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Maximum Principles for Fourth Order Nonlinear Elliptic Equations with Applications

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ABSTRACT: The paper is devoted to prove maximum principles for a certain functionals defined for solution of the fourth order nonlinear elliptic equation. The maximum principle so obtained is used to prove the nonexistence of nontrivial solutions of the fourth order nonlinear elliptic equation with some zero boundary conditions. Hopf's maximum principle is the main ingredient.

Key Words: Maximum principle, Nonlinear elliptic equations, P-functions.

Contents

1. Introduction

There are large number of papers related to study of existence of solution of second order, nonlinear elliptic boundary value problems. It is well known that maximum principle and comparison principle are the two sides of the same coin. It is popular that maximum principle, comparison principle and existence results go hand in hand (cf. [1], [2], [4]).

It is well known that there is no maximum principle for fourth order elliptic partial differential equations comparable to the maximum principle for second order elliptic equations. In [5] Miranda obtained the first such result for biharmonic equation, $\Delta^2 u = 0$, by proving that the functional

$$P = |\nabla u(x)|^2 - u\Delta u \tag{1.1}$$

is subharmonic on its domain. Since, then many researchers have employed this technique on various class of fourth order partial differential equations. In [7], for example, Schaefer utilized auxiliary functions of type (1.1) to study semilinear equations of the form

$$\Delta^2 u + \rho(x, y) f(u) = 0,$$

2000 *Mathematics Subject Classification:* 35B50, 35J40, 35J60 in a plane domain. Recently Dhaigude and Gosavi [3] extended a maximum principle due to Schaefer [7] for a class of fourth order semilinear elliptic equations to a more general fourth order semilinear elliptic equation of the form

$$\Delta^2 u + a(x,y)\Delta u + b(x,y)f(u) = 0.$$

In this paper, we study the existence problem for fourth order nonlinear elliptic equation of the form

$$\Delta^2 u + a(x,y)g(\Delta u) + b(x,y)f(u) = 0.$$

For simplicity, we use the summation convention and denote partial derivatives $\dfrac{\partial u}{\partial x_i}$ by $u_{,i}$ and $\dfrac{\partial^2 u}{\partial x_i^2}$ by $u_{,ii}$.

This paper is organized as follows. In section 2 we develop a maximum principle for a class of fourth order nonlinear elliptic equations. The maximum principle will be used to deduce the non-existence of non-trivial solutions of the boundary value problem under consideration in the last section.

2. Maximum Principle

Suppose Ω is a plane domain bounded with smooth boundary $\partial\Omega$. The following Lemma [8] is useful to prove our results.

Lemma 2.1. *For a sufficiently smooth function v the inequality*

$$N v_{,ik} v_{,ik} \geq (\Delta v)^2$$

satisfied in some domain $D \subset R^N$.

Now, we prove the following maximum principles for the function P denoted by $P = |\nabla u(x)|^2 - u\Delta u$, which will be the main result of this paper.

Theorem 2.2. *Let $u \in C^4(\Omega) \bigcap C^2(\bar{\Omega})$ be a solution of*

$$\Delta^2 u + a(x,y)g(\Delta u) + b(x,y)f(u) = 0 \tag{2.1}$$

where $a(x,y), b(x,y)$ are bounded in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)u(x,y)g(\Delta u) \geq 0 \quad in \quad \Omega. \tag{2.2}$$

Then the function

$$P = |\nabla u(x)|^2 - u\Delta u \tag{2.3}$$

assume its maximum on $\partial\Omega$.

Proof: By straightforward computations

$$P_{,k} = 2u_{,i}u_{,ik} - u_{,k}\Delta u - u(\Delta u)_{,k} \tag{2.4}$$

$$P_{,kk} = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u\Delta^2 u. \tag{2.5}$$

Using (2.1) in (2.5), we get

$$\Delta P = 2u_{,ik}u_{,ik} - (\Delta u)^2 + aug(\Delta u) + buf. \tag{2.6}$$

By Lemma 2.1 and assumption (2.2), we see that the right hand side of (2.6) is non-negative.
Thus

$$\Delta P \geq 0 \quad in \quad \Omega.$$

By maximum principle [6], the result follows.

Remark 2.3. *If $a = 0$ and $b = 0$ in (2.1) then the maximum principle due to Miranda [5] follows.*

Remark 2.4. *If $g(\Delta u) = \Delta u$ in (2.1) then the maximum principle due to Dhaigude and Gosavi [3] follows.*

Thus, we claim that our results are more general.

