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On A class of Double Difference Sequences, their Statistical convergence in 2-normed spaces and their duals

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abstract: In this article, we determine a new class of difference double sequence spaces $\ell_2^\infty(\Delta_\nu)$, $c_2(\Delta_\nu)$ and $c_2^0(\Delta_\nu)$ by defining a double difference $\Delta_\nu = (x_{mn}\nu_{mn} - x_{m,n+1}\nu_{m,n+1}) - (x_{m+1,n}\nu_{m+1,n} - x_{m+1,n+1}\nu_{m+1,n+1})$, where $\nu = (\nu_{mn})$ is a fixed double sequence of non-zero real numbers satisfying some conditions and $m, n \in \mathbb{N}$, the set of natural numbers. Moreover, we have studied their topological properties and certain inclusion relations. We have also discussed the concept of the statistical convergence of these classes in 2-normed space and found their $p\alpha-, p\beta-, p\gamma-$duals.

Key Words: Difference double sequence space; 2-normed space; natural density; statistical convergence; $p\alpha-, p\beta-$, and $p\gamma-$ duals.

Contents

1. Introduction

Let ω^2 be the set of all double sequence spaces of real numbers and ℓ_∞, c and c_0 denote the set of linear spaces that are bounded, convergent and null sequences respectively.

A double sequence $x = (x_{mn})$ is said to be *bounded* if and only if

$$\| x \|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let ℓ_2^∞ denote the space of all bounded double sequence spaces and it is known that ℓ_2^∞ is Banach space (see [18]). A double sequence $x = (x_{mn})$ is called *convergent* (with the limit L) if and only if for every $\varepsilon > 0$, there exists a positive integer

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$n_0 = n_0(\varepsilon)$ such that $|x_{mn} - L| < \varepsilon$; for all $m, n \geq n_0$. The limit L is called double limit or *Pringsheim* limit of the double sequence $x = (x_{mn})$. By c_2 and c_2^0, we denote the space of all convergent and null double sequences.

A double sequence $x = (x_{mn})$ is called *Cauchy sequence* if and only if for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that $|x_{mn} - x_{pq}| < \varepsilon$ for all $m, n, p, q \geq n_0$. In [1] it is known that a double sequence is Cauchy if and only if it is convergent.

Throughout this paper we write $\sup_{m,n}, \lim_m, \lim_{m,n}$ and $\sum_{m,n}$ instead of $\sup_{m,n\geq 1}$, $\lim_{m\to\infty}$, $\lim_{m,n\to\infty,\infty}$ and $\sum_{m,n=1,1}^{\infty,\infty}$ respectively.

The notion of difference sequence space was first introduced by Kızmaz [15] by defining the sequence space

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}, \tag{1.1}$$

for $X = \ell_\infty, c$ and c_0, where $\Delta x = (x_k - x_{k+1})$.

Let $\nu = (\nu_{mn})$ be any fixed double sequence of non zero real numbers satisfying

$$\liminf_{m,n} |\nu_{mn}|^{\frac{1}{m+n}} = r, \quad (0 < r \leq \infty) \quad \text{and} \quad \nu_{1m} = \nu_{n1} = 0 \quad \text{for all } m, n \in \mathbb{N} \setminus \{1\}.$$

Now, we define a class of double difference sequence spaces as follows:

$$\begin{aligned}
\ell_2^\infty(\Delta_\nu) &= \Big\{x = (x_{mn}) \in \omega^2 : \sup_{m,n} |\Delta_\nu x_{mn}| < \infty\Big\},\\
c_2^0(\Delta_\nu) &= \Big\{x = (x_{mn}) \in \omega^2 : \lim_{m,n} |\Delta_\nu x_{mn}| = 0\Big\},\\
c_2(\Delta_\nu) &= \Big\{x = (x_{mn}) \in \omega^2 : \lim_{m,n} |\Delta_\nu x_{mn} - L| = 0, \text{ for some } L \in \mathbb{C}\Big\}.
\end{aligned}$$

where

$$\Delta_\nu x_{mn} = x_{mn}\nu_{mn} - x_{m,n+1}\nu_{m,n+1} - x_{m+1,n}\nu_{m+1,n} + x_{m+1,n+1}\nu_{m+1,n+1}.$$

2. Preliminaries and definitions

In this part, we give the definition of 2-normed space and investigate some relations of $\ell_2^\infty(\Delta_\nu), c_2^0(\Delta_\nu)$ and $c_2(\Delta_\nu)$ in 2-normed space by introducing the concept of statistical convergence.

