Bol. Soc. Paran. Mat. (3s.) **v. 32** 1 (2014): 145–153.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v32i1.18817

# Certain Types of Functions via $\lambda$-open Sets

Miguel Caldas, Saeid Jafari and Govindappa Navalagi

ABSTRACT: New classes of functions, called somewhat $\lambda$-continuous, somewhat $\lambda$-open and hardly $\lambda$-open functions, has been defined and studied by making use of $\lambda$-open sets. Characterizations and properties of these functions are presented.

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Key Words: Topological spaces, $\lambda$-open sets, $\lambda$-continuity, somewhat $\lambda$-continuity.

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## 1. Introduction and Preliminaries

Recent progress in the study of characterizations and generalizations of continuity has been done by means of several generalized closed sets. The first step of generalizing closed set was done by Levine in 1970 [8]. The notion of generalized closed sets has been studied extensively in recent years by many topologists. Moreover, they also suggest several new properties of topological spaces. As a generalization of closed sets, $\lambda$-closed sets were introduced and studied by Arenas et al. [1]. In this paper somewhat $\lambda$-continuous, somewhat $\lambda$-open and hardly $\lambda$-open functions are introduced. We obtain some results similar to the results obtained for somewhat continuous, somewhat open and hardly open functions.

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply, $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $(X, \tau)$, $cl(A)$, $int(A)$ and $X \backslash A$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively. Maki [9] introduced the

2000 *Mathematics Subject Classification:* 54A40

notion of $\Lambda$-sets in topological spaces. A subset $A$ of a topological space $(X, \tau)$ is called a $\Lambda$-set if it coincides with its kernel (saturated set), i.e., to the intersection of all open supersets of A. A subset $A$ of a space $(X, \tau)$ is called $\lambda$-closed [1] if $A = L \cap D$, where $L$ is a $\Lambda$-set and $D$ is a closed set. The complement of a $\lambda$-closed set is called $\lambda$-open. We denote the collection of all $\lambda$-open (resp. $\lambda$-closed) sets by $\lambda O(X, \tau)$ or $\lambda O(X)$ (resp. $\lambda C(X, \tau)$) .
A point $x \in X$ is called a $\lambda$-cluster point of a subset $A \subset X$ [2,3] if for every $\lambda$-open set $U$ of $X$ containing $x$, $A \cap U \neq \emptyset$. The set of all $\lambda$-cluster points is called the $\lambda$-closure of $A$ and is denoted by $cl_\lambda(A)$. A point $x \in X$ is said to be a $\lambda$-interior point of a subset $A \subset X$ if there exists a $\lambda$-open set $U$ containing $x$ such that $U \subset A$. The set of all $\lambda$-interior points of $A$ is said to be the $\lambda$-interior of $A$ and is denoted by $int_\lambda(A)$.

**Lemma 1.1.** *([2,3]) Let $A$ be a subset of a space $X$. Then*
*(1) $cl_\lambda(A) = \cap\{F \in \lambda C(X, \tau) : A \subset F\}$.*
*(2) $A$ is $\lambda$-closed in $X$ if and only if $A = cl_\lambda(A)$.*
*(3) $cl_\lambda(X \backslash A) = X \backslash int_\lambda(A)$.*
*(4) $cl_\lambda(A)$ is $\lambda$-closed in $X$.*

For some more information concerning $\lambda$-open sets and their applications we refer the interested reader to [1], [2], [3], [4], [5], [7] and [9].

## 2. Somewhat $\lambda$-continuous functions

**Definition 2.1.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat $\lambda$-continuous provided that if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ then there is $U \in \lambda O(X, \tau)$ of $X$ such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.*

**Definition 2.2.** *A function $f : (X, \tau) \to (Y, \sigma)$ is called somewhat continuous [6], if for $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$ there exists an open set $U$ of $X$ such that $U \neq \emptyset$ and $U \subset f^{-1}(V)$.*

It is clear that every continuous function is somewhat continuous and every somewhat continuous is somewhat $\lambda$-continuous. But the converses are not true as shows the following examples.

