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Bipartite theory on Neighbourhood Dominating and Global Dominating sets of a graph

Y.B. Venkatakrishnan and V. Swaminathan

ABSTRACT: Bipartite theory of graphs was formulated by Stephen Hedetniemi and Renu Laskar in which concepts in graph theory have equivalent formulations as concepts for bipartite graphs. We give the bipartite version of Neighbourhood sets, Line Neighbourhood sets and global dominating sets.

Key Words: Bipartite theory, Global dominating set, Neighbourhood set, Line Neighbourhood set.

Contents

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by p and q respectively. For graph theoretic terminology we refer to Harary [1].

Let $G = (V, E)$ be a graph. Let $v \in V$. The open neighbourhood and the closed neighbourhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. Let $S \subseteq V$. We denote the subgraph induced by the set of vertices S as $\langle S \rangle$.

A subset S of V is called a dominating set of G if $N[S] = V$. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. For further results on domination the reader is referred to an excellent book on fundamentals of domination [2] and a survey of advanced topics in domination [3].

Various types of domination have been defined and studied by several authors. E.Sampathkumar introduced the concept of neighbourhood set [7], line neighbourhood set [8] and global dominating set [9] in graphs. A set $S \subseteq V$ is a neighbourhood set of G, if $G = \cup_{v \in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the subgraph of G induced by v and all vertices adjacent to v. The neighbourhood number $n_0(G)$ of G is the minimum cardinality of a neighbourhood set of a graph G. For a line $x = uv$ in

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G, let $N[x] = N(u) \cup N(v)$. The set T of lines of G is a line neighbourhood set [8] of G, if $G = \cup_{x \in T} \langle N[x] \rangle$. The minimum cardinality of a line neighbourhood set is called the line neighbourhood number of G and is denoted by $n_0^1(G)$. A dominating set $D \subseteq V$ is called a global dominating set of G if D is a dominating set in both G and $\overline{G}$. The minimum cardinality of a global dominating set is called the global domination number of G and is denoted by $\gamma_g(G)$.

For any problem, say P, on an arbitrary graph G, there is a corresponding problem Q on a bipartite graph G^*, such that a solution for Q provides a solution for P. The so called bipartite theory of graphs was introduced by Hedetniemi and Laskar in [5,6]. Let $G = (X, Y, E)$ be a bipartite graph. Two vertices $u, v \in X$ are $X-$adjacent if they are adjacent to a common vertex in Y. By the $X-$neighbourhood of a vertex u of X, we mean the set $N_X(u) = \{v \in X : u \ \ and \ \ v \ \ are \ \ adjacent\}$. Let $S \subseteq X$ and $N_X(S) = \cup_{x \in S} N_X(x)$. The closed $X-$neighbourhood set of S is $N_X[S] = N_X(S) \cup S$. A set $S \subseteq X$ is an $X-$dominating set if $N_X[S] = X$. The minimum cardinality of an $X-$dominating set [5] is called the $X-$domination number of G and is denoted by $\gamma_X(G)$. A subset D of X is called an $Y-$dominating set [5] if every $y \in Y$ is adjacent to a vertex of D. The minimum cardinality of an $Y-$dominating set is called the $Y-$domination number of G and is denoted by $\gamma_Y(G)$. A subset S of X is hyper independent set if there does not exist a vertex $y \in Y$ such that $N(y) \subseteq S$. The maximum cardinality of a hyper independent set of G denoted by $\beta_h(G)$ is called the hyper indendence number of G.

Bipartite theory on domination in complement of a graph, irredundant set of a graph and dominator coloring of a graph are studied in [10,11,12,13]. Here we give the bipartite theory of neighbourhood set, line neighbourhood set and global dominating set of a graph.

1.1. Bipartite Constructions

Given an arbitrary graph G, we can construct bipartite graphs $G^1 = (X, Y, E^1)$ which represents G, in the sense that given two graphs G and H, G is isomorphic to H if and only if the corresponding bipartite graphs G^1 and H^1 are isomorphic.

We give below the bipartite constructions of $VE(G)$, $EV(G)$ as given in [5] and super duplicate graph $D^*(G)$ as defined in [4].

The bipartite graph $VE(G)$: The graph $VE(G) = (V, E, F)$ is defined by the edge set $F = \{(u, e)/e = (u, v) \in E\}$. $VE(G)$ is isomorphic to $S(G)$, where $S(G)$ denotes the subdivision graph of G namely the graph obtained from G by subdividing each edge of G exactly once.

The bipartite graph $EV(G)$: The graph $EV(G) = (E, V, J)$ is defined by the edges $J = \{(e, u)(e, v)/e = (u, v) \in E\}$.

Super Duplicate graph $D^*(G)$: The bipartite graph $D^*(G) = (V, V^1, E^{11})$ is defined by the edges $E^{11} = \{(u, v^1) : (u, v) \notin E(G)\} \cup \{(u, v^1), (u^1, v) : (u, v) \in E(G)\}$.

