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Solutions for Steklov boundary value problems involving p(x)-Laplace operators

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abstract: In this paper we study the nonlinear Steklov boundary value problem of the following form:

$$(\mathcal{S})\begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega. \end{cases}$$

Using the variational method, under appropriate assumptions on f, we establish the existence of at least three solutions of this problem.

Key Words: p(x)-Laplace operator, embedding theorem, variable exponent Sobolev space, Ricceri's variational principle.

Contents

1. Introduction

In recent years, the study of differential equations and variational problems with p(x) growth conditions has been an interesting topic. We refer to [6,7,8,10,13,14,15] for the p(x)-Laplacian equations.

This paper is motivated by recent advances in mathematical modeling of non-Newtonian fluids and elastic mechanics, in particular, the electro-rheological fluids (smart fluids). This important class of fluids is characterized by the change of viscosity which is not easy and which depends on the electric field. These fluids, which are known under the name ER fluids, have many applications in elastic mechanics, fluid dynamics etc.. For more information, the reader can refer to [12,16,17].

These physical problems was facilitated by the development of Lebesgue and Sobolev spaces with variable exponent. The existence of solutions of p(x)-Laplacian problems has been studied by several authors (see [3,6,8,9,13,14]).

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Consider the following nonlinear and inhomogeneous Steklov boundary problem,

$$(\mathcal{S})\begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu} = \lambda f(x,u) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative on $\partial\Omega$, $\lambda > 0$ is a real number, p is a continuous function on $\overline{\Omega}$ with $1 < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty$. The main interest in studying such problems arises from the presence of the $p(x)$-Laplace operator $div(|\nabla u|^{p(x)-2}\nabla u)$, which is a generalization of the classical p-Laplace operator $div(|\nabla u|^{p-2}\nabla u)$ obtained in the case when p is a positive constant.

We make the following assumptions on the function f:

(f_1) $| f(x,s) |\leq C(1+ | s |^{\alpha(x)-1}), \qquad \forall(x,s) \in \partial\Omega \times \mathbb{R},$

where $C \geq 0$ is a constant, $\alpha \in C(\partial\Omega)$ and $\alpha(x) > 1$ such that $\forall x \in \partial\Omega$,

$$\alpha(x) < p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ +\infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

(f_2) $\sup_{t\in\mathbb{R}} \int_0^t f(x,s)ds > 0$, $\forall x \in \partial\Omega$.

(f_3) There exists $q \in C(\partial\Omega)$ such that $p^+ < q^- \leq q(x) < p^\partial(x)$ and

$$\limsup_{t\to 0} \sup_{x\in\partial\Omega} \frac{\int_0^t f(x,s)ds}{|t|^{q(x)}} < +\infty.$$

Using the three critical point theorem due to Ricceri, under the above assumptions on f, we establish the existence of at least three solutions of this problem.

Theorem 1.1. *If the function f satisfies $(f_1)-(f_3)$ and the function p satisfies $p^- > \alpha^+$, then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$, $(\mathcal{S})$ has at least three solutions whose norms are less than ρ.*

This paper is divided into three sections. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces, and recall B. Ricceri's three-critical-points theorem. In Section 3, we give the proof of theorem1.1.

2. Preliminaries

To guarantee completeness of this paper, we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$. For more details, see [2,4,5].
Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$ where $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \quad \text{and} \quad \inf_{x\in\overline{\Omega}} p(x) > 1\}$. Denote by

$p^- := \inf_{x\in\overline{\Omega}} p(x)$ and $p^+ := \sup_{x\in\overline{\Omega}} p(x)$.
Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{u \mid u : \Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_\Omega |u|^{p(x)}dx < +\infty\},$$

with the norm

$$|u|_{p(x)} = inf\{\tau > 0; \int_\Omega |\frac{u}{\tau}|^{p(x)}dx \leq 1\}.$$

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega); |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$\|u\| = inf\{\tau > 0; \int_\Omega (|\frac{\nabla u}{\tau}|^{p(x)} + |\frac{u}{\tau}|^{p(x)})dx \leq 1\},$$

$$\|u\| = |\nabla u|_{p(x)} + |u|_{p(x)}.$$

We refer to [4,5,6] for the basic propreties of the variable exponent Lebesgue and Sobolev spaces.

