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# Biharmonic $\mathcal{B}-$Slant Helices According To Bishop Frame In The $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

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ABSTRACT: In this paper, we study biharmonic $\mathcal{B}-$slant helices in the $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. We characterize the biharmonic $\mathcal{B}-$slant helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations.

Key Words: Biharmonic curve, $\widetilde{\mathcal{SL}_2(\mathbb{R})}$, curvature, torsion.

**Contents**



## 1. Introduction

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between manifolds are the critical points of the energy

$$E(f) = \frac{1}{2}\int_M e(f)\, v_g, \tag{1.1}$$

where $v_g$ is the volume form on $(M, g)$ and

$$e(f)(x) := \frac{1}{2}\|df(x)\|^2_{T^*M\otimes f^{-1}TN}$$

is the energy density of f at the point $x \in M$.

Critical points of the energy functional are called harmonic maps.

The first variational formula of the energy gives the following characterization of harmonic maps: the map f is harmonic if and only if its tension field $\tau(f)$ vanishes identically, where the tension field is given by

$$\tau(f) = \text{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [6], we can define the bienergy of a map $f$ by

$$E_2(f) = \frac{1}{2}\int_M \|\tau(f)\|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

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Jiang derived the first and the second variation formula for the bienergy in [7,8], showing that the Euler–Lagrange equation associated to $E_2$ is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \operatorname{trace} R^N(df, \tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$ . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

This study is organised as follows: Firstly, we study biharmonic $\mathcal{B}-$slant helices in the $\widetilde{\mathcal{SL}_2(\mathbb{R})}$. Secondly, we characterize the biharmonic $\mathcal{B}-$slant helices in terms of their curvature and torsion. Finally, we find out their explicit parametric equations.

## 2. Riemannian Structure of $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

We identify $\widetilde{\mathcal{SL}_2(\mathbb{R})}$ with

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : z > 0\right\}$$

endowed with the metric

$$g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}} = ds^2 = (dx + \frac{dy}{z})^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, \mathbf{e}_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

The characterising properties of $g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}$ defined by

$$\begin{aligned} g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_1, \mathbf{e}_1) &= g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_2, \mathbf{e}_2) = g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_3, \mathbf{e}_3) = 1, \\ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_1, \mathbf{e}_2) &= g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_2, \mathbf{e}_3) = g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{e}_1, \mathbf{e}_3) = 0. \end{aligned}$$

The Riemannian connection $\nabla$ of the metric $g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}$ is given by

$$\begin{aligned} 2g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\nabla_X Y, Z) &= X g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(Y, Z) + Y g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(Z, X) - Z g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(X, Y) \\ &\quad - g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(X, [Y, Z]) - g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(Y, [X, Z]) + g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= 0, & \nabla_{\mathbf{e}_1}\mathbf{e}_2 &= \frac{1}{2}\mathbf{e}_3, & \nabla_{\mathbf{e}_1}\mathbf{e}_3 &= -\frac{1}{2}\mathbf{e}_2, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 &= \frac{1}{2}\mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_2 &= \mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_3 &= -\frac{1}{2}\mathbf{e}_1 - \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 &= -\frac{1}{2}\mathbf{e}_2, & \nabla_{\mathbf{e}_3}\mathbf{e}_2 &= \frac{1}{2}\mathbf{e}_1, & \nabla_{\mathbf{e}_3}\mathbf{e}_3 &= 0. \end{aligned} \tag{2.2}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices $i, j, k$ and $l$ take the values $1, 2$ and $3$

$$R_{1212} = R_{1313} = \frac{1}{4}, \quad R_{2323} = -\frac{7}{4}. \tag{2.3}$$

## 3. Biharmonic $\mathcal{B}-$Slant Helices in $\widetilde{\mathcal{SL}_2(\mathbb{R})}$

Assume that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$\begin{aligned} g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{T}) &= 1, \ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{N}, \mathbf{N}) = 1, \ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{B}, \mathbf{B}) = 1, \\ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{N}) &= g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{B}) = g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{T}) &= 1, \ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{M}_1, \mathbf{M}_1) = 1, \ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{M}_1) &= g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{T}, \mathbf{M}_2) = g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{M}_1, \mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\mathfrak{Y}(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \mathfrak{Y}'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s) \cos \mathfrak{Y}(s), \\ k_2 &= \kappa(s) \sin \mathfrak{Y}(s). \end{aligned}$$

