Characterization Of Inextensible Flows Of Spacelike Curves With Sabban Frame In $S^{2}_{1}$

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ABSTRACT: In this paper, we study inextensible flows of spacelike curves on $S^{2}_{1}$. We obtain partial differential equations in terms of inextensible flows of spacelike curves according to Sabban frame on $S^{2}_{1}$.

Key Words: Inextensible flows, Sabban Frame.

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1. Introduction

Physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications.

This study is organised as follows: Firstly, we study inextensible flows of spacelike curves on $S^{2}_{1}$. Secondly, we obtain partial differential equations in terms of inextensible flows of spacelike curves according to Sabban frame on $S^{2}_{1}$.

2. Preliminaries

The Minkowski 3-space $E^{3}$ provided with the standard flat metric given by

$$\langle , \rangle = -dx_{1}^{2} + dx_{2}^{2} + dx_{3}^{2},$$

where $(x_{1}, x_{2}, x_{3})$ is a rectangular coordinate system of $E^{3}_{1}$. Recall that, the norm of an arbitrary vector $a \in E^{3}_{1}$ is given by $\|a\| = \sqrt{\langle a, a \rangle}$. $\gamma$ is called a unit speed curve if velocity vector $v$ of $\gamma$ satisfies $\|a\| = 1$.

Denote by $\{T, N, B\}$ the moving Frenet–Serret frame along the spacelike curve $\gamma$ in the space $E^{3}_{1}$. For an arbitrary spacelike curve $\gamma$ with first and second curvature,
κ and τ in the space $E^3_1$, the following Frenet–Serret formulae is given

\[
\begin{align*}
T' &= \kappa N \\
N' &= -\kappa T + \tau B \\
B' &= \tau N,
\end{align*}
\] (2.1)

where

\[
\begin{align*}
\langle T, T \rangle &= 1, \langle N, N \rangle = 1, \langle B, B \rangle = -1, \\
\langle T, N \rangle &= \langle T, B \rangle = \langle N, B \rangle = 0.
\end{align*}
\]

Here, curvature functions are defined by \(\kappa = \kappa(s) = \|T'(s)\|\) and \(\tau(s) = -\langle N, B' \rangle\).

Torsion of the spacelike curve \(\gamma\) is given by the aid of the mixed product

\[\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.\]

Now we give a new frame different from Frenet frame. Let \(\alpha : I \longrightarrow S^2_1\) be unit speed spherical curve. We denote \(\sigma\) as the arc-length parameter of \(\alpha\). Let us denote \(t_\sigma = \alpha'(\sigma)\), and we call \(t_\sigma\) a unit tangent vector of \(\alpha\). We now set a vector \(s_\sigma = \alpha(\sigma) \times t_\sigma\) along \(\alpha\). This frame is called the Sabban frame of \(\alpha\) on \(S^2_1\). Then we have the following spherical Frenet-Serret formulae of \(\alpha\):

\[
\begin{align*}
\alpha' &= t, \\
t' &= -\alpha - \kappa_g s, \\
s' &= -\kappa_g t,
\end{align*}
\] (2.2)

where \(\kappa_g\) is the geodesic curvature of the spacelike curve \(\alpha\) on the \(S^2_1\) and

\[
\begin{align*}
g(t, t) &= 1, g(\alpha, \alpha) = 1, g(s, s) = -1, \\
g(t, \alpha) &= g(t, s) = g(\alpha, s) = 0.
\end{align*}
\]

3. Inextensible Flows of Curves According to Sabban Frame in \(S^2_1\)

Let \(\alpha(u, t)\) is a one parameter family of smooth spacelike curves in \(S^2_1\). The arclength of \(\alpha\) is given by

\[
\sigma(u) = \int_0^u \left| \frac{\partial \alpha}{\partial u} \right| du,
\] (3.1)

where

\[
\left| \frac{\partial \alpha}{\partial u} \right| = \left( \left| \frac{\partial \alpha}{\partial u} \cdot \frac{\partial \alpha}{\partial u} \right| \right)^{1/2}. \] (3.2)
The operator $\frac{\partial}{\partial \sigma}$ is given in terms of $u$ by

$$\frac{\partial}{\partial \sigma} = \frac{1}{\nu} \frac{\partial}{\partial u},$$

where $\nu = \left| \frac{\partial \alpha}{\partial u} \right|$ and the arclength parameter is $d\sigma = vdu$.

Any flow of $\alpha$ can be represented as

$$\frac{\partial \alpha}{\partial t} = f_1^S \alpha + f_2^S t + f_3^S s.$$ (3.3)

Letting the arclength variation be

$$\sigma(u, t) = \int_0^u vdu.$$

In the $S_1^2$ the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t} \sigma(u, t) = \int_0^u \frac{\partial v}{\partial t} du = 0,$$ (3.4)

for all $u \in [0, l]$.

