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On Some New Modular Sequence Spaces

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ABSTRACT: Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the sequence space ℓ_M which is called an Orlicz sequence space. Another generalization of Orlicz sequence spaces is due to Woo [9]. An important subspace of $\ell(\mathcal{M})$, which is an AK-space, is the space $h(\mathcal{M})$. We define the sequence spaces $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ and $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$, where $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ are sequences of Orlicz functions such that M_k and N_k be mutually complementary for each k. We also examine some topological properties of these spaces. We give the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence space $h(\mathcal{M})$ and $\alpha-$ duals of the squence spaces $\ell(\mathcal{M},\lambda)$ and $\ell(\mathcal{N},\lambda)$.

Key Words: Difference sequence spaces, Orlicz function, $\alpha-$, $\beta-$ and $\gamma-$ duals.

Contents

1. Introduction

The difference sequence spaces was introduced by Kızmaz [6] and the concept was generalized by Et and Çolak [2]. After, Et and Esi [3] extended the difference sequence spaces to the sequence spaces

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $v = (v_k)$ be any fixed sequence of non-zero complex numbers and $(\Delta_v^m x_k) = \left(\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right)$, $\Delta_v^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for all $k \in \mathbb{N}$.

The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |v_i x_i| + \|\Delta_v^m x\|_\infty .$$

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An Orlicz function is a function $M:[0,\infty)\rightarrow[0,\infty)$ which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\rightarrow\infty$ as $x\rightarrow\infty$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the Orlicz sequence space

$$\ell_M=\left\{x=(x_k):\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty,\text{ for some }\rho>0\right\}$$

which is a Banach space with the norm

$$\|x\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\leq 1\right\}.$$

It is well known that if M is a convex function and $M(0)=0$, then $M(\lambda x)\leq \lambda M(x)$ for all λ with $0\leq\lambda\leq 1$.

Let λ be a sequence space and defined

$$\begin{aligned}\lambda^{\alpha}&=\{a=(a_k):\sum_{k=1}^{\infty}|a_kx_k|<\infty,\text{ for all }x\in\lambda\},\\ \lambda^{\beta}&=\{a=(a_k):\sum_{k=1}^{\infty}a_kx_k\text{ converges, for all }x\in\lambda\},\\ \lambda^{\gamma}&=\{a=(a_k):\sup_n\left|\sum_{k=1}^{n}a_kx_k\right|<\infty,\text{ for all }x\in\lambda\}\ [5].\end{aligned}$$

Then λ^{α}, λ^{β}, λ^{γ} are called $\alpha-$, $\beta-$, $\gamma-$dual spaces of λ, respectively. It is easy to show that $\varnothing\subset\lambda^{\alpha}\subset\lambda^{\beta}\subset\lambda^{\gamma}$. If $\lambda\subset\mu$, then $\mu^{\eta}\subset\lambda^{\eta}$ for $\eta=\alpha,\beta,\gamma$. We shall write $\lambda^{\alpha\alpha}=(\lambda^{\alpha})^{\alpha}$ [5].

Definition 1.1. *Let λ be a sequence space. Then λ is called*

(i) Solid (or normal), if $(\alpha_kx_k)\in\lambda$ whenever $(x_k)\in\lambda$ for all sequences (α_k) of scalars with $|\alpha_k|\leq 1$.

(ii) Monotone, if provided λ contains the canonical preimages of all its stepspaces.

(iii) Perfect, if $\lambda=\lambda^{\alpha\alpha}$ [5].

Proposition 1.2. *λ is perfect $\Rightarrow$ λ is normal $\Rightarrow$ λ is monotone [5].*

Proposition 1.3. *Let λ be a sequence space. If λ is monotone, then $\lambda^{\alpha}=\lambda^{\beta}$, and if λ is normal, then $\lambda^{\alpha}=\lambda^{\gamma}$.*

A Banach sequence space (λ,S) is called a $BK-$space if the topology S of λ is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps $P_i:\lambda\rightarrow K$, $P_i(x)=x_i$, $i\geq 1$ are continuous, where K is the scalar field $\mathbb{R}$ (the set of all reals) or $\mathbb{C}$ (the complex plane). For $x=(x_1,...,x_n,...)$ and $n\in\mathbb{N}$ (the set of natural numbers), we write the n^{th} section of x as $x^{(n)}=(x_1,...,x_n,0,0,...)$. If $\{x^{(n)}\}$ tends to x in (λ,S) for each $x\in\lambda$, we say that (λ,S) is an $AK-$space. The norm $\|.\|_{\lambda}$ generating the topology S of λ is said to be monotone if $\|x\|_{\lambda}\leq\|y\|_{\lambda}$ for $x=\{x_i\}$, $y=\{y_i\}\in\lambda$ with $|x_i|\leq|y_i|$, for all $i\geq 1$ [4].

