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On Solutions of Generalized Kinetic Equations of Fractional Order

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Abstract: The object of the present paper is to derive the solution of generalized kinetic equations of fractional order involving the Wright generalized Bessel function or Bessel-Mitland function. Results obtained by Chaurasia and Pandey [24] are derived more precisely through results obtained in the present paper in terms of K_4 - function obtained by Sharma [12] belived to be new. Special case, involving the F-function is considered.

Key Words: Fractional kinetic equations, Fractional Calculus, K_4 - function, Laplace transform and Bessel-Mitland function.

Contents

1. Introduction

Fractional Calculus and special functions have contributed a lot to mathematical physics and its various branches. The great use of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of astrophysics/physics.The fractional kinetic equations discussed here can be used to investigate a wide class of known fractional kinetic equations.A spherically symmetric non-rotating, self-gravitating model of star like the sun is assumed to be in thermal equilibrium and hydro static equilibrium. The star is characterized by its mass, luminosity effective surface temperature , radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of nuclear energy generation rate and the opacity.

Consider an arbitrary reaction characterized by a time dependent quantity$N = N(t)$. It is possible to calculate the rate of change by the equation $\frac{dN}{dt} = -d+p$. In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself: $d = d(N)$ or $p = p(N)$. This dependence is complicated since the destruction or production at time t depend not only on N(t)

2000 *Mathematics Subject Classification:* 33C20, 44A10 but also on the past history$N(\tau), \tau < t$, of the variable N. This may be represented by Haubold and Mathai [10]

$$\frac{dN}{dt} = -d(N_t) + p(N_i) \tag{1.1}$$

where N_t denotes the function defined by $N_t(t^*) = N(t - t^*), t^* > 0$.

Haubold and Mathai [10] studied a special case of this equation, when spatial fluctuation or inhomogenities in quantities N(t) are neglected, is given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{1.2}$$

with the initial condition $N_i(t = 0) = N_0$ is the number of density of speices i at time $t = 0$; constant $c_i > 0$, known as standard kinetic equation. A detailed discussion of the above equation is given in Kourganoff [25]. The solution of (1.2) is given by

$$N_i(t) = N_0 e^{-c_i t} \tag{1.3}$$

An alternative form of this equation can be obtained on integration:

$$N(t) - N_0 = c_0 D_t^{-1} N(t), \tag{1.4}$$

where ${}_0D_t^{-1}$ is the standard fractional integral operator. Haubold and Mathai [10] have given the fractional generalization of the standard kinetic equation (1.2) as

$$N(t) - N_0 = c_0^{\nu} D_t^{-\nu} N(t), \tag{1.5}$$

where ${}_0D_t^{-\nu}$ is the well known Riemann-Liouville fractional integral operator (Oldham and Spanier [11]; Samko, Kilbas and Marichev [20]; Miller and Ross [14]) defined by

$${}_0D_t^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^t (t - u)^{\nu - 1} f(u)\, du, R(\nu) > 0. \tag{1.6}$$

The solution of fractional kinetic equation (1.5) is given by (see Haubold and Mathai [10])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}. \tag{1.7}$$

Further Saxena and Mathai and Haubold [15] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which extented the work of Haubold and Mathai [10]. In an another paper, Saxena and Mathai and Haubold [16] developed the solutions for fractional kinetic equations associated with the generalized Mittag-Leffler function and R-function.

The fractional kinetic equations are also studied by many authors viz. Hille and Tamarkin [6], Glockle and Nonnenmacher [26], Saichev and Zaslavsky [3], Saxena et al. [15,16,17], Zaslavsky [7], Saxena and Kalla [19], Chaurasia and Pandey [23,24], Chaurasia and kumar [22] etc. for their importance in the solution of certain physical problems. Recently, Saxena et al. [18] investigated the solution of fractional reaction equation and the fractional diffusion equation. Laplace transform technique is used.

In the present article we introduce and investigate the further computable extensions of the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the Wright generalized Bessel function, are written in compact and easily computable form.

The wright generalized Bessel function and its relationship with some other functions

$$J_\nu^\delta(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\delta k + \nu + 1)}. \tag{1.8}$$

where $z \in c$, $\delta > 0$ and $\nu > -1$. The generalized Wright function yields the following relationships with various classical special functions:

Wright function

$$J_{\nu-1}^\delta(z) = W(-z;\delta;\nu) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\delta k + \nu)}. \tag{1.9}$$

Mittag-Leffler function(Mittag [8] and [9])

$$J_0^\delta(z) = \frac{1}{k!} E_\delta(-z) = \frac{1}{k!} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\delta k + 1)}. \tag{1.10}$$