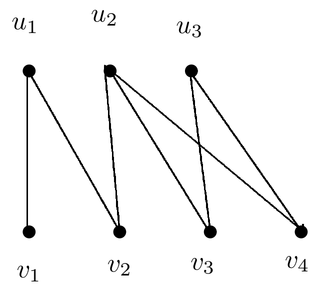
2. $X-$ Neighbourhood set

The X-induced subgraph of $N_X[x]$ denoted by $\langle N_X[x]\rangle$ is defined as the subgraph of G having vertex set $V_1 = N_X[x] \cup \{y \in Y : y$ is adjacent to x or y is adjacent to at least two vertices of $N_X[x]\}$ and the edge set E_1 as the set of edges induced by V_1 of G.

If $G = \bigcup_{x\in S} \langle N_X[x]\rangle$, then S is called a X-neighbourhood set of G. The cardinality of a smallest X-neighbourhood set of G is called the X-neighbourhood number of G and is denoted by $n_X(G)$.

Example 2.1.

Consider the graph



The set $S = \{u_1, u_2\}$ is a $X-$neighbourhood set. Since $V_1 = \{u_1, u_2, u_3, v_1, v_2, v_3, v_4\}$ and $G = \bigcup_{x\in S} \langle N_X[x]\rangle$.

Proposition 2.2. *In a bipartite graph G, every X-neighbourhood set is a X-dominating set.*

Proof: Let S is a X-neighbourhood set. Then $G = \bigcup_{x\in S} \langle N_X[x]\rangle$. For every $u \in X - S$, there exists $v \in S$ such that $u \in N_X[v]$. Therefore, S is a X-dominating set. □

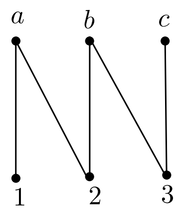
Proposition 2.3. *Every Y-dominating set of G (without isolates) is a X-neighbourhood set of G.*

Proof: Let S be a Y-dominating set. Every $y \in Y$ is adjacent to some $x \in S$. Every vertex in $X - S$ is X-adjacent to some vertex in S. Therefore, $N_X[x] = X$. Therefore, $G = \bigcup_{x\in s} \langle N_X[x]\rangle$. Hence, S is a X-neighbourhood set. □

Corollary 2.4. *In a bipartite graph G, $\gamma_X(G) \leq n_X(G) \leq \gamma_Y(G)$.*

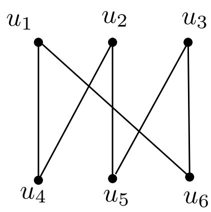
Remark 2.5.

The converse of the proposition 2.2 need not be true. Consider the graph given below, $S = \{b\}$ is a X-dominating set. $N_X[b] = \{a, b, c\}$ and $V_1 = \{a, b, c, 3, 2\}$. $G \neq \bigcup_{x\in s} \langle N_X[b]\rangle$. $S = \{b\}$ is not a X-neighbourhood set.



Remark 2.6.

The converse of the proposition 2.3 need not be true. Consider the graph given below. The set $S = \{u_2\}$ is a X-neighbourhood set. $V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $G = \langle N_X[u_2]\rangle$. S is not a Y-dominating set. Since, u_6 is not adjacent to u_2.



Theorem 2.7. *[11] In a bipartite graph G, A subset D of X is a $Y-$dominating set if and only if $X - D$ is a hyper independent set.*

Proposition 2.8. *For any bipartite graph G without isolated points, $n_X(G) = \gamma_Y(G)$ if and only if there exists a minimum n_X- set S such that $X - S$ is hyper independent set.*

Proof: Let $X - S$ be hyper independent set. S is a Y-dominating set. Hence, $\gamma_Y(G) \leq |S| = n_X(G)$. But, $n_X(G) \leq \gamma_Y(G)$. Therefore, $n_X(G) = \gamma_Y(G)$.

Conversely, let B be a γ_Y- set. Then, B is a n_X- set. Since $\gamma_Y = n_X$, B is a minimum n_X- set. Since B is a γ_Y- set. $X - B$ is hyper independent set. □

3. Bipartite theory of $X-$neighbourhood set

Theorem 3.1. *For any graph G, $n_0(G) = n_X(VE(G))$.*

Proof: Let S be a $n_0(G)-$ set. $G = \bigcup_{x\in s} \langle N_X[x]\rangle$. $N[S] = V$ and every edge of G is induced by $N[S]$ of G. In graph $VE(G) = (X, Y, E^1)$, $N_X[S] = X$. Let $e = uv \in E(G)$. Then $u, v \in X$ and $e \in Y$. If e is adjacent to u and $u \in S$, then e will be in V_1. If $u \notin S$ and e is adjacent to v. If $v \in S$ then $e \in V_1$ otherwise, e is adjacent to vertices in $N_Y(x)$. $e \in V_1$. Therefore, S is a X-neighbourhood set in $VE(G)$. Therefore, $n_X(VE(G)) \leq |S| = n_0(G)$.

