Bol. Soc. Paran. Mat. (3s.) v. 31 2 (2013): 47–53.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v31i2.15957

Characterization Of Inextensible Flows Of Spacelike Curves With Sabban Frame In $\mathbb{S}_1^2$

Mahmut Ergüt, Essin Turhan and Talat Körpınar

abstract: In this paper, we study inextensible flows of spacelike curves on $\mathbb{S}_1^2$. We obtain partial differential equations in terms of inextensible flows of spacelike curves according to Sabban frame on $\mathbb{S}_1^2$.

Key Words: Inextensible flows, Sabban Frame.

Contents

1. Introduction

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications.

This study is organised as follows: Firstly, we study inextensible flows of spacelike curves on $\mathbb{S}_1^2$. Secondly, we obtain partial differential equations in terms of inextensible flows of spacelike curves according to Sabban frame on $\mathbb{S}_1^2$.

2. Preliminaries

The Minkowski 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}_1^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the spacelike curve γ in the space $\mathbb{E}_1^3$. For an arbitrary spacelike curve γ with first and second curvature,

2000 *Mathematics Subject Classification:* 53A35

κ and τ in the space $\mathbb{E}_1^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N} \\ \mathbf{N}' &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' &= \tau\mathbf{N}, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= 1, \langle \mathbf{N}, \mathbf{N} \rangle = 1, \langle \mathbf{B}, \mathbf{B} \rangle = -1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}'(s)\|$ and $\tau(s) = -\left\langle \mathbf{N}, \mathbf{B}' \right\rangle$.

Torsion of the spacelike curve γ is given by the aid of the mixed product

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \longrightarrow \mathbb{S}_1^2$ be unit speed spherical curve. We denote σ as the arc-length parameter of α . Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on $\mathbb{S}_1^2$. Then we have the following spherical Frenet-Serret formulae of α :

$$\begin{aligned} \alpha' &= \mathbf{t}, \\ \mathbf{t}' &= -\alpha - \kappa_g \mathbf{s}, \\ \mathbf{s}' &= -\kappa_g \mathbf{t}, \end{aligned} \tag{2.2}$$

where κ_g is the geodesic curvature of the spacelike curve α on the $\mathbb{S}_1^2$ and

$$\begin{aligned} g(\mathbf{t}, \mathbf{t}) &= 1, \; g(\alpha, \alpha) = 1, \; g(\mathbf{s}, \mathbf{s}) = -1, \\ g(\mathbf{t}, \alpha) &= g(\mathbf{t}, \mathbf{s}) = g(\alpha, \mathbf{s}) = 0. \end{aligned}$$

3. Inextensible Flows of Curves According to Sabban Frame in $\mathbb{S}_1^2$

Let $\alpha(u, t)$ is a one parameter family of smooth spacelike curves in $\mathbb{S}_1^2$.

The arclength of α is given by

$$\sigma(u) = \int_0^u \left| \frac{\partial \alpha}{\partial u} \right| du, \tag{3.1}$$

where

$$\left| \frac{\partial \alpha}{\partial u} \right| = \left| \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial u} \right\rangle \right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\frac{\partial}{\partial\sigma}$ is given in terms of u by

$$\frac{\partial}{\partial\sigma}=\frac{1}{\nu}\frac{\partial}{\partial u},$$

where $v=\left|\frac{\partial\alpha}{\partial u}\right|$ and the arclength parameter is $d\sigma=vdu$.

Any flow of α can be represented as

$$\frac{\partial\alpha}{\partial t}=\mathfrak{f}_1^{\mathbb{S}}\alpha+\mathfrak{f}_2^{\mathbb{S}}\mathbf{t}+\mathfrak{f}_3^{\mathbb{S}}\mathbf{s}. \tag{3.3}$$

Letting the arclength variation be

$$\sigma(u,t)=\int_0^u vdu.$$

In the $\mathbb{S}_1^2$ the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}\sigma(u,t)=\int_0^u\frac{\partial v}{\partial t}du=0, \tag{3.4}$$

for all $u\in[0,l]$.

Definition 3.1. *The flow* $\frac{\partial\alpha}{\partial t}$ *in* $\mathbb{S}_1^2$ *are said to be inextensible if*

$$\frac{\partial}{\partial t}\left|\frac{\partial\alpha}{\partial u}\right|=0.$$

Lemma 3.2. *Let* $\frac{\partial\alpha}{\partial t}=\mathfrak{f}_1^{\mathbb{S}}\alpha+\mathfrak{f}_2^{\mathbb{S}}\mathbf{t}+\mathfrak{f}_3^{\mathbb{S}}\mathbf{s}$ *be a smooth flow of the spacelike curve* α. *The flow is inextensible if and only if*

$$\frac{\partial v}{\partial t}-\frac{\partial\mathfrak{f}_2^{\mathbb{S}}}{\partial u}=\mathfrak{f}_1^{\mathbb{S}}v-\mathfrak{f}_3^{\mathbb{S}}vk_g. \tag{3.5}$$

Proof: Suppose that $\frac{\partial\alpha}{\partial t}$ be a smooth flow of the spacelike curve α.