Generalized Mittag-Leffler function (Wiman [2])

$$J_{\nu-1}^\delta(z) = \frac{1}{k!} E_{\delta,\nu}(-z) = \frac{1}{k!} \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(\delta k + \nu)}. \tag{1.11}$$

Miller and Ross function [14]

$$J_\nu^1(az) = \frac{1}{k!} z^\nu E_z(\nu, -a) = \frac{z^\nu}{k!} \sum_{k=0}^{\infty} \frac{(-az)^k}{\Gamma(k + \nu + 1)}. \tag{1.12}$$

2. Extensions of generalized fractional kinetic equations

Theorem 2.1. *If $\delta > 0$, $\mu + 1 > 0$, $\nu > 0$ and $c > 0$, then for the solution of the equation*

$$N(t) - N_0 J_\mu^\delta(t) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{2.1}$$

there holds the formula

$$N(t) = N_0 E_{\delta,\mu+1}(-t) E_{\nu,k+1}(-c^\nu t^\nu). \tag{2.2}$$

Proof: We know that (Erdélyi et al. [1]) the Laplace transform of the Riemann-Liouville fractional integral is given by

$$L\{{}_0D_t^{-\sigma} f(t); p\} = p^{-\sigma} F(p), \tag{2.3}$$

where

$$F(p) = \int_{u=0}^{\infty} e^{-pu} f(u) du. \tag{2.4}$$

Now taking the Laplace transform of both sides of (2.1), we have

$$L\{N(t); p\} - N_0 L\{J_\mu^\delta(t); p\} = -c^\nu L\{{}_0D_t^{-\nu} N(t); p\}$$

$$N(p) - N_0 \int_0^\infty e^{-pt} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!\Gamma(\delta k + \mu + 1)} dt = -c^\nu p^{-\nu} N(p),$$

$$N(p)[1 + c^\nu p^{-\nu}] = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\delta k + \mu + 1)} \int_0^\infty e^{-pt} t^{k+1-1} dt$$

By virtue of the relationship, we have

$$L^{-1}\{p^{-\rho}\} = \frac{t^{\rho-1}}{\Gamma\rho}, R(\rho) > 0. \tag{2.5}$$

$$N(p)[1 + c^\nu p^{-\nu}] = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma k + 1}{k!\Gamma(\delta k + \mu + 1) p^{k+1}}$$

$$N(p)[1 + c^\nu p^{-\nu}] = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta k + \mu + 1) p^{k+1}}$$

$$N(p) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta k + \mu + 1)} \{p^{-(k+1)} \sum_{r=0}^{\infty} (1)_r \frac{[-(\frac{p}{c})^{-\nu}]^r}{(r)!}\}$$

Taking inverse Laplace transform, we have

$$L^{-1}\{N(p)\} = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta k + \mu + 1)} \{L^{-1} \sum_{r=0}^{\infty} (-1)^r (c^{\nu r}) p^{-(k+\nu r+1)}\}$$

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta k + \mu + 1)} \{\sum_{r=0}^{\infty} (-1)^r (c^{\nu r}) \frac{t^{k+\nu r}}{\Gamma(\nu r + k + 1)}\}$$

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{\Gamma(\delta k + \mu + 1)} \{\sum_{r=0}^{\infty} (-1)^r (c^{\nu r}) \frac{t^{\nu r}}{\Gamma(\nu r + k + 1)}\}$$

$$N(t) = N_0 E_{\delta,\mu+1}(-t) E_{\nu,k+1}(-c^{\nu} t^{\nu}). \tag{2.6}$$

Then theorem is, thus, completely proved. □

Corollary 2.2. *If $c > 0$, $b = 0$, $\delta > 0$,$\nu > 0$ $\mu > 0$, $\delta\nu - \mu > 0$, then for the solution of the equation*

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$$N(t) - N_0 K_4^{(\nu,\mu,\delta),(-c^{-\nu},0):(p;q)}(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t), \tag{2.7}$$

there holds the formula

$$N(t) = N_0 K_4^{(\nu,\mu+\nu n,\delta+n),(-c^{-\nu},0):(p;q)}(t), \tag{2.8}$$

provided both sides of (2.8) exist.