The idea of 2-normed spaces was first introduced by Gahler [7,8,9]. Later on concept of double sequence spaces and 2-normed spaces have been studied and extended by many authors such as [11,12,13,14,16,17,25] etc.

Let X be a real vector space of dimension d, where $2 \leq d \leq \infty$. A mapping on X, defined by $\|.,.\| : X \times X \to \mathbb{R}$ which satisfies the following four conditions:

(i) $\|x_1, x_2\| = 0$ if and only if x_1 and x_2 are linearly dependent;

(ii) $\|x_1, x_2\| = \|x_2, x_1\|$;

(iii) $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$ for any $\alpha \in \mathbb{R}$ and

(iv) $\|x_1 + x_1', x_2\| \leq \|x_1, x_2\| + \|x_1', x_2\|$.

The pair $(X, \|.,.\|)$ is called 2-normed space. Standard examples of 2-normed space are $\mathbb{R}^2$ equipped with the following 2-norm

- $\|x, y\| := |x_1 y_2 - x_2 y_1|$, where $x = (x_1, x_2), y = (y_1, y_2)$,
- $\|x, y\|$:=the area of the triangle having vertices $0, x$ and y (see [11]).

In this case, we have the following observations:

1. $\|x, y\| \geq 0$;
2. $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$;
3. $\|x, y + z\| = \|x, y\| + \|x, z\|$ if x, y, z are linearly independent with dimension $d = 2$.

The notion of statistical convergence was introduced by Fast [5] and studied by various authors [2,4,6,19,20,21,22,23,24]. We recall some concepts connecting with statistical convergence. Let K be a subset of $\mathbb{N}$, set of natural numbers and K_n be a set i.e.

$$K_n = \{k \in K : k < n\},$$

then the natural density of K is given by $\delta(K) = \lim_{n \to \infty} \frac{|K_n|}{n}$, provided the limit exists, where $|K_n|$ denotes the number of elements in K_n. Finite subsets have natural density zero.

Definition 2.1. *A sequence $x = (x_k)$ is said to be statistically convergent to a number L, if for every $\epsilon > 0$*

$$\lim_n \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0.$$

Equivalently, the natural density of the given set i.e. $\delta(\{k < n : |x_k - L| \geq \epsilon\}) = 0$. In this case we write $st - \lim_k x_k = L$ or $x_k \to L(S)$ and

$$S = \{x \in \omega : st - \lim_k x_k = L, \text{for some } L\}$$

Definition 2.2. *A double sequence $x = (x_{mn})$ in 2-normed space $(X, \|.,.\|)$, is said to be double statistically convergent to a number L, if for every $\epsilon > 0$ and $z \in X$*

$$\lim_{m,n} \frac{1}{mn} |\{p \leq m, q \leq n : \|x_{pq} - L, z\| \geq \epsilon\}| = 0.$$

Equivalently, the natural density of the given set i.e.

$$\delta(\{(p,q) \in \mathbb{N} \times \mathbb{N} : \|x_{pq} - L, z\| \geq \epsilon\}) = 0.$$

In this case we write $st_2 - \lim_{mn} \|x_{mn}, z\| = \|L, z\|$ or $x_{mn} \to LS(X, \|.,.\|)$.

Definition 2.3. *A double sequence $x = (x_{mn})$ in 2-normed space $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$, is said to be double Δ_ν-statistically convergent to a number L, if for every $\epsilon > 0$ and $z \in \ell_2^\infty(\Delta_\nu)$*

$$\lim_{m,n} \frac{1}{mn} |\{p \leq m, q \leq n : \|\Delta_\nu x_{pq} - L, z\| \geq \epsilon\}| = 0.$$

Equivalently, the natural density of the given set i.e.

$$\delta(\{(p,q) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{pq} - L, z\| \geq \epsilon\}) = 0.$$

In this case we write $st_2 - \lim_{mn} \|\Delta_\nu x_{mn}, z\| = \|L, z\|$ or $x_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|)$.