The relation matrix may be expressed as

$$\begin{gathered} \mathbf{T} = \mathbf{T}, \\ \mathbf{N} = \cos \mathfrak{Y}(s)\, \mathbf{M}_1 + \sin \mathfrak{Y}(s)\, \mathbf{M}_2, \\ \mathbf{B} = -\sin \mathfrak{Y}(s)\, \mathbf{M}_1 + \cos \mathfrak{Y}(s)\, \mathbf{M}_2. \end{gathered}$$

On the other hand, using above equation we have

$$\mathbf{T} = \mathbf{T},$$
$$\mathbf{M}_1 = \cos\mathfrak{Y}(s)\,\mathbf{N} - \sin\mathfrak{Y}(s)\,\mathbf{B}$$
$$\mathbf{M}_2 = \sin\mathfrak{Y}(s)\,\mathbf{N} + \cos\mathfrak{Y}(s)\,\mathbf{B}.$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3. \end{aligned} \tag{3.5}$$

**Theorem 3.1.** $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \textit{constant} \neq 0, \\ k_1'' - \left[k_1^2 + k_2^2\right]k_1 &= -k_1\left[\frac{15}{4}M_2^1 - \frac{1}{4}\right] - 2k_2M_1^1M_2^1, \\ k_2'' - \left[k_1^2 + k_2^2\right]k_2 &= 2k_1M_1^1M_2^1 - k_2\left[\frac{15}{4}M_1^1 - \frac{1}{4}\right]. \end{aligned} \tag{3.6}$$

**Definition 3.2.** *A regular curve* $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *is called a slant helix provided the unit vector* $\mathbf{M}_1$ *of the curve* $\gamma$ *has constant angle* $\mathfrak{W}$ *with some fixed unit vector* $u$, *that is*

$$g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}\left(\mathbf{M}_1(s), u\right) = \cos\mathfrak{W} \textit{ for all } s \in I. \tag{3.7}$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. The slant helices can be identified by a simple condition on natural curvatures.

To separate a slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathcal{B}$-slant helix.

We shall also use the following lemma.

**Lemma 3.3.** *Let* $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *be a unit speed biharmonic curve. Then* $\gamma$ *is a biharmonic* $\mathcal{B}$*-slant helix if and only if*

$$k_1 = -k_2\tan\mathfrak{W}. \tag{3.8}$$

**Theorem 3.4.** *Let* $\gamma : I \longrightarrow \widetilde{\mathcal{SL}_2(\mathbb{R})}$ *be a unit speed non-geodesic biharmonic*

*$\mathcal{B}-$slant helix. Then, the parametric equations of $\gamma$ are*

$$\begin{aligned}
\boldsymbol{x}(s) &= \frac{1}{\mathfrak{Q}_1}\cos\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2]+\frac{1}{\mathfrak{Q}_1}\cos\mathfrak{W}\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]+\mathfrak{Q}_4,\\
\boldsymbol{y}(s) &= -\frac{\mathfrak{Q}_3}{\mathfrak{Q}_1^2+\sin^2\mathfrak{W}}\cos\mathfrak{W}e^{-\sin\mathfrak{W}s}(\mathfrak{Q}_1\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\\
&\quad+\sin\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2])+\mathfrak{Q}_5,\\
\boldsymbol{z}(s) &= \mathfrak{Q}_3 e^{-\sin\mathfrak{W}s},
\end{aligned}\tag{3.9}$$

*where $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3,\mathfrak{Q}_4,\mathfrak{Q}_5$ are constants of integration.*

**Proof:** We suppose that $\gamma$ is a unit speed non-geodesic biharmonic $\mathcal{B}-$slant helix. Since $\gamma$ is biharmonic $\mathcal{B}-$slant helix without loss of generality, we take

$$g_{\widetilde{\mathcal{SL}_2(\mathbb{R})}}(\mathbf{M}_1,\mathbf{e}_3)=M_1^3=\cos\mathfrak{W},\tag{3.10}$$

where $\mathfrak{W}$ is constant angle.

