Bol. Soc. Paran. Mat. (3s.) **v. 31** 2 (2013): 83–99.
©SPM –ISSN-2175-1188 ON LINE ISSN-00378712 IN PRESS
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i2.15938

Review article on χ^2 sequence spaces defined by modulus and fuzzy numbers

N. Subramanian and K. Balasubramanian

Key Words: analytic sequence, modulus function, double sequences, Cesàro mean, AK-space, BK-space.

Contents

1. Introduction to single χ sequence spaces

A complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply x. Let w be the set of all sequences $x = (x_k)$ and ϕ be the set of all finite sequences. Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. In respect of ℓ_∞, c, c_0 we have
$\|x\| = \overset{sup}{k} \, |x_k|$, where $x = (x_k) \in c_0 \subset c \subset \ell_\infty$. A sequence $x = \{x_k\}$ is said to be analytic if $sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence x is called entire sequence if $lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ.χ was discussed in Kamthan

2000 *Mathematics Subject Classification:* 40A05,40C05,40D05

. Matrix transformations involving χ were characterized by Sridhar and Sirajiudeen . Let χ be the set of all those sequences $x = (x_k)$ such that $(k! |x_k|)^{1/k} \to 0$ as $k \to \infty$. Then χ is a metric space with the metric

$$d(x, y) = sup_k \left\{ (k! |x_k - y_k|)^{1/k} : k = 1, 2, 3, \cdots \right\}$$

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence $x^{(n)} = \{x_1, x_2, ..., x_n, 0, 0, ...\}$ $\delta^{(n)} = (0, 0, ..., 1, 0, 0, ...)$, 1 in the n^{th} place and zero's else where; and $s^{(n)} = (0, 0, ..., 1, -1, 0, ...)$, 1 in the n^{th} place,-1 in the $(n+1)^{th}$place and zero's else where. An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$ $(k = 1, 2, 3, \ldots)$ are continuous. We recall the following definitions .

An FK-space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. An metric-space (X, d) is said to have AK (or sectional convergence) if and only if $d\left(x^{(n)}, x\right) \to 0$ as $n \to \infty$. The space is said to have AD (or) be an AD space if ϕ is dense in X. We note that AK implies AD.
If X is a sequence space, we define
(i)$X^{'}$ = the continuous dual of X.
(ii)$X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \, for each \, x \in X\}$;
(iii)$X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \, is \, convergent, \, for each \, x \in X\}$;
(iv)$X^{\gamma} = \left\{a = (a_k) : \overset{sup}{n} |\sum_{k=1}^{n} a_k x_k| < \infty, \, for each \, x \in X\right\}$;
(v)Let X be an FK-space$\supset \phi$. Then $X^f = \left\{ f(\delta^{(n)}) : f \in X^{'} \right\}$.
$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the $\alpha-$(or Kö the-T öeplitz)dual of X, $\beta-$ (or generalized Kö the-T öeplitz)dual of X, $\gamma-$dual of X. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta, or \, \gamma$.

Theorem 1.1. *The dual space of* χ_M *is* Λ. *Inother words* $(\chi_M)^* = \Lambda$

Theorem 1.2. *Let* Y *be any FK-space*$\supset \phi$. *Then* $Y \supset \chi_M$ *if and only if the sequence* s^k *is weakly analytic*

Theorem 1.3. χ_M *is a complete metric space under the metric* $d(x, y) = sup_k \left\{ M\left(\frac{(k!|x_k - y_k|)^{1/k}}{\rho}\right) : k = 1, 2, 3, \cdots \right\}$ *where* $x = (x_k) \in \chi_M$ *and* $y = (y_k) \in \chi_M$.

2. Introduction to matrix transformation on single χ sequence spaces

Let χ denote the space of all gai sequences and Λ the space of all analytic sequences. First we show that the set $E = \left\{ s^{(k)} : k = 1, 2, 3, \cdots \right\}$ is a determining set for χ_M. The set of all finite matrices transforming χ_M into FK-space Y denoted by $(\chi_M : Y)$. We characterize the classes $(\chi_M : Y)$ when $Y = (c_0)_{\pi}, c_{\pi}, \chi_M, \ell_{\pi}, \ell_s, \Lambda_{\pi}, h_{\pi}$. In summary we have the following table:

$\nearrow$	$(c_0)_\pi$	c_π	χ_M	ℓ_π	ℓ_s	Λ_π	h_π
χ_M	Necessary and sufficient condition on the matrix are obtained						

But the approach to obtain these result in the present paper is by determining set for χ_M. First, we investigate a determining set for χ_M and then we characterize the classes of matrix transformations involving χ_M and other known sequence spaces.