Lemma 2.1. *(see [5]) Both $(L^{p(x)}(\Omega), |.|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|.\|)$ are separable and uniformly convex Banach spaces.*

Lemma 2.2. *(see [5]) Hölder inequality holds, namely*

$$\int_\Omega |uv|dx \leq 2 \mid u \mid_{p(x)} \mid v \mid_{q(x)} \qquad \forall u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

Lemma 2.3. *(see [5]) Let $I(u) = \int_\Omega (\mid \nabla u \mid^{p(x)} + \mid u \mid^{p(x)})dx$, for $u \in W^{1,p(x)}(\Omega)$ we have*

- $\|u\| < 1 (= 1, > 1) \Leftrightarrow I(u) < 1 (= 1, > 1)$.
- $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq I(u) \leq \|u\|^{p^-}$.
- $\|u\| \geq 1 \Rightarrow \|u\|^{p^-} \leq I(u) \leq \|u\|^{p^+}$.

Lemma 2.4. *(see [4]) Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$, then there is a compact embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ +\infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

Lemma 2.5. *(see [5]) If $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function and*

$$| f(x,s) | \leq a(x) + b \mid s \mid^{\frac{p_1(x)}{p_2(x)}}, \qquad \forall (x,s) \in \overline{\Omega} \times \mathbb{R},$$

where $p_1, p_2 \in C_+(\overline{\Omega})$, $a \in L^{p_2(x)}(\Omega)$, $a(x) \geq 0$ and $b \geq 0$ is a constant, then the Nemytskii operator from $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$ defined by $N_f(u)(x) = f(x, u(x))$ is a continuous and bounded operator.

Let $a : \partial\Omega \to \mathbb{R}$ be measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) = \{u \mid u : \partial\Omega \to \mathbb{R} \quad \text{is measurable and} \quad \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma < +\infty\},$$

with the norm

$$|u|_{(p(x),a(x))} = inf\{\tau > 0; \int_{\partial\Omega} |a(x)| \mid \frac{u}{\tau} \mid^{p(x)} d\sigma \leq 1\},$$

where $d\sigma$ is the measure on the boundary. Then $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space. In particular, when $a \in L^\infty(\partial\Omega)$, $L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

Lemma 2.6. *(see [2]) Let $\rho(u) = \int_{\partial\Omega} |a(x)||u|^{p(x)} d\sigma$ for $u \in L^{p(x)}_{a(x)}(\partial\Omega)$ we have*

- $|u|_{(p(x),a(x))} \geq 1 \Rightarrow |u|^{p^-}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),a(x))}$.
- $|u|_{(p(x),a(x))} \leq 1 \Rightarrow |u|^{p^+}_{(p(x),a(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),a(x))}$.

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x\in A} p(x)$, $p^+(A) = \sup_{x\in A} p(x)$. Define

$$p^\partial(x) = (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ \infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

$$p^\partial_{r(x)}(x) := \frac{r(x)-1}{r(x)} p^\partial(x),$$

where $x \in \partial\Omega$, $r \in C(\partial\Omega, \mathbb{R})$ and $r(x) > 1$.

Lemma 2.7. *(see [2]) Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$ with $p^- > 1$. Suppose that $a \in L^{r(x)}(\partial\Omega)$, $r \in C(\partial\Omega)$ with $r(x) > \frac{p^\partial(x)}{p^\partial(x)-1}$ for all $x \in \partial\Omega$. If $q \in C(\partial\Omega)$ and $1 \leq q(x) < p^\partial_{r(x)}(x)$, $\forall x \in \partial\Omega$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$.*
In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$, where $1 \leq q(x) < p^\partial(x)$, $\forall x \in \partial\Omega$.