Proof: TheK_4 - function [12] is defined as

$$\begin{aligned} K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(a_1,.....,a_p;b_1,.....b_q;x) &= K_4^{(\alpha,\beta,\gamma),(a,c):(p;q)}(x) \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{a^n(\gamma)_n)(x-c)^{(n+\gamma)\alpha-\beta-1}}{n!\Gamma((n+\gamma)\alpha-\beta)} \end{aligned}$$

where $R(\alpha\gamma - \beta) > 0$ and $(a_i)_n$ $(i = 1, 2, ..., p)$ and $(b_j)_n$ $(j = 1, 2, ..., q)$ are the pochhammer symbols and none of the parameters $b_j s$ is a negative integer or zero. Taking Laplace transform of both sides of the equation (2.7), we have.

$$L\{N(t)\} - L\{N_0 K_4^{(\nu,\mu,\delta),(-c^{-\nu},0):(p;q)}(t)\} = L\{-\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t)\}$$

or

$$N(p) = N_0 \frac{p^{\mu-\delta\nu}}{(1+p^{-\nu}c^{\nu})^{\delta+n}} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}. \tag{2.9}$$

Finally taking the inverse Laplace transform of equation (2.9), we have

$$L^{-1}\{N(p)\} = L^{-1}\{N_0 \frac{p^{\mu-\delta\nu}}{(1+p^{-\nu}c^{\nu})^{\delta+n}} \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}\}.$$

or

$$N(t) = N_0 K_4^{(\nu,\mu+\nu n,\delta+n),(-c^{-\nu},0):(p;q)}(t)$$

□

Corollary 2.3. *If $c > 0$, $b \geq 0$, $\delta > 0$,$\nu > 0$ $\mu > 0$, $\delta\nu - \mu > 0$, then for the solution of the equation*

$$N(t) - N_0 K_4^{(\nu,(\delta\nu-\mu),\delta),(-c^{-\nu},b):(p;q)}(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}{}_0D_t^{-r\nu} N(t), \tag{2.10}$$

there holds the formula

$$N(t) = N_0 K_4^{[\nu,((\delta+n)\nu-\mu),(\delta+n)],(-c^{-\nu},b):(p;q)}(t), \tag{2.11}$$

provided both sides of (2.11) exist.

Corollary 2.4. *If $c > 0$, $b = 0$, $\delta > 0$,$\nu > 0$ $\mu > 0$, $\delta\nu - \mu > 0$, then for the solution of the equation*

$$N(t) - N_0 K_4^{(\nu,(\delta\nu-\mu),\delta),(-c^{-\nu},0):(p;q)}(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}{}_0D_t^{-r\nu} N(t), \tag{2.12}$$

there holds the formula

$$N(t) = N_0 K_4^{[\nu,((\delta+n)\nu-\mu),(\delta+n)],(-c^{-\nu},0):(p;q)}(t), \tag{2.13}$$

provided both sides of (2.13) exist.

Corollary 2.5. *If $c > 0$, $b \geq 0$, $\delta > 0$,$\nu > 0$ $\mu > 0$, $\nu - \mu > 0$, then for the solution of the equation*

$$N(t) - N_0 K_4^{(\nu,\mu,1),(-c^{-\nu},b):(p;q)}(t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}{}_0D_t^{-r\nu} N(t), \tag{2.14}$$

there holds the formula

$$N(t) = N_0 K_4^{[\nu,(\mu+\nu n),(1+n)],(-c^{-\nu},b):(p;q)}(t), \tag{2.15}$$

provided both sides of (2.15) exist.

2.1. Special Cases

A known result can be obtained as the special case of Theorem 2.1 [13].

Corollary 2.6. *If $c > 0$, $b \geq 0$, $\delta > 0$,$\nu > 0$ $\mu > 0$, $\delta\nu - \mu > 0$, then for the solution of the equation*

$$N(t) - N_0 K_4^{(\nu,\mu,\delta),(-c^{-\nu},b):(p;q)}(t) = -c^{\nu}{}_0D_t^{-\nu} N(t), \tag{2.1.1}$$

there holds the formula

$$N(t) = N_0 K_4^{[\nu,(\mu+\nu),(\delta+1)],(-c^{-\nu},b):(p;q)}(t), \tag{2.1.2}$$

provided both sides of (2.1.2) exist.

Corollary 2.7. *If $c > 0$, $\nu > 0$ then for the solution of the equation*

$$N(t) - N_0 F_\nu[-c^\nu, t] = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t), \tag{2.1.3}$$

there holds the formula

$$N(t) = N_0 t^{\nu-1} E_{\nu,\nu}^{n+1}(-c^\nu t^\nu). \tag{2.1.4}$$

If we take $n = 1$ in the above equation (2.1.3), we obtain the following knowing result given by the Saxena, Mathai and Haubold [16].

if $c > 0$, $\nu > 1$ then for the solution of the equation

$$N(t) - N_0 F_\nu[-c^\nu, t] = -c^\nu {}_0D_t^{-\nu} N(t), \tag{2.1.5}$$

there holds the formula

$$N(t) = N_0 \frac{t^{\nu-1}}{\nu}[E_{\nu,\nu-1}(-c^\nu t^\nu) + E_{\nu,\nu}(-c^\nu t^\nu)]. \tag{2.1.6}$$

3. Conclusion

In this paper, we have derived a solution of generalized fractional kinetic equation in terms of the Bessel- Mitland function by the use of Laplace transform technique. The solution of fractional kinetic equations in the series forms of the K_4 -Function is also discussed.

