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Characterizing Of Dual Focal Curves In $\mathbb{D}^3$

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ABSTRACT: In this paper, we study dual focal curves in the Dual Lorentzian 3-space $\mathbb{D}_1^3$. We characterize dual focal curves in terms of their dual focal curvatures.

Key Words: Frenet frame, Dual 3-space, Focal curve.

Contents

1. Introduction

The application of dual numbers to the lines of the 3-space is carried out by the principle of transference which has been formulated by Study and Kotelnikov. It allows a complete generalization of the mathematical expression for the spherical point geometry to the spatial line geometry by means of dual-number extension, i.e. replacing all ordinary quantities by the corresponding dual-number quantities.

In this paper, we study dual focal curves in the Dual Lorentzian 3-space $\mathbb{D}_1^3$. We characterize dual focal curves in terms of their focal curvatures.

2. Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varepsilon\varphi^*$

2000 *Mathematics Subject Classification:* 53A04, 53A10 in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

By a dual number $\hat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathbb{R}$. Let the set $\mathbb{R} \times \mathbb{R}$ be denoted as $\mathbb{D}$. Two inner operations and an equality on $\mathbb{D} = \{(x, x^*) : x, x^* \in \mathbb{R}\}$ are defined as follows:

$(i) \oplus : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\hat{x} = (x, x^*), \hat{y} = (y, y^*)$ defined as

$$\hat{x} \oplus \hat{y} = (x, x^*) \oplus (y, y^*) = (x + y, x^* + y^*)$$

is called the addition in $\mathbb{D}$.

$(ii) \otimes : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ for $\hat{x} = (x, x^*), \hat{y} = (y, y^*)$ defined as

$$\hat{x} \otimes \hat{y} = (x, x^*) \otimes (y, y^*) = (xy, xy^* + x^*y)$$

is called the multiplication in $\mathbb{D}$.

The set $\mathbb{D}$ of dual numbers is a commutative ring.

(iii) If $x = y$, $x^* = y^*$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*) \in \mathbb{D}$, $\hat{x}$ and $\hat{y}$ are equal, and it is indicated $\hat{x} = \hat{y}$.

If the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ with set of real numbers $\mathbb{R}$ are defined as above, the set $\mathbb{D}$ is called the dual numbers system and the element (x, x^*) of $\mathbb{D}$ is called a dual number. In a dual number $\hat{x} = (x, x^*) \in \mathbb{D}$, the real number x is called the real part of $\hat{x}$ and the real number x^* is called the dual part of $\hat{x}$. The dual number $(1, 0) = 1$ is called unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$. The dual number $(0, 1)$ is to be denoted with " in short, and the $(0, 1) = \varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2 = 0$. Also, the dual number $\hat{x} = (x, x^*) \in \mathbb{D}$ can be written as $\hat{x} = x + \varepsilon x^*$.

The set

$$\mathbb{D}^3 = \{\hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^*, \varphi, \varphi^* \in \mathbb{E}^3\}$$

is a module over the ring $\mathbb{D}$.

The Euclidean inner product of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\Omega}, \hat{\varphi} \right\rangle = \langle \Omega, \varphi \rangle + \varepsilon \left(\langle \Omega, \varphi^* \rangle + \langle \Omega^*, \varphi \rangle \right),$$

with the Euclidean inner product Ω and φ

$$\langle \Omega, \varphi \rangle = \Omega_1\varphi_1 + \Omega_2\varphi_2 + \Omega_3\varphi_3,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$.

For $\hat{\varphi} \neq 0$,the norm $\|\hat{\varphi}\|$ of $\hat{\varphi} = \varphi + \varepsilon\varphi^*$ is defined by

$$\langle \hat{\varphi}, \hat{\varphi} \rangle^{\frac{1}{2}} = \|\hat{\varphi}\| = \|\varphi\| \, (1 + \varepsilon \frac{\langle \varphi, \varphi^* \rangle}{\|\varphi\|^2}).$$

A dual vector $\hat{\varphi}$ with norm 1 is called a dual unit vector.
Let $\hat{\varphi} = \varphi + \varepsilon\varphi^* \in \mathbb{D}^3$. The set

$$\mathbb{S}^2 = \{\hat{\varphi} = \varphi + \varepsilon\varphi^* : \|\hat{\varphi}\| = (1, 0); \varphi, \varphi^* \in \mathbb{E}^3\}$$

is called the dual unit sphere with the center $\hat{O}$ in $\mathbb{D}^3$.