Lemma 2.8. *(see [1,9,11]) Let X be a separable and reflexive real Banach space, $\phi : X \to \mathbb{R}$ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ is a continuous Gâteaux differentiable functional whose Gâteaux derivative is compact, assume that :*

i) $\lim_{\|u\|_X\to\infty}(\phi(u) + \lambda\psi(u)) = \infty$ *for all* $\lambda > 0$,

ii) there exist $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that

$$\phi(u_0) < r < \phi(u_1),$$

iii)

$$\inf_{u\in\phi^{-1}(-\infty,r]} \psi(u) > \frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)}.$$

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ.

Theorem 2.9. *If $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a carathéodory function and*

(f_1) $| f(x, s) |\leq C(1+ | s |^{\alpha(x)-1}), \qquad \forall(x, s) \in \partial\Omega \times \mathbb{R},$

where $C \geq 0$ is a constant, $\alpha \in C_+(\partial\Omega)$ such that $\forall x \in \partial\Omega$,

$$\alpha(x) < \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ +\infty, & \text{if} \quad p(x) \geq N. \end{cases} \tag{2.1}$$

Set $X = W^{1,p(x)}(\Omega)$, $F(x, u) = \int_0^u f(x, t)dt$, $\psi(u) = -\int_{\partial\Omega} F(x, u(x))d\sigma$, then $\psi \in C^1(X, \mathbb{R})$ and

$$D\psi(u, \varphi) =< \psi'(u), \varphi >= -\int_{\partial\Omega} f(x, u(x))\varphi d\sigma,$$

moreover, the operator $\psi' : X \to X^*$ is compact.

Proof. From the Mean-value theorem, we have

$$\begin{aligned} D\psi(u, \varphi) &= \lim_{t\to 0} \frac{\psi(u + t\varphi) - \psi(u)}{t} \\ &= -\lim_{t\to 0} \int_{\partial\Omega} \frac{F(x, u(x) + t\varphi(x)) - F(x, u(x))}{t} d\sigma \\ &= -\lim_{t\to 0} \int_{\partial\Omega} f(x, u(x) + t\theta\varphi(x))\varphi(x)d\sigma, \end{aligned} \tag{2.2}$$

where $0 \leq \theta = \theta(u(x), t\varphi(x)) \leq 1$.
If $u, \varphi \in X$, then by condition (2.1) and the embedding theorem (lemma2.7), we have $u, \varphi \in L^{\alpha(x)}(\partial\Omega)$. Then there is some constant C_1 such that

$$\|w\|_{L^{\alpha(x)}(\partial\Omega)} \leq C_1\|w\|_X \qquad \forall w \in X. \tag{2.3}$$

By (f_1) and Young′s inequality, we have

$$\begin{aligned} |f(x,u(x)+t\theta\varphi(x))\varphi(x)| &\le C(1+|u(x)+t\theta\varphi(x)|^{\alpha(x)-1})|\varphi(x)| \\ &\le C\frac{\alpha(x)-1}{\alpha(x)}[1+|u(x)+t\theta\varphi(x)|^{\alpha(x)-1}]^{\frac{\alpha(x)}{\alpha(x)-1}} \\ &+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)}. \end{aligned} \tag{2.4}$$

Using the inequality

$$(a+b)^p \le 2^{p-1}(|a|^p+|b|^p), \quad p\ge 1,$$

which implies that for $|t|\le 1$,

$$\begin{aligned} &C(\frac{\alpha(x)-1}{\alpha(x)})[1+|u(x)+t\theta\varphi(x)|^{\alpha(x)-1}]^{\frac{\alpha(x)}{\alpha(x)-1}}+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)} \\ &\le C\frac{(\alpha(x)-1)}{\alpha(x)}2^{\frac{1}{\alpha(x)-1}}[1+|u(x)+t\theta\varphi(x)|^{\alpha(x)}]+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)} \\ &\le C\frac{(\alpha(x)-1)}{\alpha(x)}2^{\frac{1}{\alpha(x)-1}}[1+2^{\alpha(x)-1}[|u(x)|^{\alpha(x)}+|\varphi(x)|^{\alpha(x)}]] \\ &+\frac{1}{\alpha(x)}|\varphi(x)|^{\alpha(x)}. \end{aligned}$$