**Definition 3.1.** The flow $\frac{\partial \alpha}{\partial t}$ in $S_1^2$ are said to be inextensible if

$$\frac{\partial}{\partial t} \left| \frac{\partial \alpha}{\partial u} \right| = 0.$$

**Lemma 3.2.** Let $\frac{\partial \alpha}{\partial t} = f_1^S \alpha + f_2^S t + f_3^S s$ be a smooth flow of the spacelike curve $\alpha$. The flow is inextensible if and only if

$$\frac{\partial v}{\partial t} - \frac{\partial f_2^S}{\partial u} = f_1^S v - f_3^S v \kappa g.$$ (3.5)

**Proof:** Suppose that $\frac{\partial \alpha}{\partial t}$ be a smooth flow of the spacelike curve $\alpha$.

From (3.3), we obtain

$$v \frac{\partial v}{\partial t} = \left\langle \frac{\partial \alpha}{\partial u}, \frac{\partial}{\partial u} \left( f_1^S \alpha + f_2^S t + f_3^S s \right) \right\rangle.$$

By the formula of the Sabban, we have

$$\frac{\partial v}{\partial t} = \left\langle t, \left( \frac{\partial f_1^S}{\partial u} - f_2^S v \right) \alpha + \left( f_1^S v + \frac{\partial f_2^S}{\partial u} - f_3^S v \kappa g \right) t + \left( \frac{\partial f_3^S}{\partial u} - f_2^S v \kappa g \right) s \right\rangle.$$
Making necessary calculations from above equation, we have (3.5), which proves the lemma.

\[ \frac{\partial f_2}{\partial u} = f_3^v\kappa_g - f_1^v. \] (3.7)

**Proof:** From (3.4), we have

\[ \frac{\partial}{\partial t} \sigma(u, t) = \int_0^u \frac{\partial v}{\partial t} du = \int_0^u \left( f_1^v + \frac{\partial f_2^s}{\partial u} - f_3^v\kappa_g \right) du = 0. \] (3.8)

Substituting (3.5) in (3.8) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves. That is, \( v = 1 \) and the local coordinate \( u \) corresponds to the curve arc length \( \sigma \). We require the following lemma.

**Lemma 3.4.** Let \( \frac{\partial \alpha}{\partial t} = f_1^\alpha + f_2^t + f_3^s \) be a smooth flow of the spacelike curve \( \alpha \). Then,

\[ \frac{\partial t}{\partial t} = \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) \alpha + \left( f_3^s - f_3^s \kappa_g \right) s, \] (3.9)

\[ \frac{\partial \alpha}{\partial t} = - \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) t + \psi s, \] (3.10)

\[ \frac{\partial s}{\partial t} = \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \kappa_g \right) t - \psi \alpha, \] (3.11)

where \( \psi = \langle \frac{\partial \alpha}{\partial t}, s \rangle \).

**Proof:** Using definition of \( \alpha \), we have

\[ \frac{\partial t}{\partial t} = \frac{\partial \alpha}{\partial t} = \frac{\partial}{\partial \sigma} (f_1^\alpha + f_2^t + f_3^s). \]

Using the Sabban equations, we have

\[ \frac{\partial t}{\partial t} = \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) \alpha + \left( f_3^s + \frac{\partial f_2^t}{\partial \sigma} - f_3^s \kappa_g \right) t + \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \kappa_g \right) s. \] (3.12)
On the other hand, substituting (3.7) in (3.12), we get
\[
\frac{\partial \mathbf{t}}{\partial t} = \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) \alpha + \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \kappa_g \right) s.
\]
Since, we express:
\[
\frac{\partial f_1^s}{\partial \sigma} - f_2^s + \left< t, \frac{\partial \alpha}{\partial t} \right> = 0,
\]
\[
- \frac{\partial f_3^s}{\partial \sigma} + f_2^s \kappa_g + \left< t, \frac{\partial s}{\partial t} \right> = 0,
\]
\[
\psi + \left< \alpha, \frac{\partial s}{\partial t} \right> = 0.
\]
Then, a straightforward computation using above system gives
\[
\frac{\partial \alpha}{\partial t} = - \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) t + \psi s,
\]
\[
\frac{\partial s}{\partial t} = \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \kappa_g \right) t - \psi \alpha,
\]
where \( \psi = \left< \frac{\partial \alpha}{\partial t}, s \right> \). Thus, we obtain the theorem. \( \square \)

The following theorem states the conditions on the curvature and torsion for the flow to be inextensible.