Any Orlicz function M_k always has the integral representation

$$M_k(x) = \int_0^x p_k(t)dt,$$

where p_k, known as the kernel of M_k, is non-decreasing, is right continuous for $t > 0$, $p_k(0) = 0$, $p_k(t) > 0$ for $t > 0$ and $p_k(t) \to \infty$ as $t \to \infty$.

Given an Orlicz function M_k with kernel $p_k(t)$, define

$$q_k(s) = \sup\{t : p_k(t) \leq s,\ s \geq 0\}.$$

Then $q_k(s)$ possesses the same properties as $p_k(t)$ and the function N_k defined as

$$N_k(x) = \int_0^x q_k(s)ds$$

is an Orlicz function. The functions M_k and N_k are called mutually complementary Orlicz functions.

For a sequence $\mathcal{M} = (M_k)$ of Orlicz functions, the modular sequence class $\tilde{\ell}(\mathcal{M})$ is defined by

$$\tilde{\ell}(\mathcal{M}) = \{x = (x_k) : \sum_{k=1}^{\infty} M_k(|x_k|) < \infty\}.$$

Using the sequence $\mathcal{N} = (N_k)$ of Orlicz functions, similarly we define $\tilde{\ell}(\mathcal{N})$. The class $\ell(\mathcal{M})$ is defined by

$$\ell(\mathcal{M}) = \{x = (x_k) : \sum_{k=1}^{\infty} x_k y_k \text{ converges, for all } y \in \tilde{\ell}(\mathcal{N})\}. \tag{1}$$

For a sequence $\mathcal{M} = (M_k)$ of Orlicz functions, the modular sequence space $\ell(\mathcal{M})$ is also defined as

$$\ell(\mathcal{M}) = \{x = (x_k) : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}.$$

The space $\ell(\mathcal{M})$ is a Banach space with respect to the norm $\|x\|_{\mathcal{M}}$ defined as

$$\|x\|_{\mathcal{M}} = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) \leq 1\}.$$

These spaces were introduced by Woo [9] around the year 1973, and generalizes the Orlicz sequence space ℓ_M and the modulared sequence spaces considered earlier by Nakano in [8].

Proposition 1.4. *Let M_k and N_k be mutually complementary functions for each k. Then*

(i) *For $x, y \geq 0$, $xy \leq M_k(x) + N_k(y)$.*

(ii) *For $x \geq 0$, $xp_k(x) = M_k(x) + N_k(p_k(x))$.*

An important subspace of $\ell(\mathcal{M})$, which is an AK-space, is the space $h(\mathcal{M})$ defined as

$$h(\mathcal{M}) = \{x \in \ell(\mathcal{M}) : \sum_{k=1}^{\infty} M_k \left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0\}.$$

A sequence (M_k) of Orlicz functions is said to satisfy uniform Δ_2- condition at $'0'$ if there exist $p > 1$ and $k_0 \in \mathbb{N}$ such that for all $x \in (0,1)$ and $k > k_0$, we have $\frac{xM_k'(x)}{M_k(x)} \leq p$, or equivalently, there exist a constant $K > 1$ and $k_0 \in \mathbb{N}$ such that $\frac{M_k(2x)}{M_k(x)} \leq K$ for all $k > k_0$ and $x \in \left(0, \frac{1}{2}\right]$. If the sequence (M_k) satisfies uniform Δ_2-condition, then $h(\mathcal{M}) = \ell(\mathcal{M})$ and vice-versa [9].