**Example 2.3.** *([6], Example 1) Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a, b\}, X\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is somewhat continuous. But $f$ is not continuous.*

**Example 2.4.** *Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, X\}$. Define a function $f : (X, \tau) \to (X, \sigma)$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then $f$ is somewhat $\lambda$-continuous. But $f$ is not somewhat continuous.*

A subset $E$ of a topological space $(X, \tau)$ is said to be $\lambda$-dense in $X$ if $cl_\lambda(E) = X$, equivalently if there is no proper $\lambda$-closed set $C$ in $X$ such that $E \subset C \subset X$. Take for example X=$\{$a, b, c$\}$ with Sierpinski topology $\tau = \{X, \emptyset, \{a\}\}$. The set $\{a\}$ is dense but not $\lambda$-dense.

**Theorem 2.5.** *For a surjective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
*(i) $f$ is somewhat $\lambda$-continuous;*
*(ii) If $C$ is a closed subset of $Y$ such that $f^{-1}(C) \neq X$, then there is a proper $\lambda$-closed subset $F$ of $X$ such that $f^{-1}(C) \subset F$;*
*(iii) If $E$ is a $\lambda$-dense subset of $X$, then $f(E)$ is a dense subset of $Y$.*

**Proof:** $(i) \Rightarrow (ii)$: Let $C$ be a closed subset of $Y$ such that $f^{-1}(C) \neq X$. Then $Y \backslash C$ is an open set in $Y$ such that $f^{-1}(Y \backslash C) = X \backslash f^{-1}(C) \neq \emptyset$. By (i), there exists a $\lambda$-open set $U$ in $X$ such that $U \neq \emptyset$ and $U \subset f^{-1}(Y \backslash C) = X \backslash f^{-1}(C)$. This means that $f^{-1}(C) \subset X \backslash U$ and $X \backslash U = F$ is a proper $\lambda$-closed set in $X$.
$(ii) \Rightarrow (i)$: Let $V \in \sigma$ and $f^{-1}(V) \neq \emptyset$. Then $Y \backslash V$ is closed and $f^{-1}(Y \backslash V) = X \backslash f^{-1}(V) \neq X$. By (ii), there exists a proper $\lambda$-closed set $F$ of $X$ such that $f^{-1}(Y \backslash V) \subset F$. This implies that $X \backslash F \subset f^{-1}(V)$ and $X \backslash F \in \lambda O(X)$ with $X \backslash F \neq \emptyset$.
$(ii) \Rightarrow (iii)$: Let $E$ be a $\lambda$-dense set in $X$. Suppose that $f(E)$ is not dense in $Y$. Then there exists a proper closed set $C$ in $Y$ such that $f(E) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. By (ii), there exists a proper $\lambda$-closed subset $F$ such that $E \subset f^{-1}(C) \subset F \subset X$. This is a contradiction to the fact that $E$ is $\lambda$-dense in $X$.
$(iii) \Rightarrow (ii)$: Suppose (ii) is not true. This means that there exists a closed set $C$ in $Y$ such that $f^{-1}(C) \neq X$ but there is not proper $\lambda$-closed set $F$ in $X$ such that $f^{-1}(C) \subset F$. This means that $f^{-1}(C)$ is $\lambda$-dense in $X$. But by (iii), $f(f^{-1}(C)) = C$ must be dense in $Y$, which is a contradiction to the choice of $C$. □

**Definition 2.6.** *If $X$ is a set and $\tau$ and $\tau^*$ are topologies on $X$, then $\tau$ is said to be $\lambda$-equivalent (resp. equivalent [6]) to $\tau^*$ provided if $U \in \tau$ and $U \neq \emptyset$ then there is a $\lambda$-open (resp. open) set $V$ in $(X, \tau^*)$ such that $V \neq \emptyset$ and $V \subset U$ and if $U \in \tau^*$ and $U \neq \emptyset$ then there is a $\lambda$-open (resp. open) set $V$ in $(X, \tau)$ such that $V \neq \emptyset$ and $V \subset U$*

Now consider the identity function $f : (X, \tau) \to (X, \tau^*)$ and assume that $\tau$ and $\tau^*$ are $\lambda$-equivalent. Then $f : (X, \tau) \to (X, \tau^*)$ and $f^{-1} : (X, \tau^*) \to (X, \tau)$ are somewhat $\lambda$-continuous. Conversely, if the identity function $f : (X, \tau) \to (X, \tau^*)$ is somewhat $\lambda$-continuous in both directions, then $\tau$ and $\tau^*$ are $\lambda$-equivalent.