Theorem 2.1. *Let $\left\{s^{(k)} : k = 1, 2, 3, \cdots\right\}$ be the set of all sequences in ϕ each of whose non-zero terms ± 1. Let $E = \left\{s^{(k)} : k = 1, 2, 3, \cdots\right\}$ then E is a determining set for the space χ_M.*

Theorem 2.2. *An infinite matrix $A = (a_{nk})$ is in the class*

$$A \in (\chi_M : (c_0)_\pi) \Leftrightarrow lim_{n\to\infty} \left(\frac{a_{nk}}{\pi_n}\right) = 0 \tag{2.1}$$

$$\Leftrightarrow sup_{nk} \left|\frac{a_{n1} + \cdots + a_{nk}}{\pi_n}\right| < \infty. \tag{2.2}$$

3. Introduction to single χ difference sequence spaces

The notion of difference sequences was introduced by H. Kizmaz [Canad. Math. Bull., 24(2) (1984), 215-229]. This was generalized in two different ways by M. Et and R. Colak [Soochow J. Math., 21 (4) (1995), 377-386] and B.C. Tripathy and A. Esi [International J. Sci. Tech, 1(1) (2006) 11-14]. B.C. Tripathy; A. Esi and B.K. Tripathy [Soochow J. Math. 31(3)(2005),333-340] introduced a new type of generalized difference operator, which generalizes and unifies the above two notions of generalized difference operator.

The notion of difference spaces of single sequences was introduced by Kizmaz as follows:

$$Z(\Delta) = \{x = (x_k) : \Delta x \in Z\}$$

for $Z = \ell_\infty, c, c_0$, where $\Delta x = (\Delta x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$ and showed that these are Banach spaces with norm $\|x\| = |x_1| + \|\Delta x\|_\infty$. Later on Et and Colak generalized the notion as follows :
Let $m \in \mathbb{N}$, $Z(\Delta^m) = \{x = (x_k) : \Delta^m x \in Z\}$ for $Z = \ell_\infty, c, c_0$ where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k)_{k=1}^{\infty} = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)_{k=1}^{\infty}$.
The generalized difference has the following binomial representation:

$$\Delta^m x_k = \sum_{\gamma=0}^{m} (-1)^\gamma \binom{m}{\gamma} x_{k+\gamma},$$

They proved that these are Banach spaces with the norm

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$$

The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Baar and Altay in [42] and in the case $0 < p < 1$ by Altay and Baar in [43]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Theorem 3.1. *$\chi_M(\Delta)$ is a r convex for all $r > 0$, where $0 \leq r \leq inf p_k$. Moreover if $p_k = p \leq 1$ for all $k \in \mathbb{N}$, then $\chi_M(\Delta, p)$ is $p-$ convex.*

Theorem 3.2. $(\chi_M(\Delta))^\beta = \Lambda$

4. Introduction to χ^2 sequence spaces

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \, for\, some \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gŏkhan and Colak [21,22] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [27] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [28] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [29] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor [33] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{4.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see [1]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{allfinitesequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.
If X is a sequence space, we give the following definitions:

(i)$X^{'}$= the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for \, each \, x \in X\right\};$

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \, is \, convegent, \, foreach \, x \in X\right\};$

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\};$

(v)$let \, X \, beanFK - space \supset \phi; \, then \, X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\};$

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \, foreach \, x \in X\right\};$

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or K\ddot{o}the - Toeplitz)$dual of $X, \beta - (or \, generalized - K\ddot{o}the - Toeplitz) \, dual \, of \, X, \gamma - \, dual \, of \, X, \, \delta \, - \, dual \, of \, X \, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [20]. It is clear that $x^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

Definition 4.1. *A modulus function was introduced by Nakano [12]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that*
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from here that f is continuous on $[0, \infty)$.

Definition 4.2. *Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k, \ell-$ th term to Ax is as follows:*

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$$

such transformation is said to be nonnegative if $a_{k\ell}^{mn}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [40] and Toeplitz [41]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analog of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

Throughout the present paper we assume that $\Im = (\lambda_{mn})$ is the sequence of non-zero complex numbers. Then for a sequence space E, the multiplier sequence space $E(\Im)$ associated with the multiplier sequence $\Im$ is defined by

$$E(\Im) = \left\{x = (x_{mn}) \in w^2 : \Im x = (\lambda_{mn} x_{mn}) \in E\right\}.$$

A multiplier sequence can be used to accelarate the convergece of the sequence in some spaces.

Let $A = \left(a^{k\ell}_{mn}\right)$ be the cesàro four dimensional matrix defined by

$$\left(a^{k\ell}_{mn}\right) = \begin{cases} \frac{1}{(m+1)(k+1)} & \text{if } 0 \le n, \ell \le m, k; \\ 0 & \text{if} k, \ell > m, n \end{cases} \tag{4.2}$$

for all $m, n, k, \ell \in \mathbb{N}$.

Definition 4.3. *Let f be any modulus function. Then, we define the sets* $\tilde{\chi^2_f}(A, \Im)$ *and* $\tilde{\chi^2_f}$*, by*

$\tilde{\chi^2_f}(A, \Im) = \sum_m \sum_n f\left(\frac{\left|\sum_{i=0}^m \sum_{j=0}^n \lambda_{ij}\left((i+j)!\, x_{ij}\right)^{1/i+j}\right|}{(m+1)(n+1)}\right) \to 0\, as\, m, n \to \infty.$ *and*

$\tilde{\chi^2_f} = \sum_m \sum_n f\left(\left((m+n)!\, |x_{mn}|\right)^{1/m+n}\right) \to 0\, as\, m, n \to \infty.$