2. Main Results

Definition 2.1. *Let M_k and N_k be mutually complementary functions for each k and let $\lambda = \{\lambda_k\}$ be a sequence of strictly positive real numbers. Then we define the following sequence spaces:*

$$\ell_{\lambda}^{\mathcal{M}}(\Delta^m) = \{x = (x_k) : \sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \rho} \right) < \infty, \text{ for some } \rho > 0\}$$

and

$$\ell_{\mathcal{N}}^{\lambda}(\Delta^m) = \{x = (x_k) : \sum_{k \geq 1} N_k \left(\frac{\lambda_k |\Delta^m x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0\}.$$

The spaces $\ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ and $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$ also can be written as

$$\ell_{\lambda}^{\mathcal{M}}(\Delta^m) = \{x = (x_k) : \{\frac{\Delta^m x_k}{\lambda_k}\} \in \ell(\mathcal{M})\}$$

and

$$\ell_{\mathcal{N}}^{\lambda}(\Delta^m) = \{x = (x_k) : \{\lambda_k \Delta^m x_k\} \in \ell(\mathcal{N})\}.$$

Throughout the paper we write $M_k(1) = 1$ and $N_k(1) = 1$ for all $k \in \mathbb{N}$.

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions. Then $\ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ and $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$ are linear spaces over the field of complex numbers.*

Proof:

Let $x, y \in \ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ and $a, b \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \rho_1} \right) < \infty$$

and

$$\sum_{k \geq 1} M_k \left(\frac{|\Delta^m y_k|}{\lambda_k \rho_2} \right) < \infty.$$

Define $\rho_3 = \max(2|a|\rho_1, 2|b|\rho_2)$. Since M_k are non-decreasing and convex functions and Δ^m is linear, we have

$$\sum_{k\geq 1} M_k\left(\frac{|\Delta^m(ax_k+by_k)|}{\lambda_k\rho_3}\right) \leq \sum_{k\geq 1} M_k\left(\frac{|\Delta^m x_k|}{\lambda_k\rho_1}\right) + \sum_{k\geq 1} M_k\left(\frac{|\Delta^m y_k|}{\lambda_k\rho_2}\right) < \infty.$$

This proves that $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ is a linear space. The proof for $\ell_{\mathcal{N}}^\lambda(\Delta^m)$ is similar.

□

The proofs of the following theorems are easy and thus omitted.

Theorem 2.3. *The sequence space $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ is a normed space with norm*

$$\|x\|_\lambda^{\mathcal{M}} = \sum_{i=1}^m |x_i| + \inf\{\rho>0 : \sum_{k\geq 1} M_k\left(\frac{|\Delta^m x_k|}{\lambda_k\rho}\right) \leq 1\}.$$

Theorem 2.4. *The sequence space $\ell_{\mathcal{N}}^\lambda(\Delta^m)$ is a normed space with norm*

$$\|x\|_{\mathcal{N}}^\lambda = \sum_{i=1}^m |x_i| + \inf\{\rho>0 : \sum_{k\geq 1} N_k\left(\frac{\lambda_k|\Delta^m x_k|}{\rho}\right) \leq 1\}.$$

Theorem 2.5. *The spaces $\left(\ell_\lambda^{\mathcal{M}}(\Delta^m), \|.\|_\lambda^{\mathcal{M}}\right)$ and $\left(\ell_{\mathcal{N}}^\lambda(\Delta^m), \|.\|_{\mathcal{N}}^\lambda\right)$ are Banach spaces.*

Theorem 2.6. *The sequence spaces $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ equipped with the norm $\|.\|_\lambda^{\mathcal{M}}$ and $\ell_{\mathcal{N}}^\lambda(\Delta^m)$ equipped with the norm $\|.\|_{\mathcal{N}}^\lambda$ are BK-spaces.*

Proof:

The space $\left(\ell_\lambda^{\mathcal{M}}(\Delta^m), \|.\|_\lambda^{\mathcal{M}}\right)$ is a Banach space by Theorem 2.5. Now let

$$\|x^n - x\|_\lambda^{\mathcal{M}} \to 0$$

as $n\to\infty$. Then

$$|x_k^n - x_k| \to 0$$

as $n\to\infty$ for each $k\leq m$ and

$$\inf\{\rho>0 : \sum_{k\geq 1} M_k\left(\frac{|\Delta^m x_k^n - \Delta^m x_k|}{\lambda_k\rho}\right) \leq 1\} \to 0$$

as $n\to\infty$ for all $k\in\mathbb{N}$. If $M_k\left(\frac{|\Delta^m x_k^n - \Delta^m x_k|}{\lambda_k\|x\|_\lambda^{\mathcal{M}}}\right) \leq 1$ then $\frac{|\Delta^m x_k^n - \Delta^m x_k|}{\lambda_k\|x\|_\lambda^{\mathcal{M}}} \leq 1$ for all k. Therefore we also obtain

$$|\Delta^m x_k^n - \Delta^m x_k| \leq \lambda_k \|x^n - x\|_\lambda^{\mathcal{M}}.$$