Definition 2.4. *A double sequence $x = (x_{mn})$ in 2-normed space $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$, is said to be double Δ_ν-statistically Cauchy if for every $\epsilon > 0$, there exist $M(\epsilon), N(\epsilon)$ and $z \in \ell_2^\infty(\Delta_\nu)$ such that*

$$\delta(|\{(p,q) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu(x_{pq} - x_{MN}), z\| \geq \epsilon\}|) = 0.$$

3. Main results

In this section, we discuss some inclusion relations and basic topological properties of the spaces $\ell_2^\infty(\Delta_\nu), c_2^0(\Delta_\nu)$ and $c_2(\Delta_\nu)$. Moreover, we determine some interesting results of the space $\ell_2^\infty(\Delta_\nu)$ by introducing the concept of double statistical convergence in 2-normed space.
Now, we state the following two theorems without proof.

Theorem 3.1. $c_2^0(\Delta_\nu) \subset c_2(\Delta_\nu) \subset \ell_2^\infty(\Delta_\nu) \subset \ell_2^\infty$ *and the inclusion is strict.*

Theorem 3.2. *The spaces $\ell_2^\infty(\Delta_\nu), c_2(\Delta_\nu)$ and $c_2^0(\Delta_\nu)$ are normed linear spaces, normed by*

$$\|x\|_{\Delta_\nu} = \sup_{m,n} |\Delta_\nu x_{mn}|. \tag{3.1}$$

Theorem 3.3. *The spaces $\ell_2^\infty(\Delta_\nu), c_2(\Delta_\nu)$ and $c_2^0(\Delta_\nu)$ are complete normed linear spaces with the norm defined by equation 3.1.*

Proof: Let (x^k_{mn}) be a Cauchy sequence in $\ell^\infty_2(\Delta_\nu)$ and $x^k = (x^k_{mn}), x = (x^l_{mn})$ be two double sequences in $\ell^\infty_2(\Delta_\nu)$, for all $m, n, k, l \in \mathbb{N}$.
Suppose

$$\begin{aligned} & \|x^k - x^l\|_{\Delta_\nu} = \sup_{m,n} |\Delta_\nu(x^k_{mn} - x^l_{mn})| \to 0 \quad \text{as } k, l \to \infty. \\ \Rightarrow \quad & |\Delta_\nu x^k_{mn} - \Delta_\nu x^l_{mn}| \to 0 \quad \text{as } k, l \to \infty \text{ and for all } m, n \in \mathbb{N}. \end{aligned}$$

Then for every $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that $k, l > n_0(\epsilon)$, we have $|\Delta_\nu x^k_{mn} - \Delta_\nu x^l_{mn}| < \epsilon$, for all $m, n \in \mathbb{N}$. Therefore, $(\Delta_\nu x^k_{mn})$ is a Cauchy sequence in $\mathbb{R}$, for each fixed pair $m, n \in \mathbb{N}$. By the completeness of $\mathbb{R}$, it converges to x_{mn} in $\mathbb{R}$. i.e,

$$\lim_{k\to\infty} \Delta_\nu x^k_{mn} = x_{mn} \quad \text{for each fixed pair } m, n \in \mathbb{N}.$$

Now, for given $\epsilon > 0$,

$$\begin{aligned} \lim_{l\to\infty} \|x^k - x^l\|_{\Delta_\nu} &= \lim_{k\to\infty} \sup_{m,n} |\Delta_\nu x^k_{mn} - \Delta_\nu x^l_{mn}| \\ &= \sup_{m,n} |\Delta_\nu x^k_{mn} - x_{mn}| < \epsilon \\ \Rightarrow \quad x^k \to x, & \qquad \text{in } \mathbb{R} \text{ with respect to the norm defined in (2).} \end{aligned}$$

Now to show that $x \in \ell^\infty_2(\Delta_\nu)$, this follows from the fact that $\ell^\infty_2(\Delta_\nu)$ is a linear space and $x = x - \Delta_\nu(x^k) + \Delta_\nu(x^k)$. This complete the proof. Proofs of other spaces are done by using the similar arguments. □

Theorem 3.4. **(i)** $\ell^\infty_2(\Delta_\nu) \cap \ell^\infty_2 = c^0_2(\Delta_\nu)$,

(ii) $\ell^\infty_2(\Delta_\nu) \cap c_2 = c^0_2(\Delta_\nu)$,

(iii) $\ell^\infty_2(\Delta_\nu) \cap c^0_2 = c^0_2(\Delta_\nu)$.