Theorem 2.5. *Suppose that $u \in C^4(\Omega)\bigcap C^2(\bar{\Omega})$ is a solution of*

$$\Delta^2 u + a(x,y)g(\Delta u) + b(x,y)f(u) = 0 \tag{2.7}$$

i) $a(x,y), b(x,y)$ are bounded with $b > 0$ in Ω and

$$b(x,y)u(x,y)f(u) + a(x,y)u(x,y)g(\Delta u) \geq 0 \quad in \quad \Omega. \tag{2.8}$$

ii) $\nabla(\frac{1}{b})$ is bounded and $\Delta b \leq 0$. Then the function

$$P_b = \frac{1}{b}[|\nabla u(x)|^2 - u\Delta u] \tag{2.9}$$

assume its nonnegative maximum on $\partial\Omega$ unless $P < 0$ in Ω.

Proof: By straightforward computations

$$P_{b,k} = \frac{1}{b}[\nabla(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,k}}{b^2}(|\nabla u(x)|^2 - u\Delta u) \tag{2.10}$$

$$\begin{aligned} P_{b,kk} = &\frac{1}{b}[\Delta(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,k}}{b^2}[\nabla(|\nabla u(x)|^2 - u\Delta u)] \\ &- \frac{b_{,k}}{b^2}[\nabla(|\nabla u(x)|^2 - u\Delta u)] - \frac{b_{,kk}}{b^2}(|\nabla u(x)|^2 - u\Delta u) \\ &+ \frac{2b_{,k}b_{,k}}{b^3}(|\nabla u(x)|^2 - u\Delta u). \end{aligned} \tag{2.11}$$

Using (2.7) in (2.11) and after some rearrangements, we have

$$\Delta P_b - 2b\nabla\left(\frac{1}{b}\right)\nabla P_b + \frac{\Delta b}{b}P_b = \frac{1}{b}[\Delta(|\nabla u(x)|^2 - u\Delta u)]. \tag{2.12}$$

$$b\Delta P_b - 2b^2\nabla\left(\frac{1}{b}\right)\nabla P_b + \Delta b P_b = 2u_{,ik}u_{,ik} - (\Delta u)^2 - u\Delta^2 u. \tag{2.13}$$

Then it follows from Lemma 2.1 and assumptions (2.8) that P_b satisfies

$$b\Delta P_b - 2b^2\nabla\left(\frac{1}{b}\right)\nabla P_b + \Delta b P_b \geq 0. \tag{2.14}$$

$$b\Delta P_b + 2\nabla b.\nabla P_b + \Delta b P_b \geq 0. \tag{2.15}$$

$$\Delta\big(bP_b\big) \geq 0. \tag{2.16}$$

By maximum principle [6], the result follows.

Remark 2.6. *If $a = 0$ in (2.7) then the maximum principle in Sperb ([8], Theorem 10.2) follows.*

Remark 2.7. *If $b \equiv$ constant, then the condition "unless $P_b < 0$ in Ω" in Theorem 2.5 can be omitted.*

Remark 2.8. *If $g(\Delta u) = \Delta u$ in (2.7) then the maximum principle due to Dhaigude and Gosavi [3] follows.*

The following Lemma is useful in proving the non-existence result in the next section of the paper.

Lemma 2.9. *[7] If (2.2) is satisfied and if u is a C^4 solution of (2.1) which vanishes on $\partial\Omega$, then*

$$\int_\Omega |\nabla u(x)|^2 dxdy \leq \frac{1}{2}A|\nabla u(x)|_M^2$$

where $|\nabla u|_M^2 = max|\nabla u|^2$ and A is the area of Ω.

Proof: From Theorem 2.2 we know that P attains its maximum on the boundary. Also we know that $u = 0$ on $\partial\Omega$. Let $|\nabla u|_M^2 = max|\nabla u|^2$ on $\partial\Omega$. We have

$$|\nabla u|^2 - u\Delta u \leq |\nabla u(x)|_M^2. \tag{2.17}$$

Integrating (2.17) over Ω, we get

$$\int_\Omega |\nabla u|^2 dxdy - \int_\Omega u\Delta u dxdy \leq \int_\Omega |\nabla u(x)|_M^2 dxdy. \tag{2.18}$$

Using Green's first identity

$$\int_\Omega \Big[v\Delta u + \nabla v \cdot \nabla u \Big] dxdy = \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\sigma, \quad \text{with} \quad v = u \tag{2.19}$$