References

1. A.Erdélyi, W.Magnus, F.Oberhettinger and F.G.Tricomi, Higher transcendental functions, Vol.1, McGraw-Hill, New York-Toronto-London (1953).
2. A.Wiman, Ueber den Fundamentalsatz in der Theorie der Funkilionen $E\alpha(x)$. Acta Math. 29, 191-201 (1905a).
3. A.J.Saichev and G.M. Zaslavsky, Fractional kinetic equations: solutions and applications, Chaos, 7(1997)753-784.
4. Carl F. Lorenzo and Tom T. Hartley, Generalized Functions for the Fractional Calculus,NASA/TP- 1999-209424/REV1, (1999), 17 p.; Available electronically at http:/gltrs.grc.nasa.gov/reports/1999/TP -1999 - 209424- REV1.pdf.
5. Carl F. Lorenzo and Tom T. Hartley, R-Function Relationships for Application in the Fractional Calculus, NASA/TM 2000 210361, (2000),22p.; Available electronically at http://gltrs.grc. nasa. gov/ reports/2000/TM-2000-210361.pdf
6. E. Hille and J.D. Tamarkin, On the theory of linear integral equations, Annals of Mathematics, 31(1930), 479-528.
7. G.M.Zaslavsky, Fractional kinetic equation for Hamiltonian Chaos, Physica D, 76(1994), 110-122.

8. G.M Mittag-Leffler,C.R. Acad.Sci.Paris, Ser.2, (1903), 137, 554.

9. G.M Mittag-Leffler, Acta Math., (1905), 29, 101.

10. H.J. Haubold and A.M.Mathai, The fractional reaction equation and thermonuclear functions, Astrophysics and Space Science, 273(2000), 53-63.

11. Keith B. Oldham and Jerome Spanier, The Fractional Calculus; Theory and Applications of Differentiation and integration to Arbitrary Order, Academic Press, New York and London (1974). ISBN 0-12-525550-0.

12. K. Sharma, On Application of Fractional Differintegral Operator to the K_4 - Function, Bol. Soc. Paran. Math.30 (1), (2012), 91-97.

13. K.Sharma, On The Solutions of Generalized Fractional Kinetic Equations Involving the Functions for the Fractional Calculus, Global Journal of science Frontier research volume 11, (2011), 64-68.

14. K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York (1993).

15. R.K.Saxena, A.M.Mathai and H.J.Haubold, On fractional kinetic equations, Astrophysics and Space Science, 282(2002), 281-287.

16. R.K.Saxena, A.M.Mathai and H.J.Haubold, On generalized fractional kinetic equations, Physica A, 344(2004), 657-664.

17. R.K.Saxena, A.M.Mathai and H.J.Haubold, Unified fractional kinetic equation and a fractional diffusion equation, Astrophysics and Space Science, 290(2002), 299-310.

18. R.K.Saxena, A.M.Mathai and H.J.Haubold, solutions of the fractional reaction equation and the fractional diffusion equation, ArXiv:1001.2293v1,(2010).

19. R. K.Saxena and S.L.Kalla, On the solutions of certain fractional kinetic equations, Appl. Math. Comput., 199(2008), 504-511.

20. S.G. Samko, A. Kilbas and O. Marichev, Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach Sci. Publ., New York (1990).

21. Tom T. Hartley and Carl F. Lorenzo, A solution to the Fundamental Linear Fractional Order Differential Equations, NASA/TP-1998-208693 (1998), 16 p.; Available electronically at http://gltrs.grc. nasa. gov/ reports/1998/TP-1998-208693.pdf

22. V.B.L.Chaurasia and D. Kumar, On the solutions of Generalized Fractional Kinetic Equations, Adv. Studies Theor. Phys., (2010), 773-780.

23. V.B.L.Chaurasia and S.C. Pandey, On the new computable solutions of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and related functions, Astrophysics and Space Science, 317(2008), 213-219.

24. V.B.L.Chaurasia and S.C. Pandey, Computable extensions of generalized fractional kinetic equations in astrophysics, Research in Astron. Astrophys. 2010, Vol. 10, No. 1, 22-32.

25. V.Kourganoff, Introduction to the physics of stellar interiors, D. Reidel Publishing Company, Dordrecht, (1973).

26. W.G. Glockle and T.F.Nonnenmacher, Fractional integral operators and Fox function in the theory of viscoelasticity, Macromolecules, 24(1991), 6426-6434.

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