Proof: (i) Suppose $x \in [\ell^\infty_2(\Delta_\nu) \cap \ell^\infty_2]$, which implies $\sup_{m,n} |x_{mn}| < \infty$ and $\sup_{m,n} |\Delta_\nu x_{mn}| = \sup_{m,n} |x_{mn}\nu_{mn} - x_{m,n+1}\nu_{m,n+1} - x_{m+1,n}\nu_{m+1,n}$
$+x_{m+1,n+1}\nu_{m+1,n+1}| < \infty$. Thus there exists a l in $\mathbb{C}$ such that

$$\Delta_\nu x_{ij} = l + \varepsilon_{ij} \qquad \text{where } \varepsilon_{ij} \to 0 \text{ as } i, j \to \infty.$$

Equivalently, $l + \varepsilon_{ij} = x_{ij}\nu_{ij} - x_{i,j+1}\nu_{i,j+1} - x_{i+1,j}\nu_{i+1,j} + x_{i+1,j+1}\nu_{i+1,j+1}$.

Now,

$$
\begin{aligned}
&\sum_{i,j=1,1}^{m,n}(l+\varepsilon_{ij})\\
&=\sum_{i,j=1,1}^{m,n}(x_{ij}\nu_{ij}-x_{i,j+1}\nu_{i,j+1}-x_{i+1,j}\nu_{i+1,j}+x_{i+1,j+1}\nu_{i+1,j+1})\\
&=\sum_{i=1}^{m}(x_{i,1}\nu_{i,1}-x_{i,2}\nu_{i,2}-x_{i+1,1}\nu_{i+1,1}+x_{i+1,2}\nu_{i+1,2}+x_{i,2}\nu_{i,2}-x_{i,3}\nu_{i,3}\\
&\quad -x_{i+1,2}\nu_{i+1,2}+x_{i+1,3}\nu_{i+1,3}\cdots+x_{i,n}\nu_{i,n}-x_{i,n+1}\nu_{i,n+1}-x_{i+1,n}\nu_{i+1,n}\\
&\qquad +x_{i+1,n+1}\nu_{i+1,n+1})\\
&=\sum_{i=1}^{m}(x_{i,1}\nu_{i,1}-x_{i+1,1}\nu_{i+1,1}-x_{i,n+1}\nu_{i,n+1}+x_{i+1,n+1}\nu_{i+1,n+1})\\
&=x_{1,1}\nu_{1,1}-x_{2,1}\nu_{2,1}-x_{1,n+1}\nu_{1,n+1}+x_{2,n+1}\nu_{2,n+1}+x_{2,1}\nu_{2,1}-x_{3,1}\nu_{3,1}\\
&\quad -x_{2,n+1}\nu_{2,n+1}+x_{3,n+1}\nu_{3,n+1}\cdots+x_{m,1}\nu_{m,1}-x_{m+1,1}\nu_{m+1,1}\\
&\qquad -x_{m,n+1}\nu_{m,n+1}+x_{m+1,n+1}\nu_{m+1,n+1}\\
&=x_{1,1}\nu_{1,1}-x_{1,n+1}\nu_{1,n+1}-x_{m+1,1}\nu_{m+1,1}+x_{m+1,n+1}\nu_{m+1,n+1}\\
\Rightarrow\ & l+\frac{1}{mn}\sum_{i,j=1,1}^{m,n}\varepsilon_{ij}=\frac{1}{mn}(x_{1,1}\nu_{1,1}-x_{1,n+1}\nu_{1,n+1}-x_{m+1,1}\nu_{m+1,1}\\
&\qquad +x_{m+1,n+1}\nu_{m+1,n+1})\\
\Rightarrow\ & l=\frac{1}{mn}x_{11}\nu_{11}+\frac{1}{mn}x_{m+1,n+1}\nu_{m+1,n+1}-\frac{1}{mn}\sum_{i,j=1,1}^{m,n}\varepsilon_{ij}.
\end{aligned}
$$

Therefore, $l\to 0$ as $m,n\to\infty$ and this complete the proof. □

Proofs of (ii) and (iii) can be done by using the similar techniques.