**Theorem 2.7.** *If $f : (X, \tau) \to (Y, \sigma)$ is a somewhat $\lambda$-continuous surjection and $\tau^*$ is a topology for $X$, which is $\lambda$-equivalent to $\tau$, then $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat $\lambda$-continuous.*

**Proof:** Let $V$ be an open subset of $Y$ such that $f^{-1}(V) \neq \emptyset$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\lambda$-continuous, there exists a nonempty $\lambda$-open set $U$ in $(X, \tau)$ such that $U \subset f^{-1}(V)$. But by hypothesis $\tau^*$ is $\lambda$-equivalent to $\tau$. Therefore, there exists a nonempty $\lambda$-open set $U^*$ in $(X, \tau^*)$ such that $U^* \subset U$. But $U \subset f^{-1}(V)$. Then $U^* \subset f^{-1}(V)$; hence $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat $\lambda$-continuous. □

**Theorem 2.8.** *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat $\lambda$-continuous surjection and $\sigma^*$ be a topology for $Y$, which is equivalent to $\sigma$. Then $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat $\lambda$-continuous.*

**Proof:** Let $V^*$ be an open subset of $(Y, \sigma^*)$ such that $f^{-1}(V^*) \neq \emptyset$. Since $\sigma^*$ is equivalent to $\sigma$, there exists a nonempty open set $V$ in $(Y, \sigma)$ such that $V \subset V^*$. Now $\emptyset \neq f^{-1}(V) \subset f^{-1}(V^*)$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\lambda$-continuous, there exists a nonempty $\lambda$-open set $U$ in $(X, \tau)$ such that $U \subset f^{-1}(V)$. Then $U \subset f^{-1}(V^*)$; hence $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat $\lambda$-continuous. □

**Theorem 2.9.** *If $f : (X, \tau) \to (X, \sigma)$ is somewhat $\lambda$-continuous and $g : (X, \sigma) \to (X, \eta)$ is continuous, then $fog : (X, \tau) \to (Z, \eta)$ is somewhat $\lambda$-continuous.*

## 3. Two forms of weak openness

In this section we define and characterize two new weak forms of openness, that is, the somewhat $\lambda$-open and hardly $\lambda$-open functions.

**Definition 3.1.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat $\lambda$-open provided that if $U \in \tau$ and $U \neq \emptyset$, then there exists a $\lambda$-open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subset f(U)$.*

We have the following obvious characterization of somewhat $\lambda$-openness.

**Theorem 3.2.** *A function $f : (X, \tau) \to (Y, \sigma)$ is somewhat $\lambda$-open if and only if for any $A \subset X$, $int(A) \neq \emptyset$ implies that $int_\lambda(f(A)) \neq \emptyset$.*

**Theorem 3.3.** *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
*(i) $f$ is somewhat $\lambda$-open;*
*(ii) If $D$ is a $\lambda$-dense subset of $Y$, then $f^{-1}(D)$ is a dense subset of $X$.*

**Proof:** $(i) \Rightarrow (ii)$: Suppose $D$ is a $\lambda$-dense set in $Y$. We need to show that $f^{-1}(D)$ is a dense subset of $X$. Suppose that $f^{-1}(D)$ is not dense in $X$. Then there exists a proper closed set $B$ in $X$ such that $f^{-1}(D) \subset B \subset X$. By (i) and since $X \backslash B$ is open, there exists a nonempty $\lambda$-open subset $E$ in $Y$ such that $E \subset f(X \backslash B)$. Therefore $E \subset f(X \backslash B) \subset f(f^{-1}(Y \backslash D)) \subset Y \backslash D$. It follows that $D \subset Y \backslash E \subset Y$. Now, $Y \backslash E$ is a $\lambda$-closed set and $D \subset Y \backslash E \subset Y$. This implies that $D$ is not a $\lambda$-dense set in $Y$, which is a contradiction. Therefore, $f^{-1}(D)$ is a dense subset of $X$.
$(ii) \Rightarrow (i)$: Suppose $D$ is a nonempty open subset of $X$. We need to show that $int_\lambda(f(D)) \neq \emptyset$. Suppose $int_\lambda(f(D)) = \emptyset$. Then $cl_\lambda(Y \backslash f(D)) = Y$. Therefore, by (ii) $f^{-1}(Y \backslash f(D))$ is dense in $X$. But $f^{-1}(Y \backslash f(D)) \subset X \backslash D$. Now $X \backslash D$ is closed. Therefore $f^{-1}(Y \backslash f(D)) \subset X \backslash D$ and thus $X = cl(f^{-1}((Y \backslash f(D)) \subset X \backslash D$. This implies that $D = \emptyset$ which is contrary to $D \neq \emptyset$. Therefore $int_\lambda(f(D)) \neq \emptyset$. This proves that $f$ is somewhat $\lambda$-open. □

