} K(x,s)\tilde{K}(x,s-2\tau)ds + \int_{-x}^{x-2\tau} K(x,s)\tilde{K}(x,s+2\tau)ds]. \end{aligned}$$

We next define the function

$$w(\lambda) = u_1(\pi)\sin\beta - u_2'(\pi)\cos\beta, \tag{2.16}$$

$$u_1(\pi) = -\rho_-\sin\beta\ ,\ u_2'(\pi) = \rho_-\cos\beta. \tag{2.17}$$

The zeros of $w(\lambda)$ are the eigenvalues of L or $\tilde{L}$ subject to (1.2) and hence it has only simple zeros because of the separated boundray conditions. Using (2.10)-(2.11) and (2.16) we obtain the asymptotic formulas

$$w(\lambda) = -\sqrt{\lambda}\sin\beta\sin\sqrt{\lambda} + O\left(e^{|\mathrm{Im}\sqrt{\lambda}|x}\right). \tag{2.18}$$

The asymptotic results (2.8) and (2.9) imply that u and $\tilde{u}$ are entire functions of order $\frac{1}{2}$ and hence $w(\lambda)$ is also an entire function of order $\frac{1}{2}$. In (2.7) multiplying the first equation of (2.10) by $\tilde{u}$ and u. Hence, this equation integrating, we have

$$\int_0^\pi \left(\tilde{u}u'' - \tilde{u}''u\right)dx + \int_0^\pi (\tilde{q} - q)\,\tilde{u}u\,dx = 0. \tag{2.19}$$

But, by virtue of the boundary conditions

$$\int_0^\pi \left(\tilde{u}u'' - \tilde{u}''u\right)dx = \tilde{u}u' - \tilde{u}'u\,|_0^\pi = 0. \tag{2.20}$$

Define

$$Q(x) \equiv \tilde{q} - q, \tag{2.21}$$

$$B(\lambda) = \int_0^{\frac{\pi}{2}} Q(x)\tilde{u}u\,dx. \tag{2.22}$$

From the properties of u and $\tilde{u}$, we see that is an entire function. For $\lambda = \lambda_n$, we see that the first term in (2.20) vanishes and hence

$$B(\lambda_n) = 0. \tag{2.23}$$

But this means that the set of zeros of $w(\lambda)$ is contained in the set of zeros of $B(\lambda)$. In addition using (2.12) in (2.22), we see that for all complex λ

$$|B(\lambda)| \leqq Me^{\left(\frac{1}{2}\right)|2\tau|}, \tag{2.24}$$

for some constant M.

We next consider the quotient as following

$$\psi(\lambda) \equiv \frac{B(\lambda)}{w(\lambda)}. \tag{2.25}$$

Here $\psi(\lambda)$ is an entire function. Using (2.18) and (2.24), we see that

$$|\psi(\lambda)| = O\left(\frac{1}{\sqrt{\lambda}}\right),$$

for large $|\lambda|$. But by Liouville's theorem, we must have

$$\psi(\lambda) = 0 \text{ for all } \lambda \tag{2.26}$$

or

$$B(\lambda) = 0.$$

It should be mentioned at this time that the factor of $\frac{1}{2}$in the exponent in (2.24), which corresponds to the length of the interval over wich the functions q and $\tilde{q}$ agree, is cricial for the validity of our method, for otherwise $|\psi(\lambda)|$ might have an exponential growth at infinity.

Substituting (2.13) into (2.22), we obtain as a consequence of (2.26)

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} Q(x)\left[1+\cos\left(2\sqrt{\lambda}x\right)+\int_0^x \tilde{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)dt\right]dx = 0, \tag{2.27}$$

for all λ. This can be rewriten as

$$\int_0^{\frac{\pi}{2}} Q(x)dx + \int_0^{\frac{\pi}{2}} \cos\left(2\sqrt{\lambda}x\right)\left[Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{K}(x,\tau)dx\right]d\tau = 0. \tag{2.28}$$

Letting $\lambda \to \infty$ for real λ, we see from the Riemann-Lebesque Lemma that we have

$$\int Q(x)dx = 0,$$

and

$$\int_0^{\frac{\pi}{2}} \cos\left(2\sqrt{\lambda}x\right)\left[Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{K}(x,\tau)dx\right]d\tau = 0. \tag{2.29}$$

But from the completeness of the functions $\cos\left(2\sqrt{\lambda}x\right)$,we see that the integrand in(2.29) must also vanish identically. Therefore, we obtain

$$Q(\tau)+\int_\tau^{\frac{\pi}{2}} Q(x)\tilde{K}(x,\tau)dx = 0 \text{ for } 0<\tau<\frac{\pi}{2}. \tag{2.30}$$

But the equation in (2.30) is a homogenous Volterra integral integral equation and has only the zero solution. Thus, we have obtained almost everywhere

$$Q = \tilde{q} - q = 0.$$

Theorem is proved.

□

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Münevver Tuz
Fırat University,
Department of Mathematics