Theorem 3.5. *Let $x=(x_{mn})$ be a double sequence and Δ_ν-statistically convergent to L in $(\ell_2^\infty(\Delta_\nu),\|.,.\|)$, then L is unique.*

Proof: Suppose

$$
\begin{aligned}
x_{mn}&\to LS(\ell_2^\infty(\Delta_\nu),\|.,.\|)\qquad\text{and}\\
x_{mn}&\to L'S(\ell_2^\infty(\Delta_\nu),\|.,.\|)
\end{aligned}
$$

Assuming $L\neq L'\Rightarrow L-L'\neq 0$, there exists a non zero $z\in\ell_2^\infty(\Delta_\nu)$ such that $L-L'$ and z are linearly independent and $\epsilon>0$,

$$
\begin{aligned}
\delta(\{(p,q)\in\mathbb{N}\times\mathbb{N}:\|\Delta_\nu x_{pq}-L,z\|\geq\epsilon\})=0,\\
\delta(\{(p,q)\in\mathbb{N}\times\mathbb{N}:\|\Delta_\nu x_{pq}-L',z\|\geq\epsilon\})=0.
\end{aligned}
$$

Now, $\|L-L',z\| \le \|\Delta_\nu x_{pq}-L,z\| + \|\Delta_\nu x_{pq}-L',z\|$

$$\Rightarrow \delta(\{(p,q)\in\mathbb{N}\times\mathbb{N} : \|L-L',z\|\ge\epsilon\}) \le \delta(\{(p,q)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu x_{pq}-L,z\|\ge\epsilon\}) + \delta(\{(p,q)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu x_{pq}-L',z\|\ge\epsilon\}) = 0.$$

Hence, $L = L'$ which contradicts to the assumption. □

Theorem 3.6. *Let (x_{mn}) be a double sequence and (y_{mn}) be a double Δ_ν-statistically convergent sequence in $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$ such that $\Delta_\nu x_{mn} = \Delta_\nu y_{mn}$ for almost all m,n, then (x_{mn}) is a Δ_ν-statistically convergent sequence in $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$.*

Proof: Suppose

$$x_{mn} \in (\ell_2^\infty(\Delta_\nu), \|.,.\|) \text{ and}$$
$$y_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|) \text{ such that } \Delta_\nu x_{mn} = \Delta_\nu y_{mn} \text{ for almost all } m,n,$$

which implies $\delta(\{(p,q)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu y_{mn}-L,z\|\ge\epsilon\}) = 0$ and
$$\delta(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Delta_\nu x_{mn} \ne \Delta_\nu y_{mn}\}) = 0.$$
Now,

$$\begin{aligned} &\delta(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu x_{mn}-L,z\|\ge\epsilon\}) \\ &\le \delta(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \Delta_\nu x_{mn} \ne \Delta_\nu y_{mn}\}) + \\ &\qquad \delta(\{(m,n)\in\mathbb{N}\times\mathbb{N} : \ \|\Delta_\nu y_{mn}-L,z\|\ge\epsilon\}) \\ &= 0. \\ \Rightarrow \quad & x_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|). \end{aligned}$$

□

Theorem 3.7. *Let (x_{mn}) and (y_{mn}) be two double sequences in $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$ such that $x_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|)$, $y_{mn} \to L'S(\ell_2^\infty(\Delta_\nu), \|.,.\|)$ and $\alpha \in \mathbb{R}$*

(i) *$x_{mn} + y_{mn} \to (L+L')S(\ell_2^\infty(\Delta_\nu), \|.,.\|)$;*

(ii) *$\alpha x_{mn} \to \alpha LS(\ell_2^\infty(\Delta_\nu), \|.,.\|)$.*

Proof: (i) Suppose

$$x_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|) \text{ and } y_{mn} \to L'S(\ell_2^\infty(\Delta_\nu), \|.,.\|)$$

For every non zero $z \in \ell_2^\infty(\Delta_\nu)$ and $\epsilon > 0$ $\delta(A_1(\epsilon)) = 0$ and $\delta(A_2(\epsilon)) = 0$, where

$$A_1(\epsilon) := \{(m,n)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu x_{mn}-L,z\|\ge\epsilon/2\},$$
$$A_2(\epsilon) := \{(m,n)\in\mathbb{N}\times\mathbb{N} : \|\Delta_\nu y_{mn}-L,z\|\ge\epsilon/2\}.$$