Since $\|x^n - x\|_\lambda^{\mathcal{M}} \to 0$, then $|\Delta^m x_k^n - \Delta^m x_k| \to 0$ and

$$\left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} \left(x_{k+v}^n - x_{k+v}\right)\right| \to 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$. On the other hand, since we may write

$$\begin{aligned} \left|x_{k+m}^n - x_{k+m}\right| &\leq \left|\sum_{v=0}^{m} (-1)^v \binom{m}{v} \left(x_{k+v}^n - x_{k+v}\right)\right| \\ &+ \left|\binom{m}{0} (x_k^n - x_k)\right| + ... + \left|\binom{m}{m-1} \left(x_{k+m-1}^n - x_{k+m-1}\right)\right| \end{aligned}$$

then $|x_k^n - x_k| \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. Hence $\left(\ell_\lambda^{\mathcal{M}}(\Delta^m), \|.\|_\lambda^{\mathcal{M}}\right)$ is a BK-space.

The proof is similar for $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$. □

Theorem 2.7. *If μ is a normal sequence space containing λ, then $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ is a proper subspace of μ. In addition, if μ is equipped with the monotone norm (quasi-norm) $\|.\|_\mu$, the inclusion map $I : \ell_\lambda^{\mathcal{M}}(\Delta^m) \to \mu(\Delta^m)$ is continuous with $\|I\| \leq \|\{\lambda_k\}\|_\mu$.*

Proof:

Let $x \in \ell_\lambda^{\mathcal{M}}(\Delta^m)$. Since $\sum_{k\geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \rho}\right) < \infty$ for some $\rho > 0$, then there exists a constant $K > 0$ such that $\frac{|\Delta^m x_k|}{\lambda_k \rho} \leq K$ for all $k \in \mathbb{N}$. Since μ is a normal sequence space containing λ, we have $(\Delta^m x_k) \in \mu$ and so that $x \in \mu(\Delta^m)$. Hence $\ell_\lambda^{\mathcal{M}}(\Delta^m) \subset \mu(\Delta^m)$.

Further, since $M_k(1) = 1$ for all $k \in \mathbb{N}$, then

$$\sum_{k\geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \|x\|_\lambda^{\mathcal{M}}}\right) \leq 1$$

and so that

$$|\Delta^m x_k| \leq \lambda_k \|x\|_\lambda^{\mathcal{M}}, \text{ for all } k \in \mathbb{N}.$$

As $\|.\|_\mu$ is monotone, $\|Ix\|_\mu = \|(\Delta^m x_k)\|_\mu \leq \|\{\lambda_k\}\|_\mu \|x\|_\lambda^{\mathcal{M}}$ and hence $\|I\| \leq \|\{\lambda_k\}\|_\mu$.

□

Theorem 2.8. *If η is a normal sequence space containing $\{\frac{1}{\lambda_k}\} \equiv \lambda^{-1}$, then $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$ is a proper subspace of η. If the norm (quasi-norm) $\|.\|_\eta$ on η is monotone, then the inclusion map $J : \ell_{\mathcal{N}}^{\lambda}(\Delta^m) \to \eta(\Delta^m)$ is continuous with $\|J\| \leq \left\|\{\lambda_k^{-1}\}\right\|_\eta$.*

The proof is similar to Theorem 2.7 and therefore we omitted.