**Theorem 3.4.** *For a bijective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*
*(i) $f$ is somewhat $\lambda$-open;*
*(ii) If $C$ is a closed subset of $X$ such that $f(C) \neq Y$, then there is a $\lambda$-closed subset $F$ of $Y$ such that $F \neq Y$ and $f(C) \subset F$.*

**Proof:** $(i) \Rightarrow (ii)$: Let $C$ be any closed subset of $X$ such that $f(C) \neq Y$. Then $X \backslash C$ is an open set in $X$ and $X \backslash C \neq \emptyset$. Since $f$ is somewhat $\lambda$-open, there exists a $\lambda$-open set $V$ in $Y$ such that $V \neq \emptyset$ and $V \subset f(X \backslash C)$. Put $F = Y \backslash V$. Clearly $F$ is $\lambda$-closed in $Y$ and we claim $F \neq Y$. If $F = Y$, then $V = \emptyset$ which is a contradiction. Since $V \subset f(X \backslash C)$, $f(C) = (Y \backslash f(X \backslash C)) \subset Y \backslash V = F$.
$(ii) \Rightarrow (i)$: Let $U$ be any nonempty open subset of $X$. Then $C = X \backslash U$ is a closed set in $X$ and $f(X \backslash U) = f(C) = Y \backslash f(U)$ implies $f(C) \neq Y$. Therefore, by (ii), there is a $\lambda$-closed set $F$ of $Y$ such that $F \neq Y$ and $f(C) \subset F$. Clearly $V = Y \backslash F \in \lambda O(Y, \sigma)$ and $V \neq \emptyset$. Also $V = Y \backslash F \subset Y \backslash f(C) = Y \backslash f(X \backslash U) = f(U)$. □

**Definition 3.5.** *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be hardly $\lambda$-open provided that for each $\lambda$-dense subset $A$ of $Y$ that is contained in a proper open set, $f^{-1}(A)$ is $\lambda$-dense in $X$.*

We have the following characterizations of hardly $\lambda$-openness.

**Theorem 3.6.** *A function $f : (X, \tau) \to (Y, \sigma)$ is hardly $\lambda$-open if and only if $int_\lambda(f^{-1}(A)) = \emptyset$ for each set $A \subset Y$ having the property that $int_\lambda(A) = \emptyset$ and $A$ containing a nonempty closed set.*

**Proof:** Assume $f$ is hardly $\lambda$-open. Let $A \subset Y$ such that $int_\lambda(A) = \emptyset$ and let $F$ be a nonempty closed set contained in $A$. Since $int_\lambda(A) = \emptyset$, $Y \backslash A$ is $\lambda$-dense in $Y$. Because $F \subset A$, $Y \backslash A \subset Y \backslash F \neq Y$. Therefore $f^{-1}(Y \backslash A)$ is $\lambda$-dense in $X$. Thus $X = cl_\lambda(f^{-1}(Y \backslash A)) = cl_\lambda(X \backslash f^{-1}(A)) = X \backslash int_\lambda(f^{-1}(A))$ which implies that $int_\lambda(f^{-1}(A)) = \emptyset$.
For the converse implication assume that $int_\lambda(f^{-1}(A)) = \emptyset$ for every $A \subset Y$ having the property that $int_\lambda(A) = \emptyset$ and $A$ contains a nonempty closed set. Let $A$ be a $\lambda$-dense subset of $Y$, that is contained in the proper open set $U$. Then $int_\lambda(Y \backslash A) = \emptyset$ and $\emptyset \neq Y \backslash U \subset Y \backslash A$. Thus $Y \backslash A$ contains a nonempty closed set and hence $int_\lambda(f^{-1}(Y \backslash A)) = \emptyset$. Then $\emptyset = int_\lambda(f^{-1}(Y \backslash A)) = int_\lambda(X \backslash f^{-1}(A)) = X \backslash cl_\lambda(f^{-1}(A))$ and hence $f^{-1}(A)$ is $\lambda$-dense in $X$. □

**Theorem 3.7.** *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. If $int_\lambda(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_\lambda(A) \neq \emptyset$ and there exists a nonempty closed set $F$ for which $f^{-1}(F) \subset A$, then $f$ is hardly $\lambda$-open.*