Now consider

$$A(\epsilon) := \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{mn} + \Delta_\nu y_{mn} - (L + L'), z\| \geq \epsilon\}.$$

In order to show that $\delta(A(\epsilon)) = 0$, we need to prove that $A(\epsilon) \subset A_1(\epsilon) \cup A_2(\epsilon)$. Assume $m_0, n_0 \in A(\epsilon)$ such that

$$\|\Delta_\nu x_{mn} + \Delta_\nu y_{mn} - (L + L'), z\| \geq \epsilon \text{ where } m_0, n_0 \notin A_1(\epsilon) \cup A_1(\epsilon) \tag{3.2}$$

Therefore,

$$\|\Delta_\nu x_{m_0 n_0} - L, z\| < \epsilon/2 \text{ and } \|\Delta_\nu y_{m_0 n_0} - L', z\| < \epsilon/2.$$

By using the definition of 2-nomed space (iv), it is clear that

$$\begin{aligned} \|\Delta_\nu x_{mn} + \Delta_\nu y_{mn} - (L + L'), z\| &\leq \|\Delta_\nu x_{m_0 n_0} - L, z\| + \|\Delta_\nu y_{m_0 n_0} - L', z\| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Which contradicts to the assumption (3).

(ii) Assume $x_{mn} \to LS(\ell_2^\infty(\Delta_\nu), \|.,.\|)$ for non zero $\alpha \in \mathbb{R}$, we can write

$$\delta(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{mn} - L, z\| \geq \frac{\epsilon}{|\alpha|}\}) = 0.$$

Now consider,

$$\begin{aligned} \{m, n \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu \alpha x_{mn} - L, z\| \geq \epsilon\} &= \{m, n \in \mathbb{N} \times \mathbb{N} : |\alpha| \|\Delta_\nu y_{mn} - L, z\| \geq \epsilon\} \\ &= \{m, n \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu \alpha x_{mn} - L, z\| \geq \epsilon/|\alpha|\} \end{aligned}$$

Since the natural density of the right hand side set is zero, hence
$\alpha x_{mn} \to \alpha LS(\ell_2^\infty(\Delta_\nu), \|.,.\|)$. □

Theorem 3.8. *A double sequence (x_{mn}) is double Δ_ν-statistically convergent sequence if and only if it is double Δ_ν-statistically Cauchy in $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$.*

Proof: Let (x_{mn}) is double Δ_ν-statistically convergent to a number L, for every $\epsilon > 0$ and $z \in (\ell_2^\infty(\Delta_\nu), \|.,.\|)$ such that

$$\delta(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{mn} - L, z\| \geq \epsilon\}) = 0.$$

In particular, for $m = M, n = N$

$$\delta(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{MN} - L, z\| \geq \epsilon\}) = 0.$$

By using the definition of 2-normed space (iv),

$$\delta(|\{(p,q) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu(x_{pq} - x_{MN}), z\| \geq \epsilon\}|) = 0.$$

Conversely, assume (x_{mn}) is double Δ_ν-statistically Cauchy in $(\ell_2^\infty(\Delta_\nu), \|.,.\|)$. Then

$$\delta(\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|\Delta_\nu x_{mn} - L, z\| \geq \epsilon\}) = 0.$$

This follows from the fact that

$$\|\Delta_\nu(x_{pq} - x_{MN}), z\| = \|\Delta_\nu x_{pq} - L, z\| + \|\Delta_\nu x_{MN} - L, z\|.$$

□

4. The dual spaces

In this section, we give the definition of $p\alpha-, p\beta-$ and $p\gamma-$ duals of a nonempty subset of ω^2 and determine $p\alpha-, p\beta-$ and $p\gamma-$ duals of $\ell_2^\infty(\Delta_\nu)$, $c_2(\Delta_\nu), c_2^0(\Delta_\nu)$. We also discuss certain Lemmas and theorems associated to this concept. Duals of sequence spaces were studied by Et [3], Gökhan and Çolak [10] and many others.