Definition 4.4. *Let f and Φ be mutually complementary functions. Then, we define the set $\chi^2_f(A, \Im)$ by*

$\chi^2_f(A, \Im) = \sum_m \sum_n f\left(\left(\frac{\left|\sum_{i=0}^m \sum_{j=0}^n \lambda_{ij}\left((i+j)!\, x_{ij}\right)^{1/i+j}\right|}{(m+1)(n+1)}\right)\left((m+n)! y_{mn}\right)^{1/m+n}\right) \to$ $0\, as\, m, n \to \infty$, *for all $y = (y_{mn}) \in \chi^2_\Phi$, which is called as modulus sequence space associated with the multiplier sequence $\Im = (\lambda_{mn})$ and generated by cesàro four dimensional matrix.*

Theorem 4.5. *For any modulus function f, the inclusion $\tilde{\chi^2_f}(A, \Im) \subset \chi^2_f(A, \Im)$ holds.*

Theorem 4.6. *For each $x = (x_{mn}) \in \chi^2_f(A, \Im)$*

$sup\left\{\left|\sum_m \sum_n \left(\left(\frac{\sum_{i=0}^m \sum_{j=0}^n \lambda_{ij}\left((i+j)!\, x_{ij}\right)^{1/i+j}}{(m+1)(n+1)}\right)\left((m+n)! y_{mn}\right)^{1/m+n}\right)\right| : \delta(\Phi, y_{mn}) \le 1\right\} \to$ $0\, as\, m, n \to \infty$.

5. Introduction to χ^2 difference sequence spaces

We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

The generalized difference double notion has the following binomial representation:

$$\Delta^k x_{mn} = \sum_{i=0}^{k} \sum_{j=0}^{k} (-1)^{i+j} \binom{k}{i} \binom{m}{j} x_{m+i, n+j},$$

Theorem 5.1. *i) If* $0 < p_{mn} \leq 1$ *for each* $m, n \in \mathbb{N}$, *then* $\chi^2(\Delta^m, f, p, q) \subseteq \chi^2(\Delta^m, f, q)$;

Theorem 5.2. *If* $p_{mn} \geq 1$ *for all* $m, n \in \mathbb{N}$, *then* $\chi^2(\Delta^m, f, q) \subseteq \chi^2(\Delta^m, f, p, q)$

Theorem 5.3. $\chi^2(\Delta^m, f, p, q, s)$ *is not solid for* $m > 0$

Theorem 5.4. $\chi^2(\Delta^m, f, p, q, s)$ *is not sequence algebra*

6. Introduction to Modulus function

Orlicz [13] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p\,(1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An modulus function M is said to satisfy the Δ_2- condition for small u or at 0 if for each $k \in \mathbb{N}$, there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an modulus function M is said to satisfy the Δ_2- condition if and only if

$$lim_{u \to 0+} sup \frac{M(2u)}{M(u)} < \infty$$

Two Modulus functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b].$$

An modulus function M can always be represented in the following integral form

$$M(u) = \int_0^u \eta(t)\, dt,$$

where η, the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$ for $t > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel η associated with the modulus function$M$ and let

$$\mu(s) = sup\{t : \eta(t) \leq s\}.$$

Then μ possesses the same properties as the function η. Suppose now

$$\Phi = \int_0^x \mu(s)\, ds.$$

Then, Φ is an modulus function. The functions M and Φ are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) \tag{6.1}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{6.2}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{6.3}$$

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

7. Introduction to single χ sequence space of fuzzy numbers

Since the introduction of fuzzy set by L.A.Zadeh in 1965, the concept of fuzziness has been applied in various fields of science like Expert System, Pattern Recognition, Fuzzy Control, Decision Making, Image Processing, Cybernetics, Artificial Intelligence, Operation Research, Path Tracking Application, Projectile, Texture Analysis, E-Business, Agriculture System etc. In mathematics the application of fuzzy is found in all the branches.

Several authors introduced and investigated different classes of sequences of fuzzy real numbers and established many important results. S. Nanda, B.K. Tripathy, N.R. Das, B. Choudhary, P.V. Subrahmanyam, M. Mursaleen, F. Nuray, E. Savas, M. Basari, R. Colak, M. Et, A. Esi, Y. Altin, J. Fang, H. Hung are a few to be named and subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming.

Let $C(R^n) = \{A \subset R^n : A\, compact\, and\, convex\}$. The space $C(R^n)$ has linear structure induced by the operations $A + B = \{a + b : a \in A, b \in B\}$ and $\lambda A = \{\lambda a : a \in A\}$ for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty (A, B) = max \{ sup_{a\in A} inf_{b\in B} \|a - b\|, sup_{b\in B} inf_{a\in A} \|a - b\| \}$$