Notice that the right hand side of the above inequality is independent of t and integrable on $\partial\Omega$, then by the Lebesgue dominated convergence theorem, we have

$$D\psi(u,\varphi)=-\int_{\partial\Omega}f(x,u(x))\varphi(x)d\sigma. \tag{2.5}$$

Obviously the operator $D\psi(u,\varphi)$ is a linear operator for a given u.
We know that the Nemytskii operator $N_f : u(x)\mapsto f(x,u(x))$ is a continuous bounded operator from $L^{\alpha(x)}(\partial\Omega)$ into $L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)$.
Then by (2.3) and (2.5), we have

$$D\psi(u,\varphi)=-\int_{\partial\Omega}f(x,u(x))\varphi(x)d\sigma\le 2C_1\|f(x,u)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}\|\varphi(x)\|_X.$$

Then $D\psi(u,\varphi)$ is a linear bounded functional, therefore the Gâteaux derivative of the linear bounded functional $\psi(u)$ exists and

$$D\psi(u,\varphi)=<D\psi(u),\varphi>=-\int_{\partial\Omega}f(x,u(x))\varphi(x)d\sigma \qquad \forall u,\varphi\in X. \tag{2.6}$$

We will prove that $\psi' : X\to X^*$ is completely continuous. For $u,v,\varphi\in X$, from (2.5) and (2.6), we obtain

$$|<D\psi(u)-D\psi(v),\varphi>|\le 2C_1\|f(x,u)-f(x,v)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}\|\varphi\|_X.$$

Then

$$\|D\psi(u) - D\psi(v)\|_{X^*} \leq 2C_1 \|f(x,u) - f(x,v)\|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega)}.$$

The above inequality shows that the operator $T : L^{\frac{\alpha(x)}{\alpha(x)-1}}(\partial\Omega) \to X^*$ defined by $T(f(x,u)) = D\psi(u)$ is continuous. Then the composite operator $D\psi = ToN_f oI$: $u \to D\psi(u)$ from X into X^* is continuous. Therefore ψ is Frèchet differentiable and its Frèchet derivative is $\psi'(u) = D\psi(u)$. This shows that $\psi \in C^1(X, \mathbb{R})$, $D\psi(u, \varphi) =< \psi'(u), \varphi >= -\int_{\partial\Omega} f(x, u(x))\varphi(x)d\sigma$ and $\psi' : X \to X^*$ is compact. □

Definition 2.10. *We say that $u \in X$ is a weak solution of (S) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2}uvdx = \lambda \int_{\partial\Omega} f(x,u)vd\sigma, \qquad \textit{for all } v \in X.$$

Let

$$\phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx,$$

$$\psi(u) = -\int_{\partial\Omega} F(x,u)d\sigma,$$

where $F(x,t) = \int_0^t f(x,s)ds$.
Then under (f_1), we have

$$(\phi'(u), v) = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv)dx,$$

and

$$(\psi'(u), v) = -\int_{\partial\Omega} f(x,u)vd\sigma.$$

3. Proof of Theorem 1.1

To prove our result we use lemma 2.8.
It is well known that ϕ is a continuous convex functional, then it is weakly lower semicontinuous and its inverse derivative is continuous, from theorem 2.9 the precondition of lemma 2.8 is satisfied. In following we must verify that the conditions (i), (ii) and (iii) in lemma 2.8 are fulfilled.