3. Interrelationship Between the Spaces $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ and $\ell_{\mathcal{M}}^\lambda(\Delta^m)$

If $\lambda_k = 1$ for all $k \in \mathbb{N}$, then the sequence space $\ell_{\mathcal{M}}^\lambda(\Delta^m)$ reduces to the sequence space

$$\ell_{\mathcal{M}}(\Delta^m) = \{x = (x_k) : \sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0\}.$$

Theorem 3.1. *If $\lambda = \{\lambda_k\}$ is a bounded sequence such that $\inf \lambda_k > 0$, then $\ell_{\mathcal{M}}^\lambda(\Delta^m) = \ell_\lambda^{\mathcal{M}}(\Delta^m) = \ell_{\mathcal{M}}(\Delta^m)$.*

Proof:
Let $x \in \ell_{\mathcal{M}}(\Delta^m)$. Then $\sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\rho} \right) < \infty$ for some $\rho > 0$. Since $\lambda = \{\lambda_k\}$ is bounded, we can write $a \leq \lambda_k \leq b$ for some $b > a \geq 0$. Define $\rho_1 = \rho b$. Since M_k is increasing, it follows that $\sum_{k \geq 1} M_k \left(\frac{\lambda_k |\Delta^m x_k|}{\rho_1} \right) \leq \sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\rho} \right) < \infty$. Hence $\ell_{\mathcal{M}}(\Delta^m) \subset \ell_{\mathcal{M}}^\lambda(\Delta^m)$. The other inclusion $\ell_{\mathcal{M}}^\lambda(\Delta^m) \subset \ell_{\mathcal{M}}(\Delta^m)$ follows from the inequality $\sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\rho/a} \right) \leq \sum_{k \geq 1} M_k \left(\frac{\lambda_k |\Delta^m x_k|}{\rho} \right) < \infty$. Therefore $\ell_{\mathcal{M}}^\lambda(\Delta^m) = \ell_{\mathcal{M}}(\Delta^m)$. Similarly, one can prove $\ell_\lambda^{\mathcal{M}}(\Delta^m) = \ell_{\mathcal{M}}(\Delta^m)$.

□

Theorem 3.2. *If $\{\lambda_k\} \in \ell_\infty$ with $a = \sup_{k \geq 1} \lambda_k \geq 1$ and $\{\lambda_k^{-1}\}$ is unbounded, then $\ell_\lambda^{\mathcal{M}}(\Delta^m)$ is properly contained in $\ell_{\mathcal{M}}^\lambda(\Delta^m)$ and the inclusion map $T : \ell_\lambda^{\mathcal{M}}(\Delta^m) \to \ell_{\mathcal{M}}^\lambda(\Delta^m)$ is continuous with $\|T\| \leq a^2$.*

Proof:
For any $\rho > 0$ and $\rho' = \rho a^2$, we have

$$\sum_{k \geq 1} M_k \left(\frac{\lambda_k \, |\Delta^m x_k|}{\rho'} \right) \leq \sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \rho} \right) < \infty$$

for $x = \{x_k\}$. Hence $\ell_\lambda^{\mathcal{M}}(\Delta^m) \subset \ell_{\mathcal{M}}^\lambda(\Delta^m)$.

We now show that the containment $\ell_\lambda^{\mathcal{M}}(\Delta^m) \subset \ell_{\mathcal{M}}^\lambda(\Delta^m)$ is proper. From the unboundedness of the sequence $\{\lambda_k^{-1}\}$, choose a sequence $\{k_n\}$ of positive integers such that $\lambda_{k_n}^{-1} \geq n$. Define $\Delta^m x = \{\Delta^m x_k\}$ as follows:

$$\Delta^m x_k = \begin{cases} 1/n, & k = k_n, \quad n = 1, 2, ... \\ 0, & \text{otherwise} \end{cases}$$

Then $x \in \ell_{\mathcal{M}}^\lambda(\Delta^m)$; but $x \notin \ell_\lambda^{\mathcal{M}}(\Delta^m)$.

To prove the continuity of the inclusion map T, let us first consider the case obtained for $a = 1$. For $x \in \ell_\lambda^{\mathcal{M}}(\Delta^m)$, we write

$$A_\lambda^{\mathcal{M}}(\Delta^m x) = \left\{ \rho > 0 : \sum_{k \geq 1} M_k \left(\frac{|\Delta^m x_k|}{\lambda_k \rho} \right) \leq 1 \right\}$$

and

$$B_{\mathcal{M}}^{\lambda}\left(\Delta^m x\right)=\left\{\rho>0:\sum_{k\geq 1}M_k\left(\frac{\lambda_k\left|\Delta^m x_k\right|}{\rho}\right)\leq 1\right\}.$$

Since M_k are increasing and $a=1$, we get $A_{\lambda}^{\mathcal{M}}\left(\Delta^m x\right)\subset B_{\mathcal{M}}^{\lambda}\left(\Delta^m x\right)$.