3. Dual Focal of Curves Timelike Curves According To Dual Frenet Frame In $\mathbb{D}^3_1$

Denoting the dual focal curve by $\hat{\wp}$ we can write

$$\hat{\wp}(s) = (\hat{\gamma} + \hat{\mathfrak{q}}_1\hat{\mathbf{N}} + \hat{\mathfrak{q}}_2\hat{\mathbf{B}})(s), \tag{3.1}$$

where the coefficients $\hat{\mathfrak{q}}_1$, $\hat{\mathfrak{q}}_2$ are smooth functions of the parameter of the curve $\hat{\gamma}$, called the first and second dual focal curvatures of $\hat{\gamma}$, respectively.

The formula (3.1) is separed into the real and dual part, we have

$$\begin{aligned} \wp(s) &= (\gamma + \mathfrak{m}_1\mathbf{N} + \mathfrak{m}_2\mathbf{B})(s), \qquad (3.2)\\ \wp^*(s) &= (\gamma^* + \mathfrak{m}_1\mathbf{N}^* + \mathfrak{m}_1^*\mathbf{N} + \mathfrak{m}_2\mathbf{B}^* + \mathfrak{m}_2^*\mathbf{B})\,(s). \end{aligned}$$

Theorem 3.1. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}^3_1$ be a unit speed dual timelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3_1$. Then,*

$$\wp = \gamma - \frac{1}{\kappa}\mathbf{N} + \frac{\kappa'}{\kappa^2\tau}\mathbf{B}, \tag{3.3}$$

$$\begin{aligned} \wp^* = \; & \gamma^* + \frac{1}{\kappa}\mathbf{N}^* - \frac{\kappa^*}{\kappa^2}\mathbf{N} + \frac{\kappa'}{\kappa^2\tau}\mathbf{B}^* + \qquad (3.4)\\ & \left(\frac{-(\kappa^*)'\kappa^2 + 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}\right)\mathbf{B}. \end{aligned}$$

Proof: Assume that $\hat{\gamma}$ is a unit speed dual curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}^3_1$.
So, by differentiating of the formula (3.1), we get

$$\hat{\wp}(s)' = (1 + \hat{\kappa}\hat{\mathfrak{m}}_1)\hat{\mathbf{T}} + (\hat{\mathfrak{m}}_1' - \hat{\tau}\hat{\mathfrak{m}}_2)\hat{\mathbf{N}} + (\hat{\tau}\hat{\mathfrak{m}}_1 + \hat{\mathfrak{m}}_2')\hat{\mathbf{B}}. \tag{3.5}$$

Using above equation, the first 2 components vanish, we have

$$\begin{aligned} \kappa\mathfrak{m}_1 &= -1,\\ \kappa\mathfrak{m}_1^* + \kappa_1^*\mathfrak{m} &= 0,\\ \mathfrak{m}_1' - \tau\mathfrak{m}_2 &= 0,\\ (\mathfrak{m}_1^*)' - \tau\mathfrak{m}_2^* - \tau_2^*\mathfrak{m} &= 0. \end{aligned}$$

Considering equations above system, we have

$$\begin{aligned}\mathfrak{m}_1 &= -\frac{1}{\kappa},\\ \mathfrak{m}_1^* &= -\frac{\kappa^*}{\kappa^2},\\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau},\\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2 - 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}.\end{aligned}$$

By means of obtained equations, we express (3.3) and (3.4). This completes the proof. □

Corollary 3.2. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual timelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then, the dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned}\mathfrak{m}_1 &= -\frac{1}{\kappa},\\ \mathfrak{m}_1^* &= \frac{\kappa^*}{\kappa^2},\\ \mathfrak{m}_2 &= \frac{\kappa'}{\kappa^2\tau},\\ \mathfrak{m}_2^* &= \frac{(\kappa^*)'\kappa^2 - 2\kappa\kappa^*\kappa'}{\kappa^4\tau} - \frac{\tau^*\kappa'}{\kappa^2\tau^2}.\end{aligned}$$

In the light of Theorem 3.1, we express the following corollary without proof:

Corollary 3.3. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual timelike curve and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. If $\hat{\kappa}$ and $\hat{\tau}$ are constant then, the dual focal curvatures of $\hat{\wp}$ are*

$$\begin{aligned}\mathfrak{m}_1 &= constant \neq 0,\\ \mathfrak{m}_1^* &= constant \neq 0,\\ \mathfrak{m}_2 &= 0,\\ \mathfrak{m}_2^* &= 0.\end{aligned}$$

Corollary 3.4. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a unit speed dual timelike helix and $\hat{\wp}$ its dual focal curve on $\mathbb{D}_1^3$. Then,*

$$\hat{\wp}(s) = \hat{\gamma}(s) + \hat{\mathfrak{m}}_1\hat{\mathbf{N}}(s),$$

where $\hat{\mathfrak{m}}_1$ is first dual focal curvatures of $\hat{\gamma}$.

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