Lemma 4.1. *Let x be an element in $\ell_2^\infty(\Delta_\nu)$, then*

$$\sup_{m,n} \left\{(mn)^{-1} |x_{mn}\nu_{mn}|\right\} < \infty.$$

Proof:

Let $\quad x \in \ell_2^\infty(\Delta_\nu)$

$\Rightarrow \quad \sup_{m,n\geq 1} |\Delta_\nu x_{mn}| < \infty,$

i.e. $\quad |x_{mn}\nu_{mn} - x_{m,n+1}\nu_{m,n+1} - x_{m+1,n}\nu_{m+1,n} + x_{m+1,n+1}\nu_{m+1,n+1}| < \infty,$

for all $m, n = 1, 2, 3...$

Now consider,

$$\begin{aligned}
|x_{11}\nu_{11}+x_{mn}\nu_{mn}| &= |x_{11}\nu_{11}-x_{1n}\nu_{1n}-x_{m1}\nu_{m1}+x_{mn}\nu_{mn}|\\
&= \left|\sum_{i,j=1,1}^{m-1,n-1}(x_{ij}\nu_{ij}-x_{i,j+1}\nu_{i,j+1}-x_{i+1,j}\nu_{i+1,j}+x_{i+1,j+1}\nu_{i+1,j+1})\right|\\
&\leq \sum_{i,j=1,1}^{m-1,n-1}|x_{ij}\nu_{ij}-x_{i,j+1}\nu_{i,j+1}-x_{i+1,j}\nu_{i+1,j}+x_{i+1,j+1}\nu_{i+1,j+1}|\\
&< O(mn)\\
\Rightarrow\quad & (mn)^{-1}|x_{mn}\nu_{mn}|<\infty,\quad \text{for all } m,n=1,2,3...\\
\Rightarrow\quad & \sup_{m,n}\left\{(mn)^{-1}|x_{mn}\nu_{mn}|\right\}<\infty.\ \text{ This completes the proof.}
\end{aligned}$$

□

Definition 4.2. *([3]) Let X be a nonempty subset of ω^2, and $p\geq 1$, then we define*

$$\begin{aligned}
X^{p\alpha} &= \left\{(y_{mn})\in\omega^2:\sum_{m,n}|x_{mn}y_{mn}|^p<\infty \text{ for every } x\in X\right\},\\
X^{p\beta} &= \left\{(y_{mn})\in\omega^2:\sum_{m,n}(x_{mn}y_{mn})^p \text{ converges for every } x\in X\right\},\\
X^{p\gamma} &= \left\{(y_{mn})\in\omega^2:\sup_{M,N\geq 1}\left|\sum_{m,n=1}^{M,N}(x_{mn}y_{mn})^p\right|<\infty \text{ for every } x\in X\right\}.
\end{aligned}$$

We call $X^{p\alpha}, X^{p\beta}$ and $X^{p\gamma}$ are the $p\alpha-, p\beta-$ and $p\gamma-$ duals of X respectively. For $p=1$, X^α is called Kothe-Toeplitz dual of of X and it is clear that $X^\alpha\subset X^\beta$ and $X^\alpha\subset X^\gamma$, but $X^\beta\subset X^\gamma$ does not hold since the sequence of partial sums of a double convergent series need not be bounded.

Theorem 4.3. *For $p\geq 1$,*

$$\left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}=\left[c_2(\Delta_\nu)\right]^{p\beta}=\left[c_2^0(\Delta_\nu)\right]^{p\gamma}=M_1^2.$$

where

$$M_1^2=\bigcap_{N\in\mathbb{N}\setminus\{1\}}\left\{x=(x_{mn}):\sum_{m,n}(mn)^p|x_{mn}\nu_{mn}^{-1}|^p.N^p<\infty\right\}.$$

Proof: For first inclusion, we have to show that $M_1^2 \subset \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$.
Let $x \in M_1^2$ and $y \in \ell_2^\infty(\Delta_\nu)$.
By Lemma 4.1, there exists a positive integer N such that

$$\sup_{m,n}\{(mn)^{-1}|y_{mn}\nu_{mn}|\} < N < \infty.$$

Hence, $\sum_{m,n}|x_{mn}y_{mn}|^p \leq \sum_{m,n}|x_{mn}|^p(mn)^p|\nu_{mn}^{-1}|^p.N^p = \sum_{m,n}(mn)^p|x_{mn}\nu_{mn}^{-1}|^p.N^p < \infty,$
which implies $x \in \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$. Therefore, $M_1^2 \subset \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$.
For the second part, let $x \in \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$ and $x \notin M_1^2$.
Then there exists a positive integer $N > 1$ such that

$$\sum_{m,n}(mn)^p|x_{mn}\nu_{mn}^{-1}|^p.N^p = \infty.$$

Define $y \in \ell_2^\infty(\Delta_\nu)$, by

$$y_{mn} = \frac{(mn).N}{\nu_{mn}} \cdot \text{sgn } x_{mn}; \quad m, n = 1, 2, 3 \ldots$$