From (3.3), we obtain

$$v\frac{\partial v}{\partial t}=\left\langle\frac{\partial\alpha}{\partial u},\frac{\partial}{\partial u}\left(\mathfrak{f}_1^{\mathbb{S}}\alpha+\mathfrak{f}_2^{\mathbb{S}}\mathbf{t}+\mathfrak{f}_3^{\mathbb{S}}\mathbf{s}\right)\right\rangle.$$

By the formula of the Sabban, we have

$$\frac{\partial v}{\partial t}=\left\langle\mathbf{t},\left(\frac{\partial\mathfrak{f}_1^{\mathbb{S}}}{\partial u}-\mathfrak{f}_2^{\mathbb{S}}v\right)\alpha+\left(\mathfrak{f}_1^{\mathbb{S}}v+\frac{\partial\mathfrak{f}_2^{\mathbb{S}}}{\partial u}-\mathfrak{f}_3^{\mathbb{S}}v\kappa_g\right)\mathbf{t}+\left(\frac{\partial\mathfrak{f}_3^{\mathbb{S}}}{\partial u}-\mathfrak{f}_2^{\mathbb{S}}v\kappa_g\right)\mathbf{s}\right\rangle.$$

Making necessary calculations from above equation, we have (3.5), which proves the lemma.

□

Theorem 3.3. *Let* $\frac{\partial \alpha}{\partial t} = \mathfrak{f}_1^{\mathbb{S}} \alpha + \mathfrak{f}_2^{\mathbb{S}} \mathbf{t} + \mathfrak{f}_3^{\mathbb{S}} \mathbf{s}$ *be a smooth flow of the spacelike curve* α. *The flow is inextensible if and only if*

$$\frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial u} = \mathfrak{f}_3^{\mathbb{S}} v \kappa_g - \mathfrak{f}_1^{\mathbb{S}} v. \tag{3.7}$$

Proof: From (3.4), we have

$$\frac{\partial}{\partial t} \sigma(u,t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left(\mathfrak{f}_1^{\mathbb{S}} v + \frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial u} - \mathfrak{f}_3^{\mathbb{S}} v \kappa_g \right) du = 0. \tag{3.8}$$

Substituting (3.5) in (3.8) complete the proof of the theorem.

□

We now restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arc length σ. We require the following lemma.

Lemma 3.4. *Let* $\frac{\partial \alpha}{\partial t} = \mathfrak{f}_1^{\mathbb{S}} \alpha + \mathfrak{f}_2^{\mathbb{S}} \mathbf{t} + \mathfrak{f}_3^{\mathbb{S}} \mathbf{s}$ *be a smooth flow of the spacelike curve* α. *Then,*

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \right) \alpha + \left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \kappa_g \right) \mathbf{s}, \tag{3.9}$$

$$\frac{\partial \alpha}{\partial t} = -\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \right) \mathbf{t} + \psi \mathbf{s}, \tag{3.10}$$

$$\frac{\partial \mathbf{s}}{\partial t} = \left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \kappa_g \right) \mathbf{t} - \psi \alpha, \tag{3.11}$$

where $\psi = \left\langle \frac{\partial \alpha}{\partial t}, \mathbf{s} \right\rangle$.

Proof: Using definition of α, we have

$$\frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \alpha}{\partial \sigma} = \frac{\partial}{\partial \sigma} (\mathfrak{f}_1^{\mathbb{S}} \alpha + \mathfrak{f}_2^{\mathbb{S}} \mathbf{t} + \mathfrak{f}_3^{\mathbb{S}} \mathbf{s}).$$

Using the Sabban equations, we have

$$\frac{\partial \mathbf{t}}{\partial t} = \left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \right) \alpha + \left(\mathfrak{f}_1^{\mathbb{S}} + \frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_3^{\mathbb{S}} \kappa_g \right) \mathbf{t} + \left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma} - \mathfrak{f}_2^{\mathbb{S}} \kappa_g \right) \mathbf{s}. \tag{3.12}$$

On the other hand, substituting (3.7) in (3.12), we get

$$\frac{\partial \mathbf{t}}{\partial t}=\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)\alpha+\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{s}.$$

Since, we express :

$$\begin{aligned}\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}+\left\langle \mathbf{t},\frac{\partial \alpha}{\partial t}\right\rangle &= 0,\\ -\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}+\mathfrak{f}_2^{\mathbb{S}}\kappa_g+\left\langle \mathbf{t},\frac{\partial \mathbf{s}}{\partial t}\right\rangle &= 0,\\ \psi+\left\langle \boldsymbol{\alpha},\frac{\partial \mathbf{s}}{\partial t}\right\rangle &= 0.\end{aligned}$$

Then, a straight forward computation using above system gives

$$\begin{aligned}\frac{\partial \boldsymbol{\alpha}}{\partial t} &= -\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)\mathbf{t}+\psi\mathbf{s},\\ \frac{\partial \mathbf{s}}{\partial t} &= \left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{t}-\psi\alpha,\end{aligned}$$

where $\psi=\left\langle \frac{\partial \boldsymbol{\alpha}}{\partial t},\mathbf{s}\right\rangle$. Thus, we obtain the theorem.