Hence

$$\|x\|_{\mathcal{M}}^{\lambda}=\inf B_{\mathcal{M}}^{\lambda}\left(\Delta^m x\right)\leq\inf A_{\lambda}^{\mathcal{M}}\left(\Delta^m x\right)=\|x\|_{\lambda}^{\mathcal{M}} \tag{2}$$

i.e, $\|T(x)\|_{\mathcal{M}}^{\lambda}\leq\|x\|_{\lambda}^{\mathcal{M}}$. Thus T is continuous with $\|T\|\leq 1=a^2$.

If $a\neq 1$, define $\beta_k=\frac{\lambda_k}{a}$, $k\in\mathbb{N}$. Then $\beta_k\leq 1$ and from (2), it follows that

$$\|x\|_{\mathcal{M}}^{\beta}\leq\|x\|_{\beta}^{\mathcal{M}}\text{ for }x\in\ell_{\lambda}^{\mathcal{M}}(\Delta^m). \tag{3}$$

Hence from (3)

$$\|T(x)\|_{\mathcal{M}}^{\lambda}=\|x\|_{\mathcal{M}}^{\lambda}\leq a^2\,\|x\|_{\lambda}^{\mathcal{M}},$$

i.e., T is continuous with $\|T\|\leq a^2$. This completes the proof. □

Theorem 3.3. *If $\{\lambda_k\}$ is unbounded with $\sup_{k\geq 1}\lambda_k^{-1}=d\geq 1$, $\lambda_k>0$ for all k, then $\ell_{\mathcal{M}}^{\lambda}(\Delta^m)$ is properly contained in $\ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ and the inclusion map $U:\ell_{\mathcal{M}}^{\lambda}(\Delta^m)\to\ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ is continuous with $\|U\|\leq d^2$.*

Proof: The proof of the theorem is similar to that of Theorem 3.2 and so is omitted. □

4. Dual Spaces of $h(\mathcal{M})$, $\ell(\mathcal{M},\lambda)$ and $\ell(\mathcal{N},\lambda)$

In this section we give the $\alpha-$, $\beta-$ and $\gamma-$ duals of the sequence space $h(\mathcal{M})$ and $\alpha-$ duals of $\ell(\mathcal{M},\lambda)$ and $\ell(\mathcal{N},\lambda)$.

Theorem 4.1. *Let the functions M_k and N_k, for each k be mutually complementary Orlicz functions. Then $[h(\mathcal{M})]^{\beta}=\ell(\mathcal{N})$ where*

$$\ell(\mathcal{N})=\{a=(a_k):\sum_{k=1}^{\infty}N_k\left(\frac{|a_k|}{\rho'}\right)<\infty,\ \textit{for some }\rho>0\textit{ and }\rho'=\frac{1}{\rho}\}.$$

Proof:

Let $a=(a_k)\in\ell(\mathcal{N})$ and hence $\sum_{k=1}^{\infty}N_k\left(\frac{|a_k|}{\rho'}\right)<\infty$ for some $\rho'>0$. Take any $x=(x_k)$ in $h(\mathcal{M})$. Then we have

$$\sum_{k=1}^{\infty}|a_k x_k|\leq\sum_{k=1}^{\infty}M_k\left(\frac{|x_k|}{\rho}\right)+\sum_{k=1}^{\infty}N_k\left(\frac{|a_k|}{\rho'}\right)<\infty.$$

Hence $\sum_{k=1}^{\infty} a_k x_k$ converges and $a \in [h(\mathcal{M})]^{\beta}$.

On the other hand, suppose $a \in [h(\mathcal{M})]^{\beta}$. Using (1), we find $[h(\mathcal{M})]^{\beta} \subset \ell(\mathcal{N})$. Thus $[h(\mathcal{M})]^{\beta} = \ell(\mathcal{N})$.