Proof: For $u \in X$ such that $\|u\|_X \geq 1$, we have

$$\begin{aligned}
\psi(u) &= -\int_{\partial\Omega} F(x,u)d\sigma = -\int_{\partial\Omega}[\int_0^{u(x)} f(x,t)dt]d\sigma \\
&\leq C\int_{\partial\Omega}[|u(x)| + \frac{1}{\alpha(x)}|u|^{\alpha(x)}]d\sigma \\
&\leq C\int_{\partial\Omega}|u(x)|d\sigma + \frac{C}{\alpha^-}\int_{\partial\Omega}|u|^{\alpha(x)}d\sigma.
\end{aligned}$$

By embedding theorem, we have $u \in L^{\alpha(x)}(\partial\Omega)$, therefore

$$\begin{aligned}\int_{\partial\Omega} |u|^{\alpha(x)} d\sigma &\leq \max\{\|u\|^{\alpha^+}_{L^{\alpha(x)}(\partial\Omega)}, \|u\|^{\alpha^-}_{L^{\alpha(x)}(\partial\Omega)}\} \\ &\leq C'\|u\|^{\alpha^+}_X .\end{aligned}$$

Then

$$|\psi(u)| \leq C_2\|u\|_X + C_3\|u\|^{\alpha^+}_X .$$

On the other hand,

$$\begin{aligned}\phi(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx \\ &\geq \frac{1}{p^+}\|u\|^{p^-}_X ,\end{aligned}$$

which implies that for any $\lambda > 0$,

$$\phi(u) + \lambda\psi(u) \geq \frac{1}{p^+}\|u\|^{p^-}_X - \lambda C_2\|u\|_X - \lambda C_3\|u\|^{\alpha^+}_X .$$

For $p^- > \alpha^+$ we have,

$$\lim_{\|u\|_X \to \infty} (\phi(u) + \lambda\psi(u)) = \infty,$$

then (i) of lemma 2.8 is verified.
lt remains to show (ii) and (iii) of this lemma (Ricceri).
Let $u_0 = 0$, we can easily have,

$$\phi(u_0) = \psi(u_0) = 0.$$

Now we claim that there exist $r > 0$ and $u_1 \in X$ such that $\phi(u_0) > r$ and

$$\inf_{u \in \phi^{-1}((-\infty, r])} \psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}.$$

If $\|u\| \geq 1$, we have

$$\frac{1}{p^+}\|u\|^{p^-} \leq \phi(u) \leq \frac{1}{p^-}\|u\|^{p^+}.$$

If $\|u\| < 1$, we have

$$\frac{1}{p^+}\|u\|^{p^+} \leq \phi(u) \leq \frac{1}{p^-}\|u\|^{p^-}.$$

From (f_3), there exist $\eta \in [0,1]$ and $C_4 > 0$ such that

$$F(x,t) \leq C_4|t|^{q(x)} \leq C_4|t|^{q^-}, \quad \forall t \in [-\eta, \eta] \quad \text{uniformly for} \quad x \in \partial\Omega.$$

In view of (f_1), if we put

$$C_5 = \max\{C_4, \sup_{\eta\leq|t|<1} \frac{A(1+|t|^{\alpha^-})}{|t|^{q^-}}, \sup_{|t|\geq 1} \frac{A(1+|t|^{\alpha^+})}{|t|^{q^-}}\},$$

where A is a positive constant, then we have

$$F(x,t) \leq C_5|t|^{q^-}, \quad \forall t \in \mathbb{R} \quad \text{uniformly for} \quad x \in \partial\Omega.$$

Consequently, fix r such that $0 < r < 1$. When $\frac{1}{p^+}\|u\|^{p^+} \leq r < 1$, then by the Sobolev embedding theorem ($X \hookrightarrow L^{q^-}(\partial\Omega)$ is continuous), we have

$$\int_{\partial\Omega} F(x,u)d\sigma \leq C_5 \int_{\partial\Omega} |u|^{q^-} d\sigma \leq C_6\|u\|^{q^-} \leq C_7 r^{(\frac{q^-}{p^+})},$$

where C_6 and C_7 are two positive constants.
Since $q^- > p^+$, we have

$$\lim_{r\to 0^+} \frac{\sup_{\frac{1}{p^+}\|u\|^{p^+}\leq r} \int_{\partial\Omega} F(x,u)d\sigma}{r} = 0. \tag{3.1}$$