And using $u = 0$ on the boundary in (2.19), we obtain

$$\int_\Omega |\nabla u|^2 dxdy \leq \frac{1}{2} A |\nabla u|_M^2.$$

3. Applications

In this section as an application of our maximum principle we prove non-existence of nontrivial solutions $u(x)$ of the following boundary value problem

$$\Delta^2 u + a(x,y) g(\Delta u) + b(x,y) f(u) = 0, \quad \text{in} \quad \Omega \tag{3.1}$$

$$u(x,y) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

and

$$\Delta^2 u + a(x,y) g(\Delta u) + b(x,y) f(u) = 0, \quad \text{in} \quad \Omega \tag{3.3}$$

$$u(x,y) = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.4}$$

Theorem 3.1. *If* (2.2) *is satisfied in a convex domain* Ω *then no non-trivial solution of* (3.1) - (3.2) *exists.*

Proof: It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.1) - (3.2) exists. We have P as defined in (2.3). Now, Theorem 2.2 and boundary condition (3.2) gives

$$u_{,i} u_{,i} - u \Delta u \leq 0. \tag{3.5}$$

Further integrating (3.5) over Ω, we have

$$\int_\Omega \Big[u_{,i} u_{,i} - u \Delta u \Big] dxdy \leq 0. \tag{3.6}$$

Using Green's first identity

$$\int_\Omega \Big[v\Delta u + \nabla v \cdot \nabla u \Big] dxdy = \int_{\partial\Omega} v \frac{\partial u}{\partial n} d\sigma, \quad \text{with} \quad v = u \tag{3.7}$$

and (3.2) in (3.7), we get

$$\int_\Omega |\nabla u|^2 dxdy \leq 0. \tag{3.8}$$

Consequently $|\nabla u| = 0$ in Ω and by continuity $u \equiv 0$ in $\Omega \cup \partial\Omega$. This is a contradiction. Hence there is no nontrivial solution of (3.1) - (3.2).

Theorem 3.2. *If* (2.2) *is satisfied in a convex domain* Ω *then no nontrivial solution of* (3.3) - (3.4) *exists.*

Proof: It is by contradiction. Assume on the contrary that a nontrivial solution u of the given BVP (3.3) - (3.4) exists. We have P as defined in (2.3). Then by Theorem 2.2, P takes its maximum on the boundary $\partial\Omega$ at a point, say Q. By Hopf's second maximum principle, either $\frac{\partial P}{\partial n}(Q) > 0$ or P is constant in $\Omega \cup \partial\Omega$.

Case I. Suppose $\frac{\partial P}{\partial n}(Q) > 0$ holds. Differentiate P partially in the normal direction and use boundary condition (3.4) to get

$$\frac{\partial P}{\partial n}(Q) = 2\frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial n^2}. \tag{3.9}$$

We know the following relation from differential geometry for curvature k,

$$\frac{\partial^2 u}{\partial n^2} + k\frac{\partial u}{\partial n} + \frac{\partial^2 u}{\partial s^2} = u_{,ii} = \Delta u \quad (see\ [8], p.46) \tag{3.10}$$

where $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ are normal and tangential derivatives respectively. The tangential component $\frac{\partial^2 u}{\partial s^2}$ is zero. Equation (3.10) becomes

$$\frac{\partial^2 u}{\partial n^2} = \Delta u - k\frac{\partial u}{\partial n}. \tag{3.11}$$

Using (3.11) and (3.4) in (3.9), we get

$$\frac{\partial P}{\partial n}(Q) = -2k\left(\frac{\partial u}{\partial n}\right)^2. \tag{3.12}$$

Since Ω is convex, $k > 0$. So $\frac{\partial P}{\partial n}(Q) > 0$ is impossible. Therefore, in this case no nontrivial solution exists.

Case II. Suppose P is a constant say c in $\Omega \cup \partial\Omega$. Then we have

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial n}\right)^2 = c \quad \text{on} \quad \partial\Omega. \tag{3.13}$$

Now as $P = c$ in $\Omega \cup \partial\Omega$, we have $\frac{\partial P}{\partial n} = 0$ on $\partial\Omega$. But from (3.12) we have

$$\frac{\partial P}{\partial n}(Q) = -2kc.$$

For a bounded convex domain with a continuously turning tangent on the boundary, $k \neq 0$. Moreover $c \neq 0$, for, if $c = 0$, then $|\nabla u|_M = 0$ and by Lemma 2.9 and reasoning as in Theorem 2.2 we are led to that $u \equiv 0$ in Ω. Thus $P = c$ is impossible. As neither case is possible, we conclude that no nontrivial solution exists.

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