**Proof:** Let $B \subset U \subset Y$, where $B$ is $\lambda$-dense and $U$ is a proper open set. Let $A = f^{-1}(Y \backslash B)$ and $F = Y \backslash U$, obviously $f^{-1}(F) = f^{-1}(Y \backslash U) \subset f^{-1}(Y \backslash B) = A$. Also $int_\lambda(f(A)) = int_\lambda(f(f^{-1}(Y \backslash B))) \subset int_\lambda(Y \backslash B) = \emptyset$. Therefore we must have that $\emptyset = int_\lambda(A) = int_\lambda(f^{-1}(Y\backslash B)) = int_\lambda(X\backslash f^{-1}(B))$ which implies that $f^{-1}(B)$ is $\lambda$-dense. It follows that $f$ is hardly $\lambda$-open. □

**Theorem 3.8.** *If $f : (X, \tau) \to (Y, \sigma)$ is hardly $\lambda$-open, then $int_\lambda(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_\lambda(A) \neq \emptyset$ and $f(A)$ contains a nonempty closed set.*

**Proof:** Let $A \subset X$ such that $int_\lambda(A) \neq \emptyset$ and let $F$ be a nonempty closed set for which $F \subset f(A)$. Suppose $int_\lambda(f(A)) = \emptyset$. Then $Y\backslash f(A)$ is $\lambda$-dense in $Y$ and $Y\backslash f(A) \subset Y\backslash F$ where $Y\backslash F$ is a proper open set. Since $f$ is hardly $\lambda$-open, $f^{-1}(Y\backslash f(A))$ is $\lambda$-dense in $X$. But $f^{-1}(Y\backslash f(A)) = X\backslash f^{-1}(f(A))$ and hence $int_\lambda(f^{-1}(f(A))) = \emptyset$. It follows that $int_\lambda(A) = \emptyset$ which is a contradiction.

The converses of Theorems 3.7 and 3.8 are true provided that $f$ is surjective. Thus we have the following characterization for surjective hardly $\lambda$-open functions. □

**Theorem 3.9.** *If $f : (X, \tau) \to (Y, \sigma)$ is surjective, then the following conditions are equivalent:*
*(i) $f$ is hardly $\lambda$-open.*
*(ii) $int_\lambda(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_\lambda(A) \neq \emptyset$ and there exists a nonempty closed set $F \subset Y$ such that $F \subset f(A)$.*
*(iii) $int_\lambda(f(A)) \neq \emptyset$ for every $A \subset X$ having the property that $int_\lambda(A) \neq \emptyset$ and there exists a nonempty closed set $F \subset Y$ such that $f^{-1}(F) \subset A$.*

**Proof:** $(i) \Rightarrow (ii)$: Theorem 3.8
$(ii) \Rightarrow (iii)$: Since $f$ is surjective $f^{-1}(F) \subset A$ implies that $F \subset f(A)$.
$(iii) \Rightarrow (i)$: Theorem 3.7. □

## 4. $\lambda$-resolvable spaces and $\lambda$-irresolvable spaces

**Definition 4.1.** *A topological space $(X, \tau)$ is said to be $\lambda$-resolvable (resp. resolvable [7]) if there exists a $\lambda$-dense (resp. dense) set $A$ in $(X, \tau)$ such that $X\backslash A$ is also $\lambda$-dense (resp. dense) in $(X, \tau)$. A space $(X, \tau)$ is called $\lambda$-irresolvable (resp. irresolvable) if it is not $\lambda$-resolvable (resp. resolvable).*

**Remark 4.2.** *Professor Maximilian Ganster in a private conversation with the second author mentioned that in every $T_1$ space $X$ the only $\lambda$-dense set is $X$ itself (since every singleton is $\lambda$-open). Hence such a space cannot be $\lambda$-resolvable. Clearly there are plenty of resolvable $T_1$ spaces (e.g. $R$).*

**Theorem 4.3.** *For a topological space $(X, \tau)$, the following statements are equivalent:*
*(i) $(X, \tau)$ is $\lambda$-resolvable;*
*(ii) $(X, \tau)$ has a pair of $\lambda$-dense sets $A$ and $B$ such that $A \subset (X\backslash B)$.*