By (f_2), we can choose a constant $b \in X \setminus \{0\}$ such that $\int_{\partial\Omega} F(x,b)d\sigma > 0$.
Fix r_0 such that $r_0 < \frac{1}{p^+}\min\{\|b\|^{p^+}, \|b\|^{p^-}, 1\}$.
When $\|b\| > 1$, we have

$$\frac{1}{p^+}\|b\|^{p^-} \leq \phi(b) \leq \frac{1}{p^-}\|b\|^{p^+}. \tag{3.2}$$

From (3.1) and (3.2), we know that when $0 < r < r_0$, then $\phi(b) > r$ and

$$\sup_{\frac{1}{p^+}\|u\|^{p^+}\leq r} \int_{\partial\Omega} F(x,u)d\sigma \leq \frac{r}{2}\frac{\int_{\partial\Omega} F(x,u)d\sigma}{\frac{1}{p^-}\|b\|^{p^+}}.$$

So

$$\sup_{\frac{1}{p^+}\|u\|^{p^+}\leq r} \int_{\partial\Omega} F(x,u)d\sigma < r\frac{\int_{\partial\Omega} F(x,u)d\sigma}{\frac{1}{p^-}\|b\|^{p^+}}.$$

Since $r < r_0$, we have

$$\phi^{-1}((-\infty, r]) \subseteq \{u \in X : \frac{1}{p^+}\|u\|^{p^+} \leq r\}.$$

Then

$$\sup_{\phi(u)\leq r} -\psi(u) < -r\frac{\psi(u_1)}{\phi(u_1)},$$

with $u_1 = b$, which implies that

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}.$$

So we can find $r > 0$, $u_1 = b$ and $\phi(b) > r$ satisfying (ii) and (iii) of lemma 2.8.

When $\|b\| \leq 1$, we have

$$\frac{1}{p^+}\|b\|^{p^+} \leq \phi(b) \leq \frac{1}{p^-}\|b\|^{p^-}. \tag{3.3}$$

From (3.1) and (3.3), we know that when $0 < r < r_0$, then $\phi(b) > r$ and

$$\sup_{\frac{1}{p^+}\|u\|^{p^+}\leq r} \int_{\partial\Omega} F(x,u)d\sigma \leq \frac{r}{2}\frac{\int_{\partial\Omega} F(x,u)d\sigma}{\frac{1}{p^-}\|b\|^{p^-}}.$$

So

$$\sup_{\frac{1}{p^+}\|u\|^{p^+}\leq r} \int_{\partial\Omega} F(x,u)d\sigma < r\frac{\int_{\partial\Omega} F(x,u)d\sigma}{\frac{1}{p^-}\|b\|^{p^-}}.$$

Since $r < r_0$, we have

$$\phi^{-1}((-\infty,r]) \subseteq \{u \in X : \frac{1}{p^+}\|u\|^{p^+} \leq r\}.$$

Then

$$\sup_{\phi(u)\leq r} -\psi(u) < -r\frac{\psi(u_1)}{\phi(u_1)},$$

with $u_1 = b$.
Therefore

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > r\frac{\psi(u_1)}{\phi(u_1)}.$$

So

$$\inf_{u\in\phi^{-1}((-\infty,r])} \psi(u) > \frac{(\phi(u_1) - r)\psi(u_0) + (r - \phi(u_0))\psi(u_1)}{\phi(u_1) - \phi(u_0)},$$

which means that condition (iii) in lemma 2.8 is verified. Since the assumptions of lemma 2.8 are verified, there exist an open interval $\Lambda \subset (0,\infty)$ and a positive constant $\rho > 0$ such that for any $\lambda \in \Lambda$ the equation $\phi'(u) + \lambda\psi'(u) = 0$ has at least three solutions in X whose norms are less than ρ. □

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