□

Proposition 4.2. *The sequence space $h(\mathcal{M})$ is normal for any sequence $\mathcal{M} = (M_k)$ of Orlicz functions.*

Proof: Let $x \in h(\mathcal{M})$ and $|y_k| \leq |x_k|$ for each $k \in \mathbb{N}$. Since M_k are non-decreasing, we have

$$\sum_{k=1}^{\infty} M_k\left(\frac{|y_k|}{\rho}\right) \leq \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) < \infty.$$

Hence $y \in h(\mathcal{M})$. Thus $h(\mathcal{M})$ is normal.

□

Theorem 4.3. *Let M_k and N_k, for each k be mutually complementary functions. Then*

$$[h(\mathcal{M})]^{\beta} = [h(\mathcal{M})]^{\alpha} = [h(\mathcal{M})]^{\gamma} = \ell(\mathcal{N}).$$

Proof is seen from Proposition 1.2, Proposition 1.3 and Proposition 4.2.

For $m = 0$, we write $\ell(\mathcal{M}, \lambda)$ and $\ell(\mathcal{N}, \lambda)$ instead of $\ell_{\lambda}^{\mathcal{M}}(\Delta^m)$ ve $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$, respectively which we define

$$\begin{aligned} \ell(\mathcal{M}, \lambda) &= \{x = (x_k) : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\lambda_k \rho}\right) < \infty, \text{ for some } \rho > 0\}, \\ \ell(\mathcal{N}, \lambda) &= \{y = (y_k) : \sum_{k=1}^{\infty} N_k\left(\frac{\lambda_k |y_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}. \end{aligned}$$

Theorem 4.4. ***(i)****If the sequence (M_k) satisfies uniform Δ_2-condition, then $[\ell(\mathcal{M}, \lambda)]^{\alpha} = \ell(\mathcal{N}, \lambda)$.*

(ii) *If the sequence (N_k) satisfies uniform Δ_2-condition, then $[\ell(\mathcal{N}, \lambda)]^{\alpha} = \ell(\mathcal{M}, \lambda)$.*

Proof: Let the sequence (M_k) satisfies uniform Δ_2-condition. Then for any $x \in \ell(\mathcal{M}, \lambda)$ and $a \in \ell(\mathcal{N}, \lambda)$, we have

$$\sum_{k=1}^{\infty} |a_k x_k| \leq \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\lambda_k \rho}\right) + \sum_{k=1}^{\infty} N_k\left(\frac{\lambda_k |a_k|}{\rho'}\right) < \infty$$

where $\rho' = 1/\rho$ and $\rho > 0$. Thus $a \in [\ell(\mathcal{M}, \lambda)]^{\alpha}$. Hence $\ell(\mathcal{N}, \lambda) \subset [\ell(\mathcal{M}, \lambda)]^{\alpha}$.

To prove the inclusion $[\ell(\mathcal{M},\lambda)]^{\alpha} \subset \ell(\mathcal{N},\lambda)$, let $a \in [\ell(\mathcal{M},\lambda)]^{\alpha}$. Then for all $\{x_k\}$ with $\left(\frac{x_k}{\lambda_k}\right) \in \ell(\mathcal{M})$ we have

$$\sum_{k=1}^{\infty} |a_k x_k| < \infty.$$

Since the sequence (M_k) satisfies uniform Δ_2-condition, then $\ell(\mathcal{M}) = h(\mathcal{M})$ and so for $(y_k) \in h(\mathcal{M})$, we get $\sum_{k=1}^{\infty} |\lambda_k y_k a_k| < \infty$ by (4). Thus $(\lambda_k a_k) \in [h(\mathcal{M})]^{\alpha} = \ell(\mathcal{N})$ and hence $(a_k) \in \ell(\mathcal{N},\lambda)$. This gives that $[\ell(\mathcal{M},\lambda)]^{\alpha} = \ell(\mathcal{N},\lambda)$.

(ii) Similarly, one can prove that $[\ell(\mathcal{N},\lambda)]^{\alpha} = \ell(\mathcal{M},\lambda)$ if the sequence (N_k) satisfies uniform Δ_2-condition.

□

Theorem 4.5. *If the sequences (N_k) and (M_k) satisfy uniform Δ_2-condition, then the sequence spaces $\ell(\mathcal{M},\lambda)$ and $\ell(\mathcal{N},\lambda)$ are perfect.*

Proof: It is immediate from Theorem 4.4. □

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