It is well known that $(C(R^n), \delta_\infty)$ is a complete metric space.
The fuzzy number is a function X from R^n to [0,1] which is normal, fuzzy convex, upper semi-continuous and the closure of $\{x \in R^n : X(x) > 0\}$ is compact. These properties imply that for each $0 < \alpha \leq 1$, the $\alpha-$level set $[X]^\alpha = \{x \in R^n : X(x) \geq \alpha\}$ is a nonempty compact convex subset of R^n, with support $X^c = \{x \in R^n : X(x) > 0\}$. Let $L(R^n)$ denote the set of all fuzzy numbers. The linear structure of $L(R^n)$ induces the addition $X + Y$ and scalar multiplication $\lambda X, \lambda \in R$, in terms of $\alpha-$ level sets, by $|X+Y|^\alpha = |X|^\alpha + |Y|^\alpha, |\lambda X|^\alpha = \lambda |X|^\alpha$ for each $0 \leq \alpha \leq 1$. Define, for each $1 \leq q < \infty$,

$$d_q(X, Y) = \left(\int_0^1 \delta_\infty (X^\alpha, Y^\alpha)^q \, d\alpha \right)^{1/q}, \, and \, d_\infty = sup_{0\leq\alpha\leq 1} \delta_\infty (X^\alpha, Y^\alpha),$$

where δ_∞ is the Hausdorff metric. Clearly $d_\infty(X, Y) = lim_{q\to\infty} d_q(X, Y)$ with $d_q \leq d_r$, if $q \leq r$ [27]. Throughout the paper, d will denote d_q with $1 \leq q \leq \infty$.

The additive identity in $L(R^n)$ is denoted by $\bar{0}$.

A metric on $L(R^n)$ is said to be translation invariant if $d(X + Z, Y + Z) = d(X, Y)$ for all $X, Y, Z \in L(R^n)$

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(R^n)$. The fuzzy number X_k denotes the value of the function at $k \in \mathbb{N}$. We denotes by $W(F)$ the set of all sequences $X = (X_k)$ of fuzzy numbers. Let P_s denote the class of subsets of $\mathbb{N}$, the natural numbers, which do not contain more than s elements. Throughout (ϕ_n) represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.
The sequence $\chi(\phi)$ for real numbers is defined as follows:

$$\Gamma(\phi) = \left\{ (X_k) : \tfrac{1}{\phi_s} (|X_k|)^{1/k} \to 0 \, as \, k, s \to \infty \, for \, k \in \sigma \in P_s \right\}$$

The generalized sequence space $\Gamma(\Delta_n, \phi)$ of the sequence space $\Gamma(\phi)$ for real numbers is defined as follows

$$\Gamma(\Delta_n, \phi) = \left\{ (X_k) : \tfrac{1}{\phi_s} (|\Delta X_k|)^{1/k} \to 0 \, as \, k, s \to \infty \, for \, k \in \sigma \in P_s \right\}$$

where $\Delta_n X_k = X_k - X_{k+n}$ for $k \in \mathbb{N}$ and fixed $n \in \mathbb{N}$.

In this article we introduce the following classes of sequences of fuzzy numbers:
Let M be an Orlicz function. Then write

$$\Lambda_M^F(\Delta^m) = \left\{ (X_k) \in W(F) : sup_k M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k} \right), \bar{0} \right)}{\rho} \right) < \infty \, for \, some \, \rho > 0 \right\}$$

$$\chi_M^F(\Delta^m) = \left\{ (X_k) \in W(F) : M\left(\frac{d\left((k! |\Delta^m X_k|)^{1/k}, \bar{0} \right)}{\rho} \right) \to 0 \, as \, k \to \infty, \, for \, some \, \rho > 0 \right\}$$

$$\Gamma_M^F(\Delta^m) = \left\{ (X_k) \in W(F) : M\left(\frac{d\left(\left(|\Delta^m X_k|^{1/k} \right), \bar{0} \right)}{\rho} \right) \to 0 \, as \, k \to \infty, \, for \, some \, \rho > 0 \right\}$$

$$\chi_M^F(\Delta^m,\phi)=\left\{(X_k)\in W(F): \frac{1}{\phi_s}M\left(\frac{d\left((k!|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right)\rightarrow 0\, as\, k,s\rightarrow\infty,\, for\, k\in\sigma\in P_s\right\}$$
$$\Gamma_M^F(\Delta^m,\phi)=\left\{(X_k)\in W(F): \frac{1}{\phi_s}M\left(\frac{d\left((|\Delta^m X_k|)^{1/k},\bar{0}\right)}{\rho}\right)\rightarrow 0\, as\, k,s\rightarrow\infty,\, for\, k\in\sigma\in P_s\right\}$$

Theorem 7.1. *If d is a translation invariant metric, the n $\Gamma_M^F(\Delta^m,\phi)$ is closed under the operations of addition and scalar multiplication.*

Theorem 7.2. *The space $\Lambda_M^F(\Delta^m)$ is a complete metric space with the metric by*

$$h(X,Y)=inf\left\{\rho>0: sup_k\left(M\left(\frac{d\left(|\Delta^m X_k-\Delta^m Y_k|^{1/k}\right)}{\rho}\right)\right)\leq 1\right\}$$

Theorem 7.3. *If $\left(\frac{\phi_s}{\psi_s}\right)\rightarrow 0\, as\, s\rightarrow\infty$ then $\Gamma_M^F(\Delta^m,\phi)\subset\Gamma_M^F(\Delta^m,\psi)$*

8. Introduction to χ^2 sequence space of fuzzy numbers

Throughout a double sequence is denoted by $\langle X_{mn}\rangle$, a double infinite array of fuzzy real numbers.