**Theorem 3.5.** Let \( \frac{\partial \alpha}{\partial t} \) is inextensible. Then, the following system of partial differential equations holds:
\[
\frac{\partial \kappa_g}{\partial \sigma} - \psi = \frac{\partial}{\partial \sigma} \left( f_2^s \kappa_g \right) - \frac{\partial^2 f_3^s}{\partial \sigma^2},
\]
\[
\kappa_g \psi = \frac{\partial^2 f_1^s}{\partial \sigma^2} - \frac{\partial f_3^s}{\partial \sigma}.
\]

**Proof:**
Using (3.9), we have
\[
\frac{\partial}{\partial \sigma} \frac{\partial \mathbf{t}}{\partial t} = \frac{\partial}{\partial \sigma} \left[ \left( \frac{\partial f_1^s}{\partial \sigma} - f_2^s \right) \alpha + \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \kappa_g \right) s \right]
\]
\[
= \left( \frac{\partial^2 f_1^s}{\partial \sigma^2} - \frac{\partial f_1^s}{\partial \sigma} \right) \alpha + \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \right) \kappa_g - \kappa_g \left( \frac{\partial f_3^s}{\partial \sigma} + f_2^s \kappa_g \right) t
\]
\[
+ \left( \frac{\partial^2 f_3^s}{\partial \sigma^2} - \frac{\partial f_3^s}{\partial \sigma} \right) \kappa_g s.
\]
On the other hand, from Sabban frame we have
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial \sigma} = \frac{\partial}{\partial t} (-\alpha - \kappa \frac{\partial s}{\partial \sigma})
\]
\[
= (\psi - \frac{\partial \kappa_g}{\partial \sigma}) s + \kappa \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \right) + \kappa \left( \frac{\partial f_3^s}{\partial \sigma} - \frac{\partial f_2^s}{\partial \sigma} \right) t + \kappa \psi \alpha.
\]

Hence we see that
\[
\frac{\partial \kappa_g}{\partial \sigma} - \psi = \frac{\partial}{\partial \sigma} \left( f_2^s \kappa_g \right) - \frac{\partial f_2^s}{\partial \sigma^2}
\]
and
\[
\kappa \psi = \frac{\partial^2 f_3^s}{\partial \sigma^2} - \frac{\partial f_2^s}{\partial \sigma}.
\]

Thus, we obtain the theorem. \(\square\)

**Theorem 3.6.** Let \(\frac{\partial \alpha}{\partial t} = f_1^s \alpha + f_2^s t + f_3^s s\) be a smooth flow of the spacelike curve \(\alpha\). Then,
\[
\kappa_g \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \right) = \left( \frac{\partial f_3^s}{\partial \sigma} + f_2^s \kappa_g \right) + \frac{\partial \psi}{\partial \sigma}.
\]

**Proof:**
Similarly, we have
\[
\frac{\partial}{\partial \sigma} \frac{\partial}{\partial t} = \frac{\partial}{\partial \sigma} \left[ \left( \frac{\partial f_3^s}{\partial \sigma} - \frac{\partial f_2^s}{\partial \sigma} \right) t - \psi \alpha \right]
\]
\[
= \left[ \left( \frac{\partial^2 f_3^s}{\partial \sigma^2} - \frac{\partial f_2^s}{\partial \sigma} \left( \frac{\partial f_3^s}{\partial \sigma} - \psi \right) \right) t - \kappa_g \left( \frac{\partial f_3^s}{\partial \sigma} - \frac{\partial f_2^s}{\partial \sigma} \right) s \right]
\]
\[
+ \left[ - \left( \frac{\partial f_3^s}{\partial \sigma} + f_2^s \kappa_g \right) - \frac{\partial \psi}{\partial \sigma} \alpha \right].
\]

On the other hand, a straightforward computation gives
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial \sigma} = \frac{\partial}{\partial t} \left( -\kappa \frac{\partial \alpha}{\partial \sigma} \right)
\]
\[
= - \frac{\partial \kappa_g}{\partial t} \kappa t - \kappa_g \left( \frac{\partial f_3^s}{\partial \sigma} - f_2^s \right) \alpha + \left( \frac{\partial f_3^s}{\partial \sigma} - \frac{\partial f_2^s}{\partial \sigma} \right) s.
\]

Combining these we obtain the corollary. \(\square\)

In the light of Theorem 3.6, we express the following corollary without proof:

**Corollary 3.7.**
\[
\psi = \frac{\partial^2 f_3^s}{\partial \sigma^2} - \frac{\partial}{\partial \sigma} \left( f_2^s \kappa_g \right) + \frac{\partial \kappa_g}{\partial t}.
\]
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