□

The following theorem states the conditions on the curvature and torsion for the flow to be inextensible.

Theorem 3.5. *Let* $\frac{\partial \alpha}{\partial t}$ *is inextensible. Then, the following system of partial differential equations holds:*

$$\begin{aligned}\frac{\partial \kappa_g}{\partial \sigma}-\psi &= \frac{\partial}{\partial \sigma}(\mathfrak{f}_2^{\mathbb{S}}\kappa_g)-\frac{\partial^2 \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma^2},\\ \kappa_g\psi &= \frac{\partial^2 \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma^2}-\frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial \sigma}.\end{aligned}$$

Proof:
Using (3.9), we have

$$\begin{aligned}\frac{\partial}{\partial \sigma}\frac{\partial \mathbf{t}}{\partial t} &= \frac{\partial}{\partial \sigma}\left[\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)\alpha+\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{s}\right]\\ &= \left(\frac{\partial^2 \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma^2}-\frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial \sigma}\right)\alpha+[\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)-\kappa_g\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}+\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)]\mathbf{t}\\ &\quad+\left(\frac{\partial^2 \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma^2}-\frac{\partial}{\partial \sigma}(\mathfrak{f}_2^{\mathbb{S}}\kappa_g)\right)\mathbf{s}.\end{aligned}$$

On the other hand, from Sabban frame we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial \mathbf{t}}{\partial \sigma} &= \frac{\partial}{\partial t}(-\alpha-\kappa_g\mathbf{s}) \\ &= (\psi-\frac{\partial \kappa_g}{\partial \sigma})\mathbf{s}+[\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)+\kappa_g\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)]\mathbf{t}+\kappa_g\psi\alpha.\end{aligned}$$

Hence we see that

$$\frac{\partial \kappa_g}{\partial \sigma}-\psi=\frac{\partial}{\partial \sigma}(\mathfrak{f}_2^{\mathbb{S}}\kappa_g)-\frac{\partial^2 \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma^2}.$$

and

$$\kappa_g\psi=\frac{\partial^2 \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma^2}-\frac{\partial \mathfrak{f}_2^{\mathbb{S}}}{\partial \sigma}.$$

Thus, we obtain the theorem. □

Theorem 3.6. *Let* $\frac{\partial \alpha}{\partial t}=\mathfrak{f}_1^{\mathbb{S}}\alpha+\mathfrak{f}_2^{\mathbb{S}}\mathbf{t}+\mathfrak{f}_3^{\mathbb{S}}\mathbf{s}$ *be a smooth flow of the spacelike curve* α*. Then,*

$$\kappa_g\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)=\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}+\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)+\frac{\partial \psi}{\partial \sigma}.$$

Proof:

Similarly, we have

$$\frac{\partial}{\partial \sigma}\frac{\partial \mathbf{s}}{\partial t}=\frac{\partial}{\partial \sigma}\left[\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{t}-\psi\alpha\right]$$

$$=[\left(\frac{\partial^2 \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\frac{\partial}{\partial \sigma}\left(\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)-\psi\right)\mathbf{t}-\kappa_g\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{s}$$

$$+[-\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}+\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)-\frac{\partial \psi}{\partial \sigma}]\alpha].$$

On the other hand, a straightforward computation gives

$$\frac{\partial}{\partial t}\frac{\partial \mathbf{s}}{\partial \sigma}=\frac{\partial}{\partial t}\left(-\kappa_g\mathbf{t}\right)$$

$$=-\frac{\partial \kappa_g}{\partial t}\mathbf{t}-\kappa_g\left[\left(\frac{\partial \mathfrak{f}_1^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\right)\alpha+\left(\frac{\partial \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)\mathbf{s}\right].$$

Combining these we obtain the corollary. □

In the light of Theorem 3.6, we express the following corollary without proof:

Corollary 3.7.

$$\psi=\frac{\partial^2 \mathfrak{f}_3^{\mathbb{S}}}{\partial \sigma}-\frac{\partial}{\partial \sigma}\left(\mathfrak{f}_2^{\mathbb{S}}\kappa_g\right)+\frac{\partial \kappa_g}{\partial t}.$$

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Mahmut Ergüt
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: `mergut@firat.edu.tr`

and

Essin Turhan
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: `essin.turhan@gmail.com`

and

Talat Körpınar
Fırat University,
Department of Mathematics,
23119 Elazığ, Turkey
E-mail address: `talatkorpinar@gmail.com`