On the other hand, the vector $\mathbf{M}_1$ is a unit vector, we have the following equation

$$\mathbf{M}_1=\sin\mathfrak{W}\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_1+\sin\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_2+\cos\mathfrak{W}\mathbf{e}_3,\tag{3.11}$$

where $\mathfrak{Q}_1,\mathfrak{Q}_2$ are constants of integration.

On the other hand, using Bishop formulas (3.3) and (2.1), we have

$$\mathbf{M}_2=\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_1-\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_2.\tag{3.12}$$

Using above equation and (3.11), we get

$$\mathbf{T}=\cos\mathfrak{W}\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_1+\cos\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\mathbf{e}_2-\sin\mathfrak{W}\mathbf{e}_3.\tag{3.13}$$

Using (2.1) in (3.13), we obtain

$$\mathbf{T}=(\cos\mathfrak{W}\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]-\cos\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2],z\cos\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2],-z\sin\mathfrak{W}).\tag{3.14}$$

By direct calculations we have (3.9), which proves our assertion. □

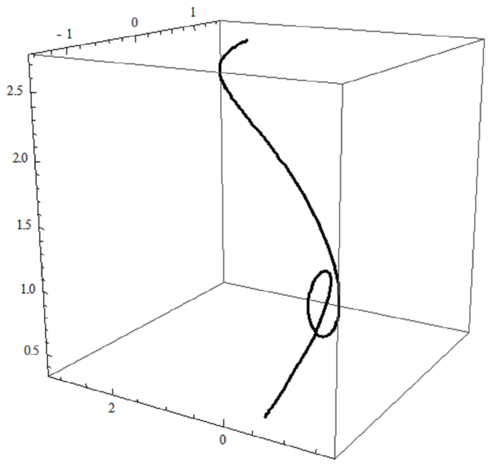
In the light of above theorem, we express the following result without proof:

**Corollary 3.5.** *Let $\gamma: I\longrightarrow\widetilde{\mathcal{SL}_2(\mathbb{R})}$ be a unit speed non-geodesic biharmonic $\mathcal{B}-$slant helix. Then, the parametric equations of $\gamma$ in terms of Bishop curvatures* *are*

$$\begin{aligned}
\boldsymbol{x}(s) &= -\frac{k_2}{k_1\mathfrak{Q}_1}\sin\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2]-\frac{k_2}{k_1\mathfrak{Q}_1}\sin\mathfrak{W}\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]+\mathfrak{Q}_4,\\
\boldsymbol{y}(s) &= \frac{k_2^3\mathfrak{Q}_3}{k_1k_2^2\mathfrak{Q}_1^2+k_1^3\cos^2\mathfrak{W}}\sin\mathfrak{W}e^{\frac{k_1}{k_2}\cos\mathfrak{W}s}(\mathfrak{Q}_1\cos[\mathfrak{Q}_1 s+\mathfrak{Q}_2]\\
&\quad -\frac{k_1}{k_2}\cos\mathfrak{W}\sin[\mathfrak{Q}_1 s+\mathfrak{Q}_2])+\mathfrak{Q}_5,\\
\boldsymbol{z}(s) &= \mathfrak{Q}_3 e^{\frac{k_1}{k_2}\cos\mathfrak{W}s},
\end{aligned}$$

*where* $\mathfrak{Q}_1,\mathfrak{Q}_2,\mathfrak{Q}_3,\mathfrak{Q}_4,\mathfrak{Q}_5$ *are constants of integration.*

We can use Mathematica in above theorem, yields

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