**Proof:** $(i) \Rightarrow (ii)$: Suppose that $(X, \tau)$ is $\lambda$-resolvable. There exists a $\lambda$-dense set $A$ in $(X, \tau)$ such that $X \backslash A$ is $\lambda$-dense in $(X, \tau)$. Set $B = X \backslash A$, then we have $A = X \backslash B$.
$(ii) \Rightarrow (i)$: Suppose that the statement (ii) holds. Let $(X, \tau)$ be $\lambda$-irresolvable. Then $X \backslash B$ is not $\lambda$-dense and $cl_\lambda(A) \subset cl_\lambda(X \backslash B) \neq X$. Hence $A$ is not $\lambda$-dense. This contradicts the assumption. □

**Theorem 4.4.** *For a topological space $(X, \tau)$, the following statements are equivalent:*
*(i) $(X, \tau)$ is $\lambda$-irresolvable (resp. irresolvable);*
*(ii) For any $\lambda$-dense (resp. dense) set $A$ in $X$, $int_\lambda(A) \neq \emptyset$ (resp $int(A) \neq \emptyset$).*

**Proof:** We prove the first statement since the proof of the second is similar.
$(i) \Rightarrow (ii)$: Let $A$ be any $\lambda$-dense set of $X$. Then we have $cl_\lambda(X \backslash A) \neq X$, hence $int_\lambda(A) \neq \emptyset$.
$(ii) \Rightarrow (i)$: Suppose that $(X, \tau)$ is a $\lambda$-resolvable space. There exists a $\lambda$-dense set $A$ in $(X, \tau)$ such that $X \backslash A$ is also $\lambda$-dense in $(X, \tau)$. It follows that $int_\lambda(A) = \emptyset$, which is a contradiction; hence $(X, \tau)$ is $\lambda$-irresolvable. □

**Definition 4.5.** *A topological space $(X, \tau)$ is said to be strongly $\lambda$-irresolvable if for a nonempty set $A$ in $X$, $int_\lambda(A) = \emptyset$ implies $int_\lambda(cl_\lambda(A)) = \emptyset$.*

**Theorem 4.6.** *If $(X, \tau)$ is an strongly $\lambda$-irresolvable space and $cl_\lambda(A) = X$ for a nonempty subset $A$ of $X$, then $cl_\lambda(int_\lambda(A)) = X$.*

**Proof:** The proof is obvious. □

**Theorem 4.7.** *If $(X, \tau)$ is an strongly $\lambda$-irresolvable space and $int_\lambda(Cl_\lambda(A)) \neq \emptyset$ for a nonempty subset $A$ of $X$, then $int_\lambda(A) \neq \emptyset$.*

**Proof:** The proof is obvious. □

**Theorem 4.8.** *Every strongly $\lambda$-irresolvable is $\lambda$-irresolvable.*

**Proof:** This follows from Theorems 4.3 and 4.4. □

However, the converse of Theorem 4.7 is not true in general as it can be seen from the following example

**Example 4.9.** *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then $(X, \tau)$ is $\lambda$-irresolvable but not strongly $\lambda$-irresolvable.*

**Theorem 4.10.** *If $f : (X, \tau) \to (Y, \sigma)$ is a somewhat $\lambda$-open function and $int_\lambda(B) = \emptyset$ for a nonempty subset $B$ of $Y$, then $int(f^{-1}(B)) = \emptyset$.*

**Proof:** Let $B$ be a nonempty set in $Y$ such that $int_\lambda(B) = \emptyset$. Then $cl_\lambda(Y\backslash B) = Y$. Since $f$ is somewhat $\lambda$-open and $Y\backslash B$ is $\lambda$-dense in $Y$, by Theorem 3.3 $f^{-1}(Y\backslash B)$ is dense in $X$. Then $cl(X\backslash f^{-1}(B)) = X$. Hence $int(f^{-1}(B)) = \emptyset$. □

**Theorem 4.11.** *If $f : (X, \tau) \to (Y, \sigma)$ be a somewhat $\lambda$-open function. If $X$ is irresolvable, then $Y$ is $\lambda$-irresolvable.*

**Proof:** Suppose that $(Y, \sigma)$ is $\lambda$-resolvable. Then there exists a $\lambda$-dense set $B$ such that $Y\backslash B$ is $\lambda$-dense. Since $f$ is somewhat $\lambda$-open, by Theorem 3.3 $f^{-1}(B)$ and $f^{-1}(Y\backslash B)$ are dense. This shows that $X$ is $\lambda$-resolvable. □

**Acknowledgments**

The authors are very grateful to the referee for his comments and suggestions which improved the quality of this paper.

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