Conversely, A be a n_X-set of $VE(G)$. $VE(G) = \bigcup_{x\in A} \langle N_X[x]\rangle$. $N_X[A] = X$ and every vertex of Y is in V_1. In graph G, $N[A] = V$ and every edge of G is induced by the subgraph $\langle N[x]\rangle$, $x \in A$. Therefore, A is a neighbourhood set of G. Hence, $n_0 \leq |A| = n_X$. Therefore, $n_0(G) = n_X(VE(G))$. □

Theorem 3.2. *For any graph G, $n_0^1(G) = n_X(EV(G))$.*

Proof: Let S be a $n_X(EV(G))$ set of the graph $EV(G) = (X, Y, E^1)$. The graph $EV(G) = \bigcup_{x\in S} \langle N_X[x]\rangle$. We have $N_X[S] = X$ and every vertex in Y is adjacent to a vertex in S or adjacent to two vertices in $X - S$. In G, $N[S] = E$ and every vertex in G is incident with an edge in S or incident with two edges in $E - S$. Since, S is an edge dominating set, there exists some edge in S which dominates the above edges. Therefore, the set S of lines of G is a line neighbourhood set of G. $n_0^1(G) \leq |S| = n_X(EV(G)$.

Conversely let A be a line neighbourhood set of G. $G = \bigcup_{x\in A} \langle N_Y[x]\rangle$. $N[A] = E$ and every vertex in G is incident with edges (e_i) in A or edges in $E - S$(say e_j). Since, $N[A] = E$, there exists an edge $e_k \in A$ adjacent to e_j. $N[e_k]$ contains vertices incident with e_j. In $EV(G)$, A is a X-neighbourhood set. Therefore, $n_X(EV(G)) \leq |A| = n_0^1(G)$. Hence, $n_X(EV(G)) = n_0^1(G)$. □

3.1. Bipartite theory of global dominating set

Now we construct a new bipartite graph called Extended super Duplicate graph from an arbitrary graph $G = (V, E)$ as follows: The bipartite graph $D_+^*(G) = (V, V^1, E^+)$ contains the edges of the super duplicate graph $D^*(G)$ together with the edges $\{(u, u^1) : u \in V\}$.

For the sake of clarity, we assign symbols to the edges of the Extended super Duplicate graph obtained from an arbitrary graph G as follows: The function f is defined as $f : E(D_+^*(G)) \longrightarrow \{-, 0, +\}$ such that uv^1 is assigned a $+$ sign if $uv \in E(G)$ and uv^1 is assigned a $-$ sign if $uv \notin E(G)$ otherwise it is assigned a sign 0.

Let $G = (X, Y, E)$ be a bipartite graph. A vertex $x \in X$ positively (negatively, neutrally) dominates $y \in Y$, if there exists an edge $e = xy$ with sign $+$ ($-$,0 respectively).

A subset D of X is a signed $Y-$dominating set if for every $y \in Y$ there exists two vertices x_1 and x_2 in D such that the edges x_1y and x_2y are of different signs. The signed $Y-$ domination number $\gamma_{sY}(G)$ of G is the minimum cardinality of a signed $Y-$ dominating set.

Theorem 3.3. *For an arbitrary graph $G = (V, E)$, $\gamma_{sY}(D_+^*(G)) = \gamma_g(G)$.*

Proof: Let S be a γ_g-set of G. Then S is a dominating set in both G and $\overline{G}$. For every vertex $v \in V - S$, there exists two vertices u_1 and u_2, $u_1 \neq u_2$ in S adjacent to v such that $u_1v \in E(G)$ and $u_2v \in E(\overline{G})$. In the graph $D_+^*(G) = (V, V^1, E^1)$, the vertex $v^1 \in V^1$ is adjacent to u_1 and u_2 in S contained in V. The edgesu_1v^1

is an edge with sign $+$ and u_2v^1 is an edge with sign $-$. Therefore, S is a signed $Y-$dominating set in $D_+^*(G)$. Hence, $\gamma_{sY}(D_+^*(G)) \leq |S| = \gamma_g(G)$.

Suppose D is $\gamma_{sY}-$ set in $D_+^*(G) = (X, Y, E^1)$. For every $y \in Y$ there exists two vertices x_1 and x_2 in D such that the edges x_1y and x_2y are of different signs. If the edge x_1y is assigned a sign $+$ and x_2y is assigned with a sign 0, then x_1y is an edge in G and $x_2 = y$ is an element of D.Hence, D is a global dominating set of G. If the edge x_1y is assigned a sign $+$ and x_2y is assigned with a sign $-$, then x_1y is an edge in G and x_2y is an edge in $\overline{G}$. Hence, D is a global dominating set of G. Similarly, we get D is a global dominating set, when the two edges get different signs. Hence, in all the cases D is a global dominating set. Therefore, $\gamma_g(G) \leq |D| = \gamma_{sY}(D_+^*(G))$. Hence, $\gamma_{sY}(D_+^*(G)) = \gamma_g(G)$. □

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Y.B.Venkatakrishnan
Department of Mathematics,
SASTRA University, Thanjavur,
India.
E-mail address: venkatakrish2@maths.sastra.edu

and

V.Swaminathan
Ramanujan Research Center,
S.N.College, Madurai,
India.
E-mail address: sulanesri@yahoo.com