Then we have $\sum_{m,n} | x_{mn}y_{mn} |^p = \sum_{m,n} | x_{mn} |^p \ (mn)^p \ | \nu_{mn}^{-1} |^p.N^p$
$= \sum_{m,n}(mn)^p|x_{mn}\nu_{mn}^{-1}|^p.N^p = \infty.$
Thus $x \notin \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$, contradicts the assumption that $x \in \left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha}$.
Hence, $\left[\ell_2^\infty(\Delta_\nu)\right]^{p\alpha} \subset M_1^2$. This completes the proof. □

Proofs of other spaces can be done by using similar techniques.

Theorem 4.4. *For $p \geq 1$ and $\eta \in \{p\alpha, p\beta, p\gamma\}$, $\left[M_1^2\right]^\eta = M_2^2$.*

where

$$M_2^2 = \bigcup_{N\in\mathbb{N}\setminus\{1\}} \left\{x = (x_{mn}) : \sup_{m,n}(mn)^{-p}|x_{mn}\nu_{mn}|^p N^{-p} < \infty\right\}.$$

Proof: We give the proof of this theorem for $\eta = p\alpha$ only and that of others follow by the similar techniques.
For the first inclusion, let $x \in M_2^2$ and $y \in M_1^2$, which implies that

$$\sum_{m,n}(mn)^p|y_{mn}\nu_{mn}^{-1}|^p.N^p < \infty.$$

Now,

$$\sum_{m,n}|x_{mn}y_{mn}|^p = \sum_{m,n}(mn)^p|y_{mn}\nu_{mn}^{-1}|^p.N^p(mn)^{-p}|x_{mn}\nu_{mn}|^p.N^{-p}$$
$$\leq \sup_{m,n}\left\{(mn)^{-p}|x_{mn}\nu_{mn}|^pN^{-p}\right\}\sum_{m,n}(mn)^p|y_{mn}\nu_{mn}^{-1}|^p.N^p$$
$$< \infty, \quad \text{by the hypothesis.}$$

Therefore, $x \in \left[M_1^2\right]^{p\alpha}$ and $M_2^2 \subset \left[M_1^2\right]^{p\alpha}$.
For the second inclusion, let $x \in \left[M_1^2\right]^{p\alpha}$ and $x \notin M_2^2$, which implies for a positive integer $N > 1$

$$\sup_{m,n}\left\{(mn)^{-p}|x_{mn}\nu_{mn}|^pN^{-p}\right\} = \infty.$$

Hence, it is clear that there exist two strictly increasing sequences $(m(i))$ and $(n(j))$ of positive integers such that

$$(m(i)n(j))^{-p}|x_{m(i)n(j)}\nu_{m(i)n(j)}|^pN^{-p} > (i+j)^p.$$

For all positive integers i, j, we define a double sequence $y = (y_{mn})$ by

$$y_{mn} = \begin{cases} (i+j)^{-1}\frac{|\nu_{m(i)n(j)}|}{(m(i)n(j))}.N^{-1}, & \text{if } m = m(i) \text{ and } n = n(j), \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 4.3, it is clear that $y \in M_1^2$. Now,

$$\sum_{m,n}|x_{mn}y_{mn}|^p = \sum_{i,j}|x_{m(i)n(j)}|^p(i+j)^{-p}\frac{|\nu_{m(i)n(j)}|^p}{(m(i)n(j))^p}.N^{-p}$$
$$> \sum_{i,j}1 = \infty, \quad \text{by the assumption.}$$

This contradicts to the fact that $x \in \left[M_1^2\right]^{p\alpha}$. Therefore $x \in M_2^2$ and $\left[M_1^2\right]^{p\alpha} \subset M_2^2$. This completes the proof. □

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