Let D denote the set of all closed and bounded intervals $X=[a_1,a_2]$ on the real line $\mathbb{R}$. For $X=[a_1,a_2]\in D$ and $Y=[b_1,b_2]\in D$, define

$$d(X,Y)=max\left(|a_1-b_1|,|a_2-b_2|\right)$$

It is known that (D,d) is a complete metric space.

A fuzzy real number X is a fuzzy set on $\mathbb{R}$, that is, a mapping $X:\mathbb{R}\rightarrow I\,(=[0,1])$ associating each real number t with its grade of membership $X(t)$.

The $\alpha-$ level set $[X]^\alpha$, of the fuzzy real number X, for $0<\alpha\leq 1$, defined by

$$[X]^\alpha=\{t\in\mathbb{R}: X(t)\geq\alpha\}.$$

The $0-$ level set is th closure of the strong $0-$ cut that is, $cl\{t\in\mathbb{R}: X(t)>0\}$.

A fuzzy real number X is called convex if $X(t)\geq X(s)\wedge X(r)=min\{X(s),X(r)\}$, where $s<t<r$. If there exists $t_0\in\mathbb{R}$ such that $X(t_0)=1$ then, the fuzzy real number X is called normal.

A fuzzy real number X is said to be upper-semi continuous if, for each $\epsilon>0, X^{-1}([0,a+\epsilon))$ is open in the usual topology of $\mathbb{R}$ for all $a\in I$.

The set of all upper-semi continuous, normal, convex fuzzy real numbers is denoted by $L(\mathbb{R})$.

The absolute value, $|X|$ of $X\in L(\mathbb{R})$ is defined by

$$|X|(t)=\begin{cases}max\{X(t),X(-t)\}, & \text{if } t\geq 0;\\ 0, & \text{if } t<0\end{cases}$$

Let $\bar{d}: L(\mathbb{R})\times L(\mathbb{R})\rightarrow\mathbb{R}$ be defined by

$$\bar{d}(X,Y)=sup_{0\leq\alpha\leq 1}d\left([X]^\alpha,[Y]^\alpha\right).$$

Then, $\bar{d}$ defines a metric on $L(\mathbb{R})$ and it is well-known that $\left(L(\mathbb{R}), \bar{d}\right)$ is a complete metric space.
A metric d on $L(\mathbb{R})$ is said to be translation invariant metric if

$$d(X+Z, Y+Z) = d(X, Y) \text{ for } X, Y, Z \in L(\mathbb{R}).$$

A sequence $\langle X_m \rangle \subset L(\mathbb{R})$ of fuzzy real numbers is said to be null to the fuzzy real number 0, such that $\bar{d}(X_m, \bar{0}) = 0$.

A double sequence $\langle X_{mn} \rangle$ of fuzzy real numbers is said to be chi in Pringsheim's sense to a fuzzy number 0 if $lim_{m,n\to\infty} ((m+n)!X_{mn})^{1/m+n} = 0$.

A double sequence $\langle X_{mn} \rangle$ is said to chi regularly if it converges in the Prinsheim's sense and the following limts zero:

$$lim_{m\to\infty} ((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } n \in \mathbb{N},$$

and

$$lim_{n\to\infty} ((m+n)!X_{mn})^{1/m+n} = 0 \text{ for each } m \in \mathbb{N}.$$

A fuzzy real-valued double sequence space E^F is said to be solid if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $|Y_{mn}| \leq |X_{mn}|$ for all $m, n \in \mathbb{N}$.

Let $K = \{(m_i, n_i) : i \in \mathbb{N}; m_1 < m_2 < m_3 \cdots and n_1 < n_2 < n_3 < \cdots\} \subseteq \mathbb{N} \times \mathbb{N}$ and E^F be a double sequence space. A $K-$step space of E^F is a sequence space $\lambda_K^E = \left\{\langle X_{m_i n_i} \rangle \in w^{2F} : \langle X_{mn} \rangle \in E^F\right\}$.

A canonical pre-image of a sequence $\langle X_{m_i n_i} \rangle \in E^F$ is a sequence $\langle Y_{mn} \rangle$ defined as follows:

$$Y_{mn} = \begin{cases} X_{mn}, & \text{if } (m,n) \in K, \\ \bar{0}, & \text{otherwise}. \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

A sequence set E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces.

A sequence set E^F is said to be symmetric if $\left\langle X_{\pi(m),\pi(n)} \right\rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$, where π is a permutation of $\mathbb{N}$.

A fuzzy real-valued sequence set E^F is said to be convergent free if $\langle Y_{mn} \rangle \in E^F$ whenever $\langle X_{mn} \rangle \in E^F$ and $X_{mn} = \bar{0}$ implies $Y_{mn} = \bar{0}$.

We define the following classes of sequences:
$\Lambda_f^{2F} = \left\{\langle X_{mn} \rangle : sup_{mn} f\left(\bar{d}\left(X_{mn}^{1/m+n}, \bar{0}\right)\right) < \infty, X_{mn} \in L(\mathbb{R})\right\}$.
$\chi_f^{2F} = \left\{\langle X_{mn} \rangle : lim_{mn\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0\right\}$.
Also, we define the classes of sequences $\chi_f^{2F^R}$ as follows :
A sequence $\langle X_{mn} \rangle \in \chi_f^{2F^R}$ if $\langle x_{mn} \rangle \in \chi_f^{2F}$ and the following limits hold

$$lim_{m\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0 \text{ for each } n \in \mathbb{N}.$$

$$lim_{n\to\infty} f\left(\bar{d}\left(((m+n)!X_{mn})^{1/m+n}, \bar{0}\right)\right) = 0 \text{ for each } m \in \mathbb{N}.$$

Theorem 8.1. *Let* $N_1 = min\Big\{n_0 : sup_{mn\geq n_0} f\Big(\bar{d}\Big(((m+n)!(X_{mn}-Y_{mn}))^{1/m+n}, \bar{0}\Big)\Big)^{P_{mn}} < \infty\Big\}$, $N_2 = min\left\{n_0 : sup_{mn\geq n_0} P_{mn} < \infty\right\}$ *and* $N = max\left(N_1, N_2\right)$.
(i) $\chi_{f_p}^{2F^R}$ *is not a paranormed space with*

$$g\left(X\right) = lim_{N\to\infty} sup_{mn\geq N} f\left(\bar{d}\left(((m+n)!\left(X_{mn}-Y_{mn}\right))^{1/m+n}, \bar{0}\right)\right)^{P_{mn}/M} \tag{8.1}$$

if and only if $\mu > 0$, *where* $\mu = lim_{N\to\infty} inf_{mn\geq N} P_{mn}$ *and* $M = max\left(1, sup_{mn\geq N} P_{mn}\right)$
(ii) $\chi_{f_p}^{2F^R}$ *is complete with the paranorm (3.1).*

Theorem 8.2. *The class of sequences* Λ_f^{2F} *is symmetric but the classes of sequences* χ_f^{2F} *and* $\chi_f^{2F^R}$ *are not sysmmetric.*

9. Difference sequence space related to the space ℓ_p

W.L.C. Sargent introduced the crisp set sequence space m(ϕ) and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as m(ϕ) by D. Rath, B.C. Tripathy, T. Bilgin, A. Esi, M. Sen and others

10. Statistically convergent difference χ^2 sequence spaces

In order to extend the notion of convergence of sequences, statistical convergence of sequences was introduced by H. Fast in 1951, R.C. Buck in 1953 and I.J. Schoenberg in 1959 independently. It is also found in A. Zygmund. Later on it was studied from sequence space point of view and linked with summability theory by J.A. Fridy, T. Šalát, J.S. Connor, I.J. Maddox, D. Rath, M. Mursaleen, B.C. Tripathy, M. Et, F. Nuray, R. Colak, A. Esi and many others. The notion of statistical convergence depends on the notion of asymptotic density of subsets of the set N of natural numbers (refer to Niven, Zuckerman and Montgomery).

Definition 10.1. *A double sequence* $X = (X_{mn})$ *of fuzzy numbers is said to be* $\chi_{f\Delta}^2-$ *convergent in the Pringsheim's sense or* $P_{\chi_{f\Delta}^2}-$ *convergent to a fuzzy number 0 then*

$$lim_{m,n\to\infty} d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \bar{0}\right) = 0$$

we denote $P - lim_{m,n\to\infty} d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}\right) = 0$. *The number 0 is called the Pringsheim limit of* ΔX. *More exactly, we say that a double sequence* (ΔX_{mn}) *converges to a finite fuzzy number 0.*

Definition 10.2. *A double sequence* $X = (X_{mn})$ *of fuzzy numbers is said to be* Λ_Δ^2- *analytic if there exists a positive number K such that if the set*

$$\left\{|X_{mn}|^{1/m+n} : m, n \in \mathbb{N}\right\}$$

We denote the set of all double $\Delta-$ analytic sequences of fuzzy numbers by $\Lambda^2(\Delta, F)$.

Definition 10.3. *A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be $\chi^2_{f\Delta}-$ statistically convergent to 0 such that*

$$P - lim_{k,\ell}\frac{1}{k,\ell}\left|\left\{(m,n) : m \le k, n \le \ell; d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \overline{0}\right)\right\}\right| = 0$$

In this case we write $S^2 - lim_{m,n\to\infty}\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}\right) = 0$. or $f\left(\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \overline{0}\right) \to 0\left(S^2(\Delta, F)\right)$ and we denote the set of all double $\Delta-$ statistically convergent sequences of fuzzy numbers by $S^2(\Delta, F)$.

Definition 10.4. *Let $\beta = (\beta_m)$ and $\mu = (\mu_n)$ be two nondecreasing sequences of positive real numbers such that each tend to infinity and $\beta_{m+1} \le \beta_m + 1, \beta_1 = 1$ and $\mu_{n+1} \le \mu_n + 1, \mu_1 = 1$. A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be strongly $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ summable if there is a fuzzy number $\overline{0}$ such that*

$$P - lim_{k,\ell}\frac{1}{\lambda_{k\ell}}\sum_{m\in I_{k\ell}}\sum_{n\in I_{k\ell}} d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \overline{0}\right) = 0.$$

where $\lambda_{k\ell} = \beta_m \cdot \beta_n$ and $I_{k\ell} = \{(mn) : k - \beta_m + 1 \le m \le k; \ell - \mu_\ell + 1 \le n \le \ell\}$. We denote the set of strongly double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ summable sequences $\left[V_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)}\right]$. If $\lambda_{k\ell} = k\ell$ for all $k, \ell \in \mathbb{N}$, then the class of strongly double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ summable sequences reduce to $[C,1,1]\,\chi^2_f(\Delta, F)$, the class of strongly double cesàro summable sequences of fuzzy numbers defined as follows

$$P - lim_{k\ell}\frac{1}{k\ell}\sum_{m=1}^{k}\sum_{n=1}^{\ell} d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \overline{0}\right) = 0.$$

Definition 10.5. *A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ statistically convergent or $s^2_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)} -$ convergent to a fuzzy number $\overline{0}$,*

$$P - lim_{k\ell}\frac{1}{\lambda_{k\ell}}\left|\left\{(mn) \in I_{k\ell} : d\left(f\left((m+n)!\Delta X_{mn}\right)^{1/m+n}, \overline{0}\right)\right\}\right| = 0$$

In this case we write $s^2_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)} - limf\left((m+n)!\Delta X_{mn}\right)^{1/m+n} = 0$ or $\left((m+n)!\Delta X_{mn}\right)^{1/m+n} \to \overline{0}\left(s^2_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)}\right)$ and we denote the set of all double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ statistically convergent sequences of fuzzy numbers by $s^2_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)}$. If $\lambda_{k\ell} = k\ell$ for all $k, \ell \in \mathbb{N}$, we write $s^2_{\chi^2_f} - limf\left((m+n)!\Delta X_{mn}\right)^{1/m+n} = \overline{0}$ $\left((m+n)!\Delta X_{mn}\right)^{1/m+n} \to \overline{0}\left(s^2_{\left(\chi^{2F}_{f\Delta}\right)}\right)$ and the set $s^2_{\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)} -$ reduces to $s^2_{\left(\chi^{2F}_{f\Delta}\right)}$.

Theorem 10.6. *A double sequence $X = (X_{mn})$ of fuzzy numbers is strongly double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ summable to the fuzzy number $\overline{0}$, then it is double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right) -$ statistically convergent to $\overline{0}$.*

Theorem 10.7. *If a double Δ^2- analytic double sequence of fuzzy numbers $X = (X_{mn})$ is double $\overline{\lambda}\left(\chi^{2F}_{f\Delta}\right)-$ statistically convergent to the fuzzy number $\overline{0}$ then it is strongly $\overline{\lambda}\left(\Lambda^{2F}_{f\Delta}\right)-$ summable to $\overline{0}$*

References

1. T.Apostol, Mathematical Analysis, *Addison-wesley , London*, 1978.
2. M.Basarir and O.Solancan, On some double sequence spaces, *J. Indian Acad. Math.*, **21(2)** (1999), 193-200.
3. C.Bektas and Y.Altin, The sequence space $\ell_M(p,q,s)$ on seminormed spaces, *Indian J. Pure Appl. Math.*, **34(4)** (2003), 529-534.
4. T.J.I'A.Bromwich, An introduction to the theory of infinite series *Macmillan and Co.Ltd.* ,New York, (1965).
5. G.H.Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86-95.
6. M.A.Krasnoselskii and Y.B.Rutickii, Convex functions and Orlicz spaces, *Gorningen, Netherlands*, **1961**.
7. J.Lindenstrauss and L.Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.
8. I.J.Maddox, Sequence spaces defined by a modulus, *Math. Proc. Cambridge Philos. Soc*, **100(1)** (1986), 161-166.
9. F.Moricz, Extentions of the spaces c and c_0 from single to double sequences, *Acta. Math. Hung.*, **57(1-2)**, (1991), 129-136.
10. F.Moricz and B.E.Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, *Math. Proc. Camb. Phil. Soc.*, **104**, (1988), 283-294.
11. M.Mursaleen,M.A.Khan and Qamaruddin, Difference sequence spaces defined by Orlicz functions, *Demonstratio Math.* , **Vol. XXXII** (1999), 145-150.
12. H.Nakano, Concave modulars, *J. Math. Soc. Japan*, **5**(1953), 29-49.
13. W.Orlicz, *Ü*ber Raume $\left(L^M\right)$ *Bull. Int. Acad. Polon. Sci. A*, (1936), 93-107.
14. S.D.Parashar and B.Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.* , **25(4)**(1994), 419-428.
15. K.Chandrasekhara Rao and N.Subramanian, The Orlicz space of entire sequences, *Int. J. Math. Math. Sci.*, **68**(2004), 3755-3764.
16. W.H.Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, *Canad. J. Math.*, **25**(1973), 973-978.
17. B.C.Tripathy, On statistically convergent double sequences, *Tamkang J. Math.*, **34(3)**, (2003), 231-237.
18. B.C.Tripathy,M.Et and Y.Altin, Generalized difference sequence spaces defined by Orlicz function in a locally convex space, *J. Anal. Appl.*, **1(3)**(2003), 175-192.
19. A.Turkmenoglu, Matrix transformation between some classes of double sequences, *J. Inst. Math. Comp. Sci. Math. Ser.*, **12(1)**, (1999), 23-31.
20. P.K.Kamthan and M.Gupta, Sequence spaces and series, Lecture notes, Pure and Applied Mathematics, *65 Marcel Dekker, In c., New York* , 1981.
21. A.Gökhan and R.olak, The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$, *Appl. Math. Comput.*, **157(2)**, (2004), 491-501.
22. A.Gökhan and R.olak, Double sequence spaces ℓ_2^∞, *ibid.*, **160(1)**, (2005), 147-153.

23. M.Zeltser, Investigation of Double Sequence Spaces by Soft and Hard Analitical Methods, Dissertationes Mathematicae Universitatis Tartuensis 25, *Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu*, **2001**.

24. M.Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, *J. Math. Anal. Appl.*, **288(1)**, (2003), 223-231.

25. M.Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 523-531.

26. M.Mursaleen and O.H.H. Edely,Almost convergence and a core theorem for double sequences, *J. Math. Anal. Appl.*, **293(2)**, (2004), 532-540.

27. B.Altay and F.Baar, Some new spaces of double sequences, *J. Math. Anal. Appl.*, **309(1)**, (2005), 70-90.

28. F.Baar and Y.Sever, The space $\mathcal{L}_p$ of double sequences, *Math. J. Okayama Univ*, **51**, (2009), 149-157.

29. N.Subramanian and U.K.Misra, The semi normed space defined by a double gai sequence of modulus function, *Fasciculi Math.*, **46**, (2010).

30. H.Kizmaz, On certain sequence spaces, *Cand. Math. Bull.*, **24(2)**, (1981), 169-176.

31. B.Kuttner, Note on strong summability, *J. London Math. Soc.*, **21**(1946), 118-122.

32. I.J.Maddox, On strong almost convergence, *Math. Proc. Cambridge Philos. Soc.*, **85(2)**, (1979), 345-350.

33. J.Cannor, On strong matrix summability with respect to a modulus and statistical convergence, *Canad. Math. Bull.*, **32(2)**, (1989), 194-198.

34. A.Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, *Math. Ann.*, **53**, (1900), 289-321.

35. H.J.Hamilton, Transformations of multiple sequences, *Duke Math. J.*, **2**, (1936), 29-60.

36. ———-, A Generalization of multiple sequences transformation, *Duke Math. J.*, **4**, (1938), 343-358.

37. ———-, Change of Dimension in sequence transformation , *Duke Math. J.*, **4**, (1938), 341-342.

38. ———-, Preservation of partial Limits in Multiple sequence transformations, *Duke Math. J.*, **4**, (1939), 293-297.

39. , G.M.Robison, Divergent double sequences and series, *Amer. Math. Soc. Trans.*, **28**, (1926), 50-73.

40. L.L.Silverman, On the definition of the sum of a divergent series, *un published thesis*, **University of Missouri studies, Mathematics series**.

41. O.Toeplitz, Über allgenmeine linear mittel bridungen, *Prace Matemalyczno Fizyczne (warsaw)*, **22**, (1911).

42. F.Baar and B.Atlay, On the space of sequences of $p-$ bounded variation and related matrix mappings, *Ukrainian Math. J.*, **55(1)**, (2003), 136-147.

43. B.Altay and F.Baar, The fine spectrum and the matrix domain of the difference operator Δ on the sequence space $\ell_p, (0 < p < 1)$, *Commun. Math. Anal.*, **2(2)**, (2007), 1-11.

44. R.olak,M.Et and E.Malkowsky, Some Topics of Sequence Spaces, Lecture Notes in Mathematics, *Firat Univ. Elazig, Turkey*, **2004**, pp. 1-63, Firat Univ. Press, (2004), ISBN: 975-394-0386-6.

45. H.Dutta, F.Baar, A Generalization of Orlicz sequence spaces by cesàro mean of order one, *Acta Math. Univ. Commenianae*, **Vol. LXXX (2)**, (2011), 185-200.

46. B.C.Tripathy and B.Sarma, Some double sequence spaces of fuzzy numbes defined by Orlicz function, *Acta Mthematica Scientia*, **(**31 B(1)), (2011), 1-7.

47. L.A.Zadeh, Fuzzy sets, *Information and Control*, **8**, (1965), 338-353.

48. S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems*, **33** (1989), 123-126.

49. F.Nuray and E.Savas, Statistical convergence of fuzzy numbers,*Math Slovaca* **45(3)**, (1995), 269-273.

50. F.Nuray, *Lacunary statistical convergence of sequences of fuzzy numbers*, *Fuzzy Sets and Systems*, **99** (1998), 353-356.

51. Ayhan Esi, *On some Double* $\overline{\lambda}(\Delta, F)-$ *statistical convergence of fuzzy numbers*, *Acta Universitatis Apulensis*, **No.25** (2011), 99-104.

N. Subramanian
Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India
E-mail address: nsmaths@yahoo.com

and

K. Balasubramanian
Department of Mathematics,
SASTRA University,
Thanjavur-613 401, India
E-mail address: k_